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Review of Multiresolution Analysis and Wavelets on Locally Compact Abelian Groups

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Abstract

This paper attempts to discuss the concept of multiresolution analysis and wavelets on a locally compact Abelian group. It reviews recent published works dealing with wavelets and its properties on a locally compact Abelian group. After revisiting the basics of a locally compact Abelian group and wavelets, the theory of multiresolution analysis and wavelets are extended from the real space to a locally compact Abelian group.

Keywords: Wavelets, multiresolution analysis, locally compact Abelian group, local field, wavelet sets, Cantor dyadic group, Vilenkin group.

1. INTRODUCTION

In December 21, 1807, Joseph Fourier claimed that, “every periodic function can be expressed as a weighted sum of sines and cosines” and introduced the Fourier theory. Eventhough Fourier analysis is useful in many areas, it has a drawback. That is, the Fourier transform lacks the resolution between the frequency and time domain. This means that although we might be able to determine all the frequencies present in the signal, we do not know when they are present. Wavelet theory is probably the most recent solution to overcome the shortcomings of the Fourier transform.

The word wavelet was first introduced by Grossmann and Morlet in the early 1980s. They used the French word *ondelette*, meaning small wave. In the past few years, the study of wavelets have brought about sweeping changes in the disciplines of pure and applied mathematics and science. As a mathematical tool, wavelets can be used to extract information from many different kinds of data, including audio signals and images. The basic idea of wavelet theory is very elementary: Try to compose a basis of Lebesgue space from integer shifts of dyadic dilates of one function, called a wavelet. The first wavelet construction is due to Haar. He presented an orthonormal basis of Lebesgue space generated by the box function. A very important notion of mutiresolution analysis was invented by Meyer and Mallat. A multiresolution analysis of the Lebesgue space consists of sequences of nested subspaces that satisfies certain self-similarity relations in time-space and scale-frequency, as well as completeness and regularity relations. Wavelets are better signal representations because of multiresolution analysis. This motivates why wavelet transforms are now being adapted for a vast number of applications. Wavelets are already recognized as a powerful new mathematical tool in signal and image processing, time series analysis, geophysics, approximation theory, and many other areas.

In this review, we summarize the concepts of multiresolution analysis and wavelets on a locally compact Abelian group. In Section 2, we gave the basic definitions of multiresolution analysis, locally compact Abelian group and local field. Previous researches in the area of wavelets on locally compact Abelian group are summarized in Section 3. In Section 4, we considered some particular locally compact Abelian groups such as Cantor dyadic group and Vilenkin group, and discussed the notion of wavelets there. Finally, conclusions are drawn and future directions are suggested in Section 5.

2. PRELIMINARY

A **wavelet system**[1] for $L^2(\mathbb{R})$ is a complete orthonormal basis of the form $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for some $\psi \in L^2(\mathbb{R})$, where $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$. The functions $\psi_{j,k},\ j,k \in \mathbb{Z}$ are called wavelets and the function ψ is called the mother wavelet.

A **multiresolution analysis** for $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\phi \in V_0$, such that the following conditions holds:

1. $\cdots V_{-1} \subset V_0 \subset V_1 \subset \cdots$
2. $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$ and $\bigcap_j V_j = \{0\}$
3. $f \in V_j \Leftrightarrow [x \longmapsto f(2x)] \in V_{j+1}$

4. $f \in V_0 \Rightarrow T_k f \in V_0, \;\; \forall k \in \mathbb{Z}$
5. $\{T_k \phi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

Assume that the conditions above are satisfied. For $j \in \mathbb{Z}$, let W_j denote the orthogonal complement of V_j in V_{j+1}. Letting Q_j denote the orthogonal projection onto W_j, it follows from (3) and (4) that each $f \in L^2(\mathbb{R})$ has a representation $f = \sum_{j\in\mathbb{Z}} Q_j f$, where $Q_j f \perp Q_{j'} f$ for each $j \neq j'$. That is,

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$$

The spaces W_j satisfies the same dilation relationship as V_j. i.e.,

$$\psi \in W_0 \Leftrightarrow [x \longmapsto \psi(2^j x)] \in W_j$$

In order to obtain an orthonormal basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, it is enough to find $\psi \in W_0$ such that $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_0; using the dilation property (3) and (4), this implies that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. Such functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ are called MRA wavelets.[1]

For a given multiresolution analysis and the corresponding orthonormal wavelet basis of $L^2(\mathbb{R})$, **wavelet packets** were constructed by Coifman, Meyer and Wickerhauser.[2] A wavelet packet ψ is a square integrable function with mean 0, that is well localized in both position and frequency. These functions are very powerful terms in the theory of wavelets, because of its flexibility in representing different kinds of signals.

A **locally compact Abelian group** is a topological group whose topology is locally compact and Hausdorff. Every locally compact Abelian group possess a Haar measure which is unique upto a multiplicative constant. The collection of all unitary homomorphisms from G (a locally compact Abelian group) to the circle group $\mathbb{T}$ are called the dual group of G, denoted by $\widehat{G}$, and such homomorphisms are called **characters** of G.[3]

Let G be a locally compact Abelian group. $L^2(G)$ is the space of square integrable functions on G, and the Fourier transform of $f \in L^2(G)$ is the function $\widehat{f} \in L^2(\widehat{G})$ formally defined as

$$\widehat{f}(\gamma) = \int_G f(x)\overline{(x,\gamma)}d\mu(x)$$

where μ denotes the Haar measure on G, $\gamma \in \widehat{G}$ and (x,γ) denotes the action of duality between G and $\widehat{G}$.[4]

A group G with discrete topology is called a **discrete group**.[5]

Let K be a field and a topological space. Then K is called a **locally compact field** or a **local field** if both K^+ and K^* are locally compact abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. So by a local field, we mean a field K which is locally compact, nondiscrete and totally disconnected.

3. WAVELETS ON LOCALLY COMPACT ABELIAN GROUP

Let G be a locally compact Abelian group and suppose that G contains a discrete countable subgroup K such that the quotient group G/K is compact. Furthermore, let us assume that there exist an automorphism A of G such that $A(K) \subset K$. Then a multiresolution analysis of $L^2(G,\mu)$ where μ denotes the Haar measure on G, is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(G,\mu)$ such that

1. $\cdots V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots$
2. $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(G,\mu)$ and $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$
3. $f(g) \in V_j \Leftrightarrow f(A(g)) \in V_{j+1}$
4. $f(g) \in V_0 \Leftrightarrow f(g \circ k) \in V_0, \ \forall\, k \in K$ where $\circ$ is the group operation of G.
5. There exists a function ϕ such that the collection $\{\phi(\cdot \circ k)\}_{k\in K}$ of translates of ϕ are stable and V_0 is the closed linear span of $\phi(\cdot \circ k)$.

Now a system of functions $\{\psi_1, \psi_2, \dots \psi_N\}$ is called a family of (mother) wavelets if the translates of $\{\psi_1, \psi_2, \dots \psi_N\}$ span the orthogonal complement W_0 of V_0 in V_1. Since the union of the spaces $\{V_j\}_{j\in\mathbb{Z}}$ is dense in $L^2(G,\mu)$ where their intersection is zero, we can see that $L^2(G,\mu)$ is spanned by all scaled and translated versions of the functions $\{\psi_1, \psi_2, \dots \psi_N\}$. Then the collection $\Psi = \{\psi_1, \psi_2, \dots \psi_N\}$ is a wavelet generator for $L^2(G,\mu)$. If $\Psi = \{\psi\}$, then ψ is a single wavelet for $L^2(G,\mu)$.

Let $\Omega_1, \Omega_2, \dots \Omega_N$ be measurable subsets of $\widehat{G}$, and let $\psi_j = \check{1}_{\Omega_j}$, for each $j = 1, 2, \dots N$. We say that $\{\Omega_1, \Omega_2, \dots \Omega_N\}$ is a **wavelet collection of sets** if $\Psi = \{\psi_1, \psi_2, \dots \psi_N\}$ is a wavelet generator for $L^2(G,\mu)$. If $N = 1$, then $\Omega = \Omega_1$ is a **wavelet set**.

A measurable subset F of $\widehat{G}$ is called a **scaling set** if $\varphi = \check{\chi}_F$ is a scaling function of a multiresolution analysis in $L^2(G)$.

A measurable subset F of $\widehat{G}$ is called a **generalized scaling set** if $|F| = 1$ and $(A(F)/F)$ is a wavelet set. (For a Cantor dyadic group G, $\widehat{G}$ is topologically isomorphic to G itself. So $A(F)$ well defined.)

The following literature review discusses the results obtained in the wavelet theory on a locally compact Abelian group. In 1994, S. Dahlke[6] generalized the concept of multiresolution analysis approximation of functions to locally compact Abelian group. He first defined the generalized B-splines on locally compact Abelian group. Most important properties of cardinal B-splines on $\mathbb{R}$ carry over to the general situation. Then he studied under what conditions the generalized B-splines generate a multiresolution analysis. He also gave some examples and suggested an outline on how the resulting generalized B-spline wavelets can be constructed.

In his paper, T. P. Lukashenko[7] proposed more general constructions of wavelets on locally compact Abelian group. He proved analogues of the Parseval-Plancheral identity and the reconstruction formula (inversion formula). Wavelets and Gabor functions are special cases of these constructions. He also mentioned examples of other families of functions for which analogues of the Parseval-Plancheral identity and the reconstruction formula are true.

F. Galindo and J. Sanz[8] also extended the definition of multiresolution analysis to a locally compact Abelian group and characterized the spaces of integrable functions $L^p(G, \mu)$ and the complex Radon measure $M(G)$ in terms of this multiresolution analysis.

J.J. Benedetto and M. Leon[9] constructed single dyadic orthonormal wavelet for a d-dimensional Euclidean space using the concept of wavelet sets. They developed an algorithm for the construction of wavelet sets and illustrated the computer implementation of these wavelet sets.

In his paper, R.L. Benedetto[10] proposed a theory of wavelets on locally compact Abelian group and constructed wavelet sets. He considered some examples of wavelets on such groups, in particular, Haar wavelet and Shannon wavelet. These two wavelets are at opposite extremes in Euclidean spaces. But, he proved that they are same on a locally compact Abelian group.

J.J. Benedetto and R.L. Benedetto[11] together developed some theories related to wavelets on a locally compact Abelian group. They defined the translation and dilation on locally compact Abelian group and characterized the wavelet collection of sets. They also provided an algorithm for the construction of wavelet sets and finally examined some examples of such wavelets. They mainly deals with the non-

Euclidean theory of wavelets.[12] Many examples are studied in both the Euclidean and non-Euclidean theory of wavelets. They discussed about Shannon wavelet set, Haar wavelet set, wedding cake set, wedding night set, wavelet sets on p-adic sets, etc.

R. A. Kamyabi Gol and R. R. Tousi[13] also contributed some new notions in this theory. They extended the definition of spectral function of a shift invariant space to a locally compact Abelian group and found the conditions under which a function generates a multiresolution analysis.[14] Also gave an alternative definition of multiresolution analysis using spectral functions. They stated the equivalent conditions for the union density and intersection triviality properties of multiresolution analysis in terms of the spectral function. The authors described the structure of shift invariant subspaces of $L^2(G, \mu)$, where G is a locally compact Abelian group.[15] It is shown that every shift invariant space can be decomposed into an orthogonal sum of spaces each of which is generated by a single function whose shifts form a Parseval frame. They also analyzed the shift preserving operators on locally compact Abelian group.

F. A. Shah and A. Wahid[16] studied about the construction of wavelet packets associated with multiresolution analysis on locally compact Abelian group. They gave the necessary and sufficient conditions for shifts of a function $\varphi \in L^2(G)$ to be an orthonormal system for $L^2(G)$. This is called **splitting lemma**. They also characterized the collection of translations and dilations of wavelet packets which form an orthonormal basis for $L^2(G)$.

In 2012, B. Behra and Q. Jahan introduced the concept of multiresolution analysis on a local field K of positive characteristic. They proved the splitting lemma on K. They also constructed[17] wavelet packets associated with such multiresolution analysis and generated an orthonormal basis for $L^2(K)$ by translations only. One of the important properties of multiresolution analysis is that the core subspace V_0 has an orthonormal basis consisting of translates of a single function. The authors showed[18] that it is enough to assume that these translates form only a Riesz basis and proved how to construct an orthonormal basis from a Riesz basis. They also found that the intersection triviality and union density conditions follows from the other properties of a multiresolution analysis with an extra condition.

Behra and Jahan generalized the concept of biorthogonal wavelets to a local field of positive characteristic.[19] They determined the conditions for a function to generate a Riesz basis and defined the projection operator associated with the multiresolution analysis. They proved that the wavelets associated with dual multiresolution analysis are biorthogonal and form a Riesz bases for $L^2(K)$. The authors also provided[20] a characterization of wavelets, tight wavelet frames and MRA wavelets of $L^2(K)$.

Recently, M. Brownik and Q. Jahan[21] considered compact Abelian groups and studied the wavelet theory there. They provided several examples of epimorphisms $A : G \to G$ with finite kernal such that $\bigcup_{j\in\mathbb{Z}_+} \ker A^j$ is dense in G and defined the concept of a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}_+}$ on compact Abelian groups. Also they proved the characterization of scaling sequences of a multiresolution analysis for $L^p(G)$, $1 \leq p < \infty$ and constructed the wavelet basis for $L^2(G)$. Finally, they proved the existence of minimally supported frequency (MSF) multiresolution analysis under the standing assumptions on an epimorphism A on G and constructed the orthonormal MSF wavelets in $L^2(G)$.

4. CANTOR DYADIC GROUPS AND VILENKIN GROUP

One of the important examples of a locally compact Abelian group is Cantor dyadic group. The Cantor dyadic group is $D = \prod_{n=-\infty}^{-1} \mathbb{Z}/(2)$ under the cartesian product topology, where $\mathbb{Z}/(2)$ is the integer modulo 2. That is we can write D as,

$$D = \{(x_j)_{j<0} : x_j \in \{0,1\} \text{ for each } j\}.$$

The negative indices are given so that we may think of $x = (x_j)_{j<0}$ as $x = 0.x_{-1}x_{-2}\cdots$, a binary fraction expansion. Now consider the map $x \to |x| : D \to [0,1]$, where $|x| = \sum_{j<0} x_j 2^j$. This induces the Haar measure on D.

Now a locally compact Cantor dyadic group can be defined as,

$$\begin{aligned} G &= \prod_{n=-\infty}^{\infty} \mathbb{Z}/(2) \\ &= \{(x_j)_{j\in\mathbb{Z}} : x_j \in \{0,1\} \text{ for all } j \text{ and } x_j = 0 \text{ for all } j > n, \text{ for some } n \in \mathbb{Z}\}. \end{aligned}$$

So we can think of $x = x_n x_{n-1}\cdots x_1 x_0 . x_{-1}x_{-2}\cdots$, and we can identify G with $[0,\infty)$ as a measure space by $x \to |x| : D \to [0,\infty)$, where $|x| = \sum_{j\in\mathbb{Z}} x_j 2^j$. This induces the Haar measure on G. Also the dual group $\widehat{G}$ of G is topologically isomorphic to G.

We define Vilenkin's group G as the group of sequences

$$x = (x_j) = (\ldots, 0, 0, x_k, x_{k+1}, x_{k+2}, \ldots),$$

where $x_j \in \{0, 1, \ldots, p-1\}$ for $j \in \mathbb{Z}$ and $x_j = 0$ for $j < k = k(x)$. The group operation on G is the coordinate-wise addition modulo p and the topology in G is introduced via the complete system of neighbourhoods of zero

$$U_l = \{(x_j) \in G : x_j = 0 \text{ for } j \leq l\}, \;\; l \in \mathbb{Z}.$$

Put $U = U_0$. The Lebesgue spaces $L^p(G,\mu)$, $1 \le p \le \infty$, are defined by the Haar measure μ on Borel's subsets of G normalized by $\mu(U) = 1$. The dual group of G is $\widehat{G}$ consisting of all sequences of the form

$$\omega = (\omega_j) = (\ldots, 0, 0, \omega_k, \omega_{k+1}, \omega_{k+2}, \ldots),$$

where $\omega_j \in \{0, 1, \ldots, p-1\}$ for $j \in \mathbb{Z}$ and $\omega_j = 0$ for $j < k = k(\omega)$. Each character on G can be defined by the formula

$$\chi(x,\omega) = \exp\left(\frac{2\pi i}{p} \sum_{j\in\mathbb{Z}} x_{-j}\omega_{j-1}\right), \qquad x \in G,$$

for some $\omega \in \widehat{G}$.

The field $\mathbb{Q}_p$ of p-adic numbers is the completion of the field $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ defined as follows:

$$|x|_p = \begin{cases} p^{-\gamma}, & \text{if } x = p^{\gamma}\frac{m}{n} \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

where $\gamma \in \mathbb{Z}$ and m, n are integers not divisible by p. The extension of this norm to $\mathbb{Q}_p$ is also denoted by $|\cdot|_p$. The norm $|\cdot|_p$ is non-Archimedean, i.e. satisfies the triangle inequality

$$|x + y|_p \le \max\{|x|_p, |y|_p\} \ \ x, y \in \mathbb{Q}_p.$$

Any p-adic number $x \neq 0$ can be uniquely written in the form

$$x = \sum_{j=\gamma}^{\infty} x_j p^j$$

where $\gamma \in \mathbb{Z}$ and $x_j \in \{0, 1, \ldots, p-1\}$ with $x_\gamma \neq 0$. The fractional part $\{x\}_p$ of the number x equals by definition $\sum_{j=\gamma}^{-1} x_j p^j$. We also set $\{0\}_p = 0$. The ring of p-adic integers $\mathbb{Z}_p$ is defined by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \{x\}_p = 0\}$. The additive character χ_p of the field $\mathbb{Q}_p$ is defined by $\chi_p(x) = e^{2\pi i\{x\}_p}$, $x \in \mathbb{Q}_p$. The field $\mathbb{Q}_p$ is locally compact. Denote by dx the normalized Haar measure on it. The Fourier transform of a function f on $\mathbb{Q}_p$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x) f(x)dx, \ \ \xi \in \mathbb{Q}_p.$$

Now we present a review of wavelets on Cantor dyadic groups, Vilenkin groups and p-adic field. W. C. Lang[22] investigated a lot in the Cantor dyadic group. He defined translation, dilation and multiresolution analysis on a locally compact Abelian group and regularity of a wavelet on locally compact Cantor dyadic group. Also gave sufficient conditions on the scaling filters for regular orthonormal wavelets to occur and constructed orthogonal wavelets on locally compact Cantor dyadic group. He described the Mallat algorithm for these wavelets and identified the orthogonal Cantor dyadic group wavelets with the multiwavelets on real line consisting of piecewise fractal functions.[23] Other multiwavelet systems with algorithm of same structure are also studied. Compactly supported orthogonal wavelets were built on the locally compact Cantor dyadic group. Lang[24] provided a necessary and sufficient conditions on a trigonometric polynomial scaling filter for a multiresolution analysis and wavelets for a basis of $L^2(G, \mu)$, to arise. He also stated a Lipchitz regularity condition and an unconditional L^p-convergence result ($p > 1$) for regular wavelet expansions. He described some examples of wavelets include the Walsh series on the real line and check all the above conditions.

Yu. A. Farkov[25] analysed the theory of wavelets on Cantor dyadic group and Vilenkin group. He extended the results of Lang[24] to Vilenkin group G, which is defined for an integer $p \geq 2$ and coincides with Cantor dyadic group when $p = 2$. He determined the scaling function ϕ in $L^2(G, \mu)$ and orthogonal wavelets with compact support are defined by such ϕ.[26] He proposed a method for estimating the modulii of continuity of functions ϕ, which leads to sharp estimates for small p and $n \geq 2$. The author illustrated a method for constructing compactly supported orthogonal wavelets on a Vilenkin group G which can be written as a weak direct product of countable sets of cyclic groups of p^{th} order.[27] He established the necessary and sufficient conditions under which the solutions of scaling equations with p^n coefficients generate a multiresolution analysis on $L^2(G)$. He showed that these p^n coefficients can be calculated using the discrete Vilenkin-Chrestenson transform. Also he found the conditions for the compactly supported scaling functions to be stable and orthonormal. He produced some examples of scaling functions.

Farkov deals with the concept of multiresolution analysis and compactly supported orthogonal wavelets on Vilenkin groups.[28] He described some important properties like Strang-Fix condition, partition of unity property, linear independence, stability, and orthonormality of integer shifts of the corresponding refinable functions. He also gave a necessary and sufficient conditions for the scaling function to generate a multiresolution analysis in the L^2 spaces on Vilenkin groups and presented several examples of such wavelets. Farkov[29] proved that the orthogonal compactly supported wavelets on

Vilenkin group can be represented as lacunary series in generalized Walsh functions. He also provided algorithms for constructing biorthogonal wavelet systems and the corresponding scaling functions whose masks are generalized Walsh polynomials. Some examples of biorthogonal compactly supported wavelets are given.

The field $\mathbb{Q}_p$ of all p-adic numbers and the ring $\mathbb{Z}_p$ of all p-adic integers are zero dimensional locally compact Abelian group and compact Abelian group respectively. In 2009, S. F. Lukomskii[30] suggested an algorithm for constructing a Haar system on a compact zero-dimensional Abelian group, and introduced[31] the concept of multiresolution analysis on locally compact Abelian group and explained a method to generate orthogonal wavelet bases. Also, he examined[32] the problem of construction of multiresolution analysis and orthogonal wavelet bases on the product $\mathbb{Q}_p^d$. The product $\mathbb{Q}_p^d$ is not a field of p-adic numbers, but is zero-dimensional locally compact Abelian group. The author explained all dilation operators on this space and then made multiresolution analysis corresponding to each of these operators. Also he described a procedure for forming orthogonal wavelet bases on this space.

In their paper, A.Yu. Khrennikov, V. M. Shelkovich and M. Skopina described[33] a large class of p-adic refinement equations and explained the formation of refinable function which generates p-adic multiresolution analysis. They suggested an algorithm to construct p-adic orthogonal wavelet basis using these refinable functions and illustrated it by examples which gives 3-adic bases. The authors discussed[34] all compactly supported orthogonal wavelet bases for $L^2(\mathbb{Q}_p)$ generated by the unique p-adic multiresolution analysis, i.e.,the Haar bases of $L^2(\mathbb{Q}_p)$. M. Skopina, together with S. Albeverio and S. Evdokimov, studied p-adic multiresolution analysis.[35] They gave a complete characterization of scaling functions and explained what types of scaling functions are orthogonal. They also suggested a technique for constructing wavelets and showed that any wavelet function generate p-adic wavelet frame.

The real case of wavelet sets is an important tool for the construction of MRA as well as non-MRA wavelets. Almost all the geometric and topological properties of wavelets can be described using these wavelet sets. The paper of P. Mahapatra and D. Singh[36] is devoted to the study of wavelet sets in Cantor dyadic group and their association with multiresolution analysis. Also they characterized the wavelet sets on Cantor dyadic group and gave examples of MRA and non-MRA wavelet sets. Scaling and generalized scaling sets are used to construct wavelet sets and wavelets and these wavelets are actually MRA wavelets. The authors established[37] the properties of both scaling and generalized scaling sets on Cantor dyadc group and described some examples.

5. CONCLUSION

In this paper, we reviewed wavelet theory, an important mathematical tool for signal and image processing, on a locally compact Abelian group. We extended the concept of multiresolution analysis to such groups, defined the spectral functions and found the conditions under which a spectral function generates a multiresolution analysis. Wavelet sets are also an important notion in this study. Wavelet sets are used to construct MRA as well as non-MRA wavelets and it describes all the geometric and topological properties of wavelets.

In literature, S. Dhalke[6] introduced the generalized B-spline function on a locally compact Abelian group and he proposed necessary and sufficient conditions for a generalized B-spline to form a multiresolution analysis and wavelets. That is, Dhalke extended the Haar wavelet to locally compact Abelian group. Similar to this, we expect that Shannon wavelet also can be generalized. But in the paper[10] of R. L. Benedetto, he proved using wavelet sets that on a locally compact Abelian group, Haar and Shannon wavelets are same. That is, he constructed Haar and Shannon wavelets on a locally compact Abelian group with the help of wavelet sets and identified that they are same.

Since both are MRA wavelets, can we define them through multiresolution analysis? Dhalke defined a scaling function for Haar wavelet on locally compact Abelian group. Similarly, can we find a scaling function corresponding to Shannon wavelet? Can we extend the Sinc function to a locally compact Abelian group? While this literature review raises the above questions, it is fruitful to pursue further studies about extended Shannon wavelets. We expect that we can answer these problems in our future research. Further studies are also needed to investigate the properties of generalized wavelets on locally compact Abelian group.

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