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NOTES ON BIPOLAR VALUED FUZZY SUBRINGS OF A RING

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Abstract

In this paper, we study some of the properties of bipolar valued fuzzy subring of a ring and prove some results on these. Using some basic definitions, we derive the some important Theorems. Intersection and product are applied into the bipolar valued fuzzy subring of a ring.

Keywords: Bipolar valued fuzzy set, bipolar valued fuzzy subring, product.

INTRODUCTION

In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar valued fuzzy sets. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar valued fuzzy subring and established some results.

1. PRELIMINARIES:

1.1 Definition:
A bipolar valued fuzzy set (BVFS) A in X is defined as an object of the form A = { < x, $A^+(x)$, $A^-(x)$ >/ $x \in X$}, where $A^+ : X \rightarrow [0, 1]$ and $A^- : X \rightarrow [-1, 0]$. The positive membership degree $A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar valued fuzzy set A and the negative membership degree $A^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued fuzzy set A. If $A^+(x) \neq 0$ and $A^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A^+(x) = 0$ and $A^-(x) \neq 0$, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that $A^+(x) \neq 0$ and $A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

1.2 Example:
A = { < a, 0.5, –0.3 >, < b, 0.1, –0.7 >, < c, 0.5, –0.4 >} is a bipolar-valued fuzzy subset of X= {a, b, c }.

1.3 Definition:
Let R be a ring. A bipolar valued fuzzy subset A of R is said to be a bipolar valued fuzzy subring of R if the following conditions are satisfied,
(i) $A^+(x-y) \geq \min\{ A^+(x), A^+(y) \}$
(ii) $A^+(xy) \geq \min\{ A^+(x), A^+(y) \}$
(iii) $A^-(x-y) \leq \max\{ A^-(x), A^-(y) \}$
(iv) $A^-(xy) \leq \max\{ A^-(x), A^-(y) \}$ for all x and y in R.

1.4 Definition:
Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be any two bipolar valued fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by A×B, is defined as A×B = { $\langle (x, y), (A\times B)^+(x, y), (A\times B)^-(x, y) \rangle$ / for all x in G and y in H }, where $(A\times B)^+(x, y) = \min \{ A^+(x), B^+(y) \}$ and $(A\times B)^-(x, y) = \max \{ A^-(x), B^-(y) \}$, for all x in G and y in H.

1.5 Definition:
Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subset in a set S, the strongest bipolar valued fuzzy relation on S, that is a bipolar valued fuzzy relation on A is V = { $\langle (x, y), V^+(x, y), V^-(x, y) \rangle$ / x and y in S} given by $V^+(x, y) = \min \{ A^+(x), A^+(y) \}$ and $V^-(x, y) = \max \{ A^-(x), A^-(y) \}$, for all x and y in S.

2. PROPERTIES:

2.1 Theorem:

Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subring of a ring R. Then $A^+(-x) = A^+(x)$ and $A^-(-x) = A^-(x)$, $A^+(x) \leq A^+(e)$ and $A^-(x) \geq A^-(e)$ for all x in R and the identity element e in R.

Proof:

Let x be in R. Now $A^+(x) = A^+(\,-(-x)\,) \geq A^+(-x) \geq A^+(x)$. Therefore $A^+(x) = A^+(-x)$ for all x in R. And $A^-(x) = A^-(\,-(-x)\,) \leq A^-(-x) \leq A^-(x)$. Therefore $A^-(-x) = A^-(x)$ for all x in R. Also $A^+(e) = A^+(x-x) \geq \min\{ A^+(x), A^+(x) \}= A^+(x)$. Therefore $A^+(e) \geq A^+(x)$ for all x in R. And $A^-(e)= A^-(x-x) \leq \max\{ A^-(x), A^-(x) \} = A^-(x)$. Therefore $A^-(e) \leq A^-(x)$ for all x in R.

2.2 Theorem:

Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subring of a ring R. Then

(i) $A^+(x-y) = A^+(e)$ implies that $A^+(x) = A^+(y)$ for x and y in R.

(ii) $A^-(x-y) = A^-(e)$ implies that $A^-(x) = A^-(y)$ for x and y in R.

Proof:

Now $A^+(x) = A^+(\,x-y+y) \geq \min\{ A^+(x-y), A^+(y) \} = \min\{ A^+(e), A^+(y) \} = A^+(y) = A^+(\,y-x+x) \geq \min\{ A^+(y-x), A^+(x) \}= \min\{ A^+(e), A^+(x) \} = A^+(x)$. Therefore $A^+(x) = A^+(y)$ for x and y in R. And $A^-(x) = A^-(x-y+y) \leq \max\{ A^-(x-y), A^-(y) \} = \max\{ A^-(e), A^-(y) \}= A^-(y) = A^-(y-x+x) \leq \max\{ A^-(y-x), A^-(x) \}= \max\{ A^-(e), A^-(x) \} = A^-(x)$. Therefore $A^-(x) = A^-(y)$ for x and y in R.

2.3 Theorem:

Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subring of a ring R.

(i) If $A^+(x-y) = 1$, then $A^+(x) = A^+(y)$ for x and y in R.

(ii) If $A^-(x-y) = -1$, then $A^-(x) = A^-(y)$ for x and y in R.

Proof:

Now $A^+(x) = A^+(x-y+y) \geq \min\{ A^+(x-y), A^+(y) \}= \min\{ 1, A^+(y) \}= A^+(y) = A^+(-y) = A^+(-x+x-y) \geq \min\{ A^+(x), A^+(x-y) \} = \min\{ A^+(x), 1 \} = A^+(x)$. Therefore $A^+(x) = A^+(y)$ for x and y in R. Hence (i) is proved. Also $A^-(x) = A^-(x-y+y) \leq \max\{A^-(x-y), A^-(y) \}= \max\{-1, A^-(y) \}= A^-(y) = A^-(-y) = A^-(-x+x-y) \leq \max\{ A^-(x), A^-(x-y) \}= \max\{ A^-(x), -1\} = A^-(x)$. Therefore $A^-(x) = A^-(y)$ for x and y in R. Hence (ii) is proved.

2.4 Theorem:

Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subring of a ring G.

(i) If $A^+(xy^{-1}) = 0$, then either $A^+(x) = 0$ or $A^+(y) = 0$, for x and y in G.

(ii) If $A^-(xy^{-1}) = 0$, then either $A^-(x) = 0$ or $A^-(y) = 0$, for x and y in G.

Proof:

Let x and y in G. (i) By the definition $A^+(xy^{-1}) \geq \min \{ A^+(x), A^+(y) \}$, which implies that $0 \geq \min \{ A^+(x), A^+(y) \}$. Therefore, either $A^+(x) = 0$ or $A^+(y) = 0$. (ii) By the definition $A^-(xy^{-1}) \leq \max \{ A^-(x), A^-(y) \}$, which implies that $0 \leq \max \{ A^-(x), A^-(y) \}$. Therefore, either $A^-(x) = 0$ or $A^-(y) = 0$.

2.5 Theorem:

If $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subring of G, then

(i) $A^+ (xy) = A^+ (yx)$ if and only if $A^+ (x) = A^+ (y^{-1}xy)$, for x and y in G.

(ii) $A^- (xy) = A^- (yx)$ if and only if $A^- (x) = A^- (y^{-1}xy)$, for x and y in G.

Proof:

Let x and y be in G. Assume that $A^+ (xy) = A^+ (yx)$, so, $A^+ (y^{-1}xy) = A^+ (y^{-1}yx) = A^+ (ex) = A^+ (x)$. Therefore $A^+ (x) = A^+ (y^{-1}xy)$, for x and y in G. Conversely, assume that $A^+ (x) = A^+ (y^{-1}xy)$, we get, $A^+ (xy) = A^+ (xyxx^{-1}) = A^+ (yx)$. Therefore $A^+ (xy) = A^+ (xy)$, for x and y in G. Hence $A^+ (xy) = A^+ (yx)$ if and only if $A^+ (x) = A^+ (y^{-1}xy)$, for x and y in G. Also assume that $A^-(xy) = A^-(yx)$, we get, $A^-(y^{-1}xy) = A^-(y^{-1}yx) = A^-(ex) = A^-(x)$. Therefore $A^- (x) = A^- (y^{-1}xy)$, for x and y in G. Conversely, assume that $A^- (x) = A^- (y^{-1}xy)$, so, $A^-(xy) = A^- (xyxx^{-1}) = A^-(yx)$. Therefore $A^- (xy) = A^- (xy)$, for x and y in G. Hence $A^- (xy) = A^- (yx)$ if and only if $A^- (x) = A^- (y^{-1}xy)$, for x and y in G.

2.6 Theorem:

If $A = \langle A^+, A^- \rangle$ is a bipolar-valued fuzzy subring of a ring G, then $H = \{ x\in G \mid A^+ (x) = 1, A^- (x) = -1 \}$ is either empty or is a subring of G.

Proof:

If no element satisfies this condition, then H is empty. If x and y in H, then $A^+ (xy^{-1}) \geq \min \{ A^+ (x), A^+ (y) \}= \min\{ 1, 1 \}= 1$. Therefore $A^+ (xy^{-1}) = 1$. And $A^- (xy^{-1}) \leq \max\{ A^- (x), A^- (y) \} = \max \{ -1, -1\}= -1$. Therefore $A^- (xy^{-1}) = -1$. That is $xy^{-1}\in H$. Hence H is a subring of G. Hence H is either empty or is a subring of G.

2.7 Theorem:

If $A = \langle A^+, A^- \rangle$ is a bipolar-valued fuzzy subring of G, then $H = \{ x\in G \mid A^+(x) = A^+ (e)$ and $A^- (x) = A^- (e) \}$ is a subring of G.

Proof:

Here $H = \{ x\in G \mid A^+ (x) = A^+ (e)$ and $A^- (x) = A^- (e) \}$, by Theorem 2.1, $A^+ (x^{-1}) = A^+ (x) = A^+ (e)$ and $A^- (x^{-1}) = A^- (x) = A^- (e)$. Therefore $x^{-1}\in H$. Now, $A^+ (xy^{-1}) \geq \min\{ A^+ (x), A^+ (y)\}= \min\{ A^+(e), A^+(e) \}= A^+(e)$, and $A^+ (e) = A^+ ((xy^{-1})(xy^{-1})^{-1}) \geq \min\{ A^+ (xy^{-1}), A^+ (xy^{-1}) \}= A^+ (xy^{-1})$. Hence $A^+ (e) = A^+ (xy^{-1})$.

Also, $A^-(xy^{-1}) \le \max\{ A^-(x), A^-(y) \} = \max\{ A^-(e), A^-(e) \} = A^-(e)$, and $A^-(e) = A^-((xy^{-1})(xy^{-1})^{-1}) \le \max\{ A^-(xy^{-1}), A^-(xy^{-1}) \} = A^-(xy^{-1})$. Therefore $A^-(e) = A^-(xy^{-1})$. Hence $A^+(e) = A^+(xy^{-1})$ and $A^-(e) = A^-(xy^{-1})$. Therefore $xy^{-1} \in H$. Hence H is a subring of G.

2.8 Theorem:
Let G be a ring. If $A = \langle A^+, A^- \rangle$ is a bipolar-valued fuzzy subring of G, then $A^+(xy) = \min\{ A^+(x), A^+(y) \}$ and $A^+(xy) = \max\{ A^-(x), A^-(y) \}$ for each x,y in G with $A^+(x) \ne A^+(y)$ and $A^-(x) \ne A^-(y)$.

Proof:
Assume that $A^+(x) > A^+(y)$ and $A^-(x) < A^-(y)$. Then $A^+(y) = A^+(x^{-1}xy) \ge \min\{ A^+(x^{-1}), A^+(xy) \} = \min\{ A^+(x), A^+(xy) \} = A^+(xy) \ge \min\{ A^+(x), A^+(y) \} = A^+(y)$. Therefore $A^+(xy) = A^+(y) = \min\{ A^+(x), A^+(y) \}$. And $A^-(y) = A^-(x^{-1}xy) \le \max\{ A^-(x^{-1}), A^-(xy) \} = \max\{ A^-(x), A^-(xy) \} = A^-(xy) \le \max\{ A^-(x), A^-(y) \} = A^-(y)$. Therefore $A^-(xy) = A^-(y) = \max\{ A^-(x), A^-(y) \}$.

2.9 Theorem:
If $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ are two bipolar-valued fuzzy subrings of a ring G, then their intersection $A \cap B$ is a bipolar-valued fuzzy subring of G.

Proof:
Let $A = \{ \langle x, A^+(x), A^-(x) \rangle / x \in G \}$, $B = \{ \langle x, B^+(x), B^-(x) \rangle / x \in G \}$. Let $C = A \cap B$ and $C = \{ \langle x, C^+(x), C^-(x) \rangle / x \in G \}$. Now, $C^+(xy^{-1}) = \min\{ A^+(xy^{-1}), B^+(xy^{-1}) \} \ge \min\{ \min\{A^+(x), A^+(y)\}, \min\{B^+(x), B^+(y)\} \} \ge \min\{ \min\{A^+(x), B^+(x)\}, \min\{A^+(y), B^+(y)\} \} = \min\{ C^+(x), C^+(y) \}$. Therefore $C^+(xy^{-1}) \ge \min\{C^+(x), C^+(y)\}$. Also, $C^-(xy^{-1}) = \max\{ A^-(xy^{-1}), B^-(xy^{-1}) \} \le \max\{ \max\{A^-(x), A^-(y)\}, \max\{B^-(x), B^-(y)\} \} \le \max\{ \max\{A^-(x), B^-(x)\}, \max\{A^-(y), B^-(y)\} \} = \max\{ C^-(x), C^-(y) \}$. Therefore $C^-(xy^{-1}) \le \max\{ C^-(x), C^-(y) \}$. Hence $A \cap B$ is a bipolar-valued fuzzy subring of G.

2.10 Theorem:
The intersection of a family of bipolar-valued fuzzy subrings of a ring G is a bipolar-valued fuzzy subring of G.

Proof:
Let $\{ V_i : i \in I \}$ be a family of bipolar-valued fuzzy subrings of a ring G and let $A = \bigcap_{i \in I} V_i$. Let x and y in G. Now, $A^+(xy^{-1}) = \inf_{i \in I} V_i^+(xy^{-1}) \ge \inf_{i \in I} \min\{ V_i^+(x), V_i^+(y) \} = \min\{ \inf_{i \in I} V_i^+(x), \inf_{i \in I} V_i^+(y) \} = \min\{ A^+(x), A^+(y) \}$. Therefore, $A^+(xy^{-1}) \ge \min\{ A^+(x), A^+(y) \}$, for all x and y in G. And, $A^-(xy^{-1}) = \sup_{i \in I} V_i^-(xy^{-1}) \le \sup_{i \in I} \max\{ V_i^-(x), V_i^-(y) \} = \max\{ \sup_{i \in I} V_i^-(x), \sup_{i \in I} V_i^-(y) \} = \max\{ A^-(x), A^-(y) \}$.

Therefore, $A^-(xy^{-1}) \leq \max\{ A^-(x), A^-(y) \}$, for all x and y in G. Hence, the intersection of a family of bipolar-valued fuzzy subrings of a ring G is a bipolar-valued fuzzy subring of G.

2.11 Theorem:
If $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ are any two bipolar-valued fuzzy subrings of the rings G_1 and G_2 respectively, then $A\times B = \langle (A\times B)^+, (A\times B)^- \rangle$ is a bipolar-valued fuzzy subring of $G_1\times G_2$.

Proof:
Let A and B be two bipolar-valued fuzzy subrings of the rings G_1 and G_2 respectively. Let x_1 and x_2 be in G_1, y_1 and y_2 be in G_2. Then (x_1, y_1) and (x_2, y_2) are in $G_1\times G_2$. Now, $(A\times B)^+[(x_1, y_1)(x_2, y_2)^{-1}] = (A\times B)^+(x_1x_2^{-1}, y_1y_2^{-1}) = \min\{A^+(x_1x_2^{-1}), B^+(y_1y_2^{-1})\} \geq \min\{ \min\{A^+(x_1), A^+(x_2)\}, \min\{ B^+(y_1), B^+(y_2) \}\} = \min\{ \min\{A^+(x_1), B^+(y_1)\}, \min\{ A^+(x_2), B^+(y_2) \}\} = \min\{(A\times B)^+(x_1, y_1), (A\times B)^+(x_2, y_2) \}$. Therefore, $(A\times B)^+[(x_1, y_1)(x_2, y_2)^{-1}] \geq \min\{ (A\times B)^+(x_1, y_1), (A\times B)^+(x_2, y_2) \}$. Also, $(A\times B)^-[(x_1, y_1)(x_2, y_2)^{-1}] = (A\times B)^-(x_1x_2^{-1}, y_1y_2^{-1}) = \max\{ A^-(x_1x_2^{-1}), B^-(y_1y_2^{-1}) \} \leq \max\{ \max\{A^-(x_1), A^-(x_2)\}, \max\{B^-(y_1), B^-(y_2)\}\} = \max\{ \max\{ A^-(x_1), B^-(y_1)\}, \max\{A^-(x_2), B^-(y_2)\}\} = \max\{ (A\times B)^-(x_1, y_1), (A\times B)^-(x_2, y_2) \}$. Therefore, $(A\times B)^-[(x_1, y_1)(x_2, y_2)^{-1}] \leq \max\{ (A\times B)^-(x_1, y_1), (A\times B)^-(x_2, y_2) \}$. Hence $A\times B$ is a bipolar-valued fuzzy subring of $G_1\times G_2$.

2.12 Theorem:
Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be any two bipolar-valued fuzzy subsets of the rings G and H respectively. Suppose that e and e^l are the identity elements of G and H respectively. If $A\times B$ is a bipolar-valued fuzzy subring of $G\times H$, then at least one of the following two statements must hold.

(i) $B^+(e^l) \geq A^+(x)$, for all x in G and $B^-(e^l) \leq A^-(x)$, for all x in G,

(ii) $A^+(e) \geq B^+(y)$, for all y in H and $A^-(e) \leq B^-(y)$, for all y in H.

Proof:
Let $A\times B$ is a bipolar-valued fuzzy subring of $G\times H$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a in G and b in H such that $A^+(a) > B^+(e^l)$, $A^-(a) < B^-(e^l)$ and $B^+(b) > A^+(e)$, $B^-(b) < A^-(e)$. We have, $(A\times B)^+(a, b) = \min\{ A^+(a), B^+(b)\} > \min\{A^+(e), B^+(e^l)\} = (A\times B)^+(e, e^l)$. Also, $(A\times B)^-(a, b) = \max\{A^-(a), B^-(b) \} < \max\{A^-(e), B^-(e^l) \} = (A\times B)^-(e, e^l)$. Thus $A\times B$ is not a bipolar-valued fuzzy subring of $G\times H$. Hence either $B^+(e^l) \geq A^+(x)$, for all x in G and $B^-(e^l) \leq A^-(x)$, for all x in G or $A^+(e) \geq B^+(y)$, for all y in H and $A^-(e) \leq B^-(y)$, for all y in H.

2.13 Theorem:
Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be any two bipolar-valued fuzzy subsets of the rings G and H, respectively and $A\times B$ is a bipolar-valued fuzzy subring of $G\times H$. Then the following are true:

(i) if $A^{+}(x) \leq B^{+}(e^{\prime})$, for all x in G and $A^{-}(x) \geq B^{-}(e^{\prime})$, for all x in G, then A is a bipolar-valued fuzzy subring of G, where $e^{\prime}$ is identity element of H.
(ii) if $B^{+}(x) \leq A^{+}(e)$ for all x in H and $B^{-}(x) \geq A^{-}(e)$, for all x in H, then B is a bipolar-valued fuzzy subring of H, where e is identity element of G.
(iii) either A is a bipolar-valued fuzzy subring of G or B is a bipolar-valued fuzzy subring of H, where e and $e^{\prime}$ are the identity elements of G and H respectively.

Proof:
Let A×B be a bipolar-valued fuzzy subring of G×H and x and y in G. Then $(x, e^{\prime})$ and $(y, e^{\prime})$ are in G×H. Now, using the property if $A^{+}(x) \leq B^{+}(e^{\prime})$, for all x in G and $A^{-}(x) \geq B^{-}(e^{\prime})$, for all x in G, where $e^{\prime}$ is identity element of H, we get, $A^{+}(xy^{-1}) = \min\{ A^{+}(xy^{-1}), B^{+}(e^{\prime}e^{\prime}) \} = (A\times B)^{+}((xy^{-1}), (e^{\prime}e^{\prime})) = (A\times B)^{+}[(x, e^{\prime})(y^{-1}, e^{\prime})] \geq \min\{ (A\times B)^{+}(x, e^{\prime}), (A\times B)^{+}(y^{-1}, e^{\prime}) \} = \min\{\min\{A^{+}(x), B^{+}(e^{\prime})\}, \min\{A^{+}(y^{-1}), B^{+}(e^{\prime})\}\} = \min\{ A^{+}(x), A^{+}(y^{-1}) \} \geq \min\{ A^{+}(x), A^{+}(y) \}$. Therefore, $A^{+}(xy^{-1}) \geq \min\{ A^{+}(x), A^{+}(y) \}$, for all x and y in G. Also, $A^{-}(xy^{-1}) = \max\{A^{-}(xy^{-1}), B^{-}(e^{\prime}e^{\prime})\} = (A\times B)^{-}((xy^{-1}), (e^{\prime}e^{\prime})) = (A\times B)^{-}[(x, e^{\prime})(y^{-1}, e^{\prime})] \leq \max\{(A\times B)^{-}(x, e^{\prime}), (A\times B)^{-}(y^{-1}, e^{\prime})\} = \max\{ A^{-}(x), B^{-}(e^{\prime}) \}, \max\{ A^{-}(y^{-1}), B^{-}(e^{\prime}) \} \} = \max\{A^{-}(x), A^{-}(y^{-1})\} \leq \max\{A^{-}(x), A^{-}(y) \}$. Therefore, $A^{-}(xy^{-1}) \leq \max\{ A^{-}(x), A^{-}(y) \}$, for all x and y in G. Hence A is a bipolar-valued fuzzy subring of G. Thus (i) is proved. Now, using the property $B^{+}(x) \leq A^{+}(e)$ for all x in H and $B^{-}(x) \geq A^{-}(e)$, for all x in H, we get, $B^{+}(xy^{-1}) = \min\{ B^{+}(xy^{-1}), A^{+}(e.e) \} = (A\times B)^{+}((e.e), (xy^{-1})) = (A\times B)^{+}[(e, x)(e, y^{-1})] \geq \min\{(A\times B)^{+}(e, x), (A\times B)^{+}(e, y^{-1})\} = \min\{\min\{A^{+}(e), B^{+}(x)\}, \min\{A^{+}(e), B^{+}(y^{-1})\}\} = \min\{ B^{+}(x), B^{+}(y^{-1}) \} \geq \min\{ B^{+}(x), B^{+}(y) \}$. Therefore, $B^{+}(xy^{-1}) \geq \min\{ B^{+}(x), B^{+}(y) \}$, for all x and y in H. Also, $B^{-}(xy^{-1}) = \max\{ B^{-}(xy^{-1}), A^{-}(ee)\} = (A\times B)^{-}((ee), (xy^{-1})) = (A\times B)^{-}[(e, x)(e, y^{-1})] \leq \max\{ (A\times B)^{-}(e, x), (A\times B)^{-}(e, y^{-1}) \} = \max\{ \max\{ A^{-}(e), B^{-}(x) \}, \max\{ A^{-}(e), B^{-}(y^{-1}) \} \} = \max\{B^{-}(x), B^{-}(y^{-1}) \} \leq \max\{ B^{-}(x), B^{-}(y) \}$. Therefore, $B^{-}(xy^{-1}) \leq \max\{ B^{-}(x), B^{-}(y) \}$, for all x and y in H. Hence B is a bipolar-valued fuzzy subring of H. Thus (ii) is proved. Hence (iii) is clear.

2.14 Theorem:
Let $A = \langle A^{+}, A^{-} \rangle$ be a bipolar-valued fuzzy subset of a ring (G, .) and $V = \langle V^{+}, V^{-} \rangle$ be the strongest bipolar-valued fuzzy relation of G. Then A is a bipolar-valued fuzzy subring of G if and only if V is a bipolar-valued fuzzy subring of G×G.

Proof:
Suppose that A is a bipolar-valued fuzzy subring of G. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in G×G. We have, $V^{+}(xy^{-1}) = V^{+}[(x_1, x_2)(y_1, y_2)^{-1}] = V^{+}(x_1y_1^{-1}, x_2y_2^{-1}) = \min\{ A^{+}(x_1y_1^{-1}), A^{+}(x_2y_2^{-1}) \} \geq \min\{ \min\{ A^{+}(x_1), A^{+}(y_1) \}, \min\{ A^{+}(x_2), A^{+}(y_2) \} \} = \min\{ \min\{ A^{+}(x_1), A^{+}(x_2) \}, \min\{ A^{+}(y_1), A^{+}(y_2)\}\} = \min\{ V^{+}(x_1, x_2), V^{+}(y_1, y_2) \} = \min\{ V^{+}(x), V^{+}(y) \}$. Therefore, $V^{+}(xy^{-1}) \geq \min\{ V^{+}(x), V^{+}(y)\}$, for all x and y in G×G. Also we have, $V^{-}(xy^{-1}) = V^{-}[(x_1, x_2)(y_1, y_2)^{-1}] = V^{-}(x_1y_1^{-1}, x_2y_2^{-1}) = \max\{ A^{-}(x_1y_1^{-1}), A^{-}(x_2y_2^{-1}) \} \leq \max\{ \max\{ A^{-}(x_1), A^{-}(y_1) \}, \max\{A^{-}(x_2), A^{-}(y_2)\}\} = \max\{ \max\{A^{-}(x_1), A^{-}(x_2)\}, \max\{ A^{-}(y_1), A^{-}(y_2) \}\} = \max\{ V^{-}(x_1, x_2), V^{-}(y_1, y_2) \} = \max\{ V^{-}(x), V^{-}(y) \}$. Therefore, $V^{-}(xy^{-1}) \leq \max$

$\{V^-(x), V^-(y)\}$, for all x and y in G×G. This proves that V is a bipolar-valued fuzzy subring of G×G. Conversely, assume that V is a bipolar-valued fuzzy subring of G×G, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in G×G, we have $\min\{ A^+(x_1y_1^{-1}), A^+(x_2y_2^{-1}) \} = V^+(x_1y_1^{-1}, x_2y_2^{-1}) = V^+[(x_1, x_2)(y_1, y_2)^{-1}] = V^+(xy^{-1}) \geq \min\{ V^+(x), V^+(y) \} = \min\{ V^+(x_1, x_2), V^+(y_1, y_2)\} = \min\{\min\{A^+(x_1), A^+(x_2)\}, \min\{ A^+(y_1), A^+(y_2) \}\}$. If we put $x_2 = y_2 = e$, we get, $A^+(x_1y_1^{-1}) \geq \min\{ A^+(x_1), A^+(y_1) \}$, for all x_1 and y_1 in G. Also we have, $\max\{ A^-(x_1y_1^{-1}), A^-(x_2y_2^{-1})\} = V^-(x_1y_1^{-1}, x_2y_2^{-1}) = V^-[(x_1, x_2)(y_1, y_2)^{-1}] = V^-(xy^{-1}) \leq \max\{V^-(x), V^-(y)\} = \max\{V^-(x_1, x_2), V^-(y_1, y_2)\} = \max\{\max\{A^-(x_1), A^-(x_2)\}, \max\{A^-(y_1), A^-(y_2)\}\}$. If we put $x_2 = y_2 = e$, we get, $A^-(x_1y_1^{-1}) \leq \max\{ A^-(x_1), A^-(y_1) \}$, for all x_1 and y_1 in G. Hence A is a bipolar-valued fuzzy subring of G.

CONCLUSION:

In the study of the structure of bipolar valued fuzzy algebraic system, we notice that bipolar valued fuzzy subring of a ring with special properties always play an important role. In this paper, we define bipolar valued fuzzy subring of a ring and investigate the relationship among these bipolar valued fuzzy subrings of a ring. Some characterization theorems of bipolar valued fuzzy subrings of a ring are obtained. We hope that the research along this direction can be continued, and in fact, this work would serve as a foundation for further study of the theory of ring, it will be necessary to carry out more theoretical research to establish a general framework for the practical application.

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