NOTES ON BIPOLAR VALUED FUZZY SUBRINGS OF A RING

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Abstract

In this paper, we study some of the properties of bipolar valued fuzzy subring of a ring and prove some results on these. Using some basic definitions, we derive the some important Theorems. Intersection and product are applied into the bipolar valued fuzzy subring of a ring.

Keywords: Bipolar valued fuzzy set, bipolar valued fuzzy subring, product.

INTRODUCTION

In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar valued fuzzy sets. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0 ) indicates that elements somewhat satisfy the implicit counter property. Bipolar valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar valued fuzzy subring and established some results.
1. PRELIMINARIES:

1.1 Definition:
A bipolar valued fuzzy set (BVFS) A in X is defined as an object of the form A = \{ < x, A^+(x), A^-(x) >/ x \in X \}, where A^+ : X \rightarrow [0, 1] and A^- : X \rightarrow [-1, 0]. The positive membership degree A^+(x) denotes the satisfaction degree of an element x to the property corresponding to a bipolar valued fuzzy set A and the negative membership degree A^-(x) denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued fuzzy set A. If A^+(x) ≠ 0 and A^-(x) = 0, it is the situation that x is regarded as having only positive satisfaction for A and if A^+(x) = 0 and A^- ≠ 0, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that A^+(x) ≠ 0 and A^- ≠ 0 when the membership function of the property overlaps that of its counter property over some portion of X.

1.2 Example:
A = \{ < a, 0.5, -0.3 >, < b, 0.1, -0.7 >, < c, 0.5, -0.4 > \} is a bipolar-valued fuzzy subset of X = \{a, b, c\}.

1.3 Definition:
Let R be a ring. A bipolar valued fuzzy subset A of R is said to be a bipolar valued fuzzy subring of R if the following conditions are satisfied,
(i) A^+(x-y) ≥ min\{ A^+(x), A^+(y) \}
(ii) A^+(xy) ≥ min\{ A^+(x), A^+(y) \}
(iii) A^-(x-y) ≤ max\{ A^-(x), A^-(y) \}
(iv) A^- ≤ max\{ A^-(x), A^-(y) \} for all x and y in R.

1.4 Definition:
Let A = \{ A^+, A^- \} and B = \{ B^+, B^- \} be any two bipolar valued fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by AxB, is defined as AxB = \{ < (x, y), (AxB)^+(x, y), (AxB)^-(x, y) > / for all x in G and y in H \}, where (AxB)^+(x, y) = min\{ A^+(x), B^+(y) \} and (AxB)^-(x, y) = max\{ A^-(x), B^- \} for all x in G and y in H.

1.5 Definition:
Let A = \{ A^+, A^- \} be a bipolar valued fuzzy subset in a set S, the strongest bipolar valued fuzzy relation on S, that is a bipolar valued fuzzy relation on A is V = \{ < (x, y), V^+(x, y), V^-(x, y) > / x and y in S \} given by V^+(x, y) = min\{ A^+(x), A^+(y) \} and V^-(x, y) = max\{ A^-(x), A^-(y) \}, for all x and y in S.
2. PROPERTIES:

2.1 Theorem:
Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subring of a ring $R$. Then $A^+(-x) = A^+(x)$ and $A^-(x) = A^-(x)$, $A^+(x) \leq A^+(e)$ and $A^-(x) \geq A^-(e)$ for all $x$ in $R$ and the identity element $e$ in $R$.

**Proof:**
Let $x$ be in $R$. Now $A^+(x) = A^+(-(-x)) \geq A^+(-x) \geq A^+(x)$. Therefore $A^+(x) = A^+(-x)$ for all $x$ in $R$. And $A^-(x) = A^-(-(-x)) \leq A^-(-x) \leq A^-(x)$. Therefore $A^-(x) = A^-(-x)$ for all $x$ in $R$. Also $A^+(e) = A^+(x-x) \geq \min \{ A^+(x), A^+(x) \} = A^+(x)$. Therefore $A^+(e) \geq A^+(x)$ for all $x$ in $R$. And $A^-(e) = A^-(x-x) \leq \max \{ A^-(x), A^-(x) \} = A^-(x)$. Therefore $A^-(e) \leq A^-(x)$ for all $x$ in $R$.

2.2 Theorem:
Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subring of a ring $R$. Then
(i) $A^+(x-y) = A^+(e)$ implies that $A^+(x) = A^+(y)$ for $x$ and $y$ in $R$.
(ii) $A^-(x-y) = A^-(e)$ implies that $A^-(x) = A^-(y)$ for $x$ and $y$ in $R$.

**Proof:**
Now $A^+(x) = A^+(x-y+y) \geq \min \{ A^+(x-y), A^+(y) \} = \min \{ A^+(e), A^+(y) \} = A^+(y) = A^+(x-y+0) \geq \min \{ A^+(y-x), A^+(x) \} = \min \{ A^+(e), A^+(x) \} = A^+(x)$. Therefore $A^+(x) = A^+(y)$ for $x$ and $y$ in $R$. And $A^-(x) = A^-(x-y+y) \leq \max \{ A^-(x-y), A^-(y) \} = \max \{ A^-(e), A^-(y) \} = A^-(y) = A^-(x-y+0) \leq \max \{ A^-(y-x), A^-(x) \} = \max \{ A^-(e), A^-(x) \} = A^-(x)$. Therefore $A^+(x) = A^+(y)$ for $x$ and $y$ in $R$.

2.3 Theorem:
Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subring of a ring $R$.
(i) If $A^+(x-y) = 1$, then $A^+(x) = A^+(y)$ for $x$ and $y$ in $R$.
(ii) If $A^-(x-y) = -1$, then $A^-(x) = A^-(y)$ for $x$ and $y$ in $R$.

**Proof:**
Now $A^+(x) = A^+(x-y+y) \geq \min \{ A^+(x-y), A^+(y) \} = \min \{ 1, A^+(y) \} = A^+(y) = A^+(-y) = A^+(x+y-x) \geq \min \{ A^+(x), A^+(x-y) \} = \min \{ 1, A^+(x) \} = A^+(x)$. Therefore $A^+(x) = A^+(y)$ for $x$ and $y$ in $R$. Hence (i) is proved. Also $A^-(x) = A^-(x-y+y) \leq \max \{ A^-(x-y), A^-(y) \} = \max \{ -1, A^-(y) \} = A^-(y) = A^-(x-y) = A^-(-x+y-x) \leq \max \{ A^-(x), A^-(x-x) \} = \max \{ A^-(x), -1 \} = A^-(x)$. Therefore $A^+(x) = A^+(y)$ for $x$ and $y$ in $R$. Hence (ii) is proved.

2.4 Theorem:
Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subring of a ring $G$.
(i) If $A^+(xy^{-1}) = 0$, then either $A^+(x) = 0$ or $A^+(y) = 0$, for $x$ and $y$ in $G$.
(ii) If $A^-(xy^{-1}) = 0$, then either $A^-(x) = 0$ or $A^-(y) = 0$, for $x$ and $y$ in $G$. 
Proof:
Let x and y in G. (i) By the definition $A^*(xy^{-1}) \geq \min \{ A^*(x), A^*(y) \}$, which implies that $0 \geq \min \{ A^*(x), A^*(y) \}$. Therefore, either $A^*(x) = 0$ or $A^*(y) = 0$. (ii) By the definition $A^-(xy^{-1}) \leq \max \{ A^-(x), A^-(y) \}$, which implies that $0 \leq \max \{ A^-(x), A^-(y) \}$. Therefore, either $A^-(x) = 0$ or $A^-(y) = 0$.

2.5 Theorem:
If $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subring of G, then
(i) $A^*(xy) = A^*(yx)$ if and only if $A^+(x) = A^+(y^{-1}xy)$, for x and y in G.
(ii) $A^-(xy) = A^-(yx)$ if and only if $A^-(x) = A^-(y^{-1}xy)$, for x and y in G.

Proof:
Let x and y be in G. Assume that $A^+(xy) = A^+(yx)$, so, $A^+(y^{-1}xy) = A^+(yx^{-1}) = A^+(x)$. Therefore $A^+(xy) = A^+(yx^{-1})$, for x and y in G. Conversely, assume that $A^+(xy) = A^+(yx^{-1})$, we get, $A^+(yx^{-1}) = A^+(yx^{-1})$, Therefore $A^+(xy) = A^+(yx^{-1})$, for x and y in G. Hence $A^+(xy) = A^+(yx^{-1})$ if and only if $A^+(x) = A^+(y^{-1}xy)$, for x and y in G. Also assume that $A^-(xy) = A^-(yx)$, we get, $A^-(yx) = A^-(yx^{-1}) = A^-(yx)$. Therefore $A^-(xy) = A^-(yx^{-1})$, for x and y in G. Conversely, assume that $A^-(xy) = A^-(yx^{-1})$, so, $A^-(yx^{-1}) = A^-(yx)$. Therefore $A^-(xy) = A^-(yx^{-1})$, for x and y in G. Hence $A^-(xy) = A^-(yx^{-1})$ if and only if $A^-(x) = A^-(y^{-1}xy)$, for x and y in G.

2.6 Theorem:
If $A = \langle A^+, A^- \rangle$ is a bipolar-valued fuzzy subring of a ring G, then $H = \{ x \in G \mid A^+(x) = 1, A^-(x) = -1 \}$ is either empty or is a subring of G.

Proof:
If no element satisfies this condition, then H is empty. If x and y in H, then $A^+(xy) \geq \min \{ A^+(x), A^+(y) \} = \min \{ 1, 1 \} = 1$. Therefore $A^+(xy) = 1$. And $A^-(xy) \leq \max \{ A^-(x), A^-(y) \} = \max \{ -1, -1 \} = -1$. Therefore $A^-(xy) = -1$. That is $xy \in H$. Hence H is a subring of G. Hence H is either empty or is a subring of G.

2.7 Theorem:
If $A = \langle A^+, A^- \rangle$ is a bipolar-valued fuzzy subring of G, then $H = \{ x \in G \mid A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e) \}$ is a subring of G.

Proof:
Here $H = \{ x \in G \mid A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e) \}$, by Theorem 2.1, $A^+(x^*) = A^+(x) = A^+(e)$ and $A^-(x^*) = A^-(x) = A^-(e)$. Therefore $x^* \in H$. Now, $A^+(xy) \geq \min \{ A^+(x), A^+(y) \} = \min \{ A^+(e), A^+(e) \} = A^+(e)$, and $A^+(e) = A^+(xy) \geq \min \{ A^+(xy), A^+(y) \} = A^+(y)$. Hence $A^+(e) = A^+(xy)$.
Also, $A^-(xy^i) \leq \max \{ A^-(x), A^-(y) \} = \max \{ A^-(e), A^-(e) \} = A^-(e)$, and $A^-(e) = A^-((xy^i)(xy^{-1})) \leq \max \{ A^-(xy^i), A^-(xy^{-1}) \} = A^-(xy^i)$. Therefore $A^-(e) = A^-((xy^i))$. Hence $A^+(e) = A^+(xy^i)$ and $A^-(e) = A^-((xy^i))$. Therefore $xy^1 \in H$. Hence $H$ is a subring of $G$.

**2.8 Theorem:**
Let $G$ be a ring. If $A = \langle A^+, A^- \rangle$ is a bipolar-valued fuzzy subring of $G$, then $A^+(xy) = \min \{ A^+(x), A^+(y) \}$ and $A^+(xy) = \max \{ A^-(x), A^-(y) \}$ for each $x,y \in G$ with $A^+(x) \neq A^+(y)$ and $A^-(x) \neq A^-(y)$.

**Proof:**
Assume that $A^+(x) > A^+(y)$ and $A^-(x) < A^-(y)$. Then $A^+(y) = A^+(x^{-1}xy) \geq \min \{ A^+(x), A^+(xy) \} = \min \{ A^+(x), A^+(xy) \} = A^+(xy) \geq \min \{ A^+(x), A^+(y) \} = A^+(y)$. Therefore $A^+(xy) = A^+(y) = \min \{ A^+(x), A^+(y) \}$. And $A^-(y) = A^-((xy^{-1}xy) \leq \max \{ A^-(x), A^-(xy) \} = \max \{ A^-(x), A^-(xy) \} = A^-((xy^{-1})) = A^-((xy)$. Therefore $A^-(xy) = A^-(y) = \max \{ A^-(x), A^-(y) \}$.

**2.9 Theorem:**
If $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ are two bipolar-valued fuzzy subrings of a ring $G$, then their intersection $A \cap B$ is a bipolar-valued fuzzy subring of $G$.

**Proof:**
Let $A = \{ x, A^+(x), A^-(x) > / x \in G \}$, $B = \{ x, B^+(x), B^-(x) > / x \in G \}$. Let $C = A \cap B$ and $C = \{ x, C^+(x), C^-(x) > / x \in G \}$. Now, $C^+(xy^i) = \min \{ A^+(xy^i), B^+(xy^i) \} \geq \min \{ \min \{ A^+(x), A^+(y) \}, \min \{ B^+(x), B^+(y) \} \} \geq \min \{ \min \{ A^+(x), B^+(x) \}, \min \{ A^+(y), B^+(y) \} \} = \min \{ C^+(x), C^+(y) \}$. Therefore $C^+(xy^i) \geq \min \{ C^+(x), C^+(y) \}$. Also, $C^-(xy^i) = \max \{ A^-(xy^i), B^-(xy^i) \} \leq \max \{ \max \{ A^-(x), A^-(y) \}, \max \{ B^-(x), B^-(y) \} \} \leq \max \{ \max \{ A^-(x), B^-(x) \}, \max \{ A^-(y), B^-(y) \} \} = \max \{ C^-(x), C^-(y) \}$. Therefore $C^-((xy^i)) \leq \max \{ C^-(x), C^-(y) \}$. Hence $A \cap B$ is a bipolar-valued fuzzy subring of $G$.

**2.10 Theorem:**
The intersection of a family of bipolar-valued fuzzy subrings of a ring $G$ is a bipolar-valued fuzzy subring of $G$.

**Proof:**
Let $\{ V_i : i \in I \}$ be a family of bipolar-valued fuzzy subrings of a ring $G$ and let $A = \bigcap_{i \in I} V_i$. Let $x$ and $y$ in $G$. Now, $A^+(xy^i) = \inf V_i^+(xy^i) \geq \inf \{ V_i^+(x), V_i^+(y) \} = \min \{ V_i^+(x), V_i^+(y) \} = \min \{ A^+(x), A^+(y) \}$. Therefore, $A^+(xy^i) \geq \min \{ A^+(x), A^+(y) \}$, for all $x$ and $y \in G$. And, $A^-(xy^i) = \sup V_i^-(xy^i) \leq \sup \{ V_i^-(x), V_i^-(y) \} = \sup \{ V_i^-(x), V_i^-(y) \}$. Therefore, $A^-(xy^i) \leq \sup \{ V_i^-(x), V_i^-(y) \} = \max \{ A^-(x), A^-(y) \}$. 

Notes
Therefore, $A^-(xy^{-1}) \leq \max\{ A^-(x), A^-(y) \}$, for all $x$ and $y$ in $G$. Hence, the intersection of a family of bipolar-valued fuzzy subrings of a ring $G$ is a bipolar-valued fuzzy subring of $G$.

2.11 Theorem:

If $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ are any two bipolar-valued fuzzy subrings of the rings $G_1$ and $G_2$ respectively, then $A \times B = \langle (A \times B)^+, (A \times B)^- \rangle$ is a bipolar-valued fuzzy subring of $G_1 \times G_2$.

Proof:

Let $A$ and $B$ be two bipolar-valued fuzzy subrings of the rings $G_1$ and $G_2$ respectively. Let $x_1$ and $x_2$ be in $G_1$, $y_1$ and $y_2$ be in $G_2$. Then $(x_1, y_1)$ and $(x_2, y_2)$ are in $G_1 \times G_2$. Now, $(A \times B)^+[(x_1, y_1)(x_2, y_2)] = (A \times B)^+(x_1x_2^{-1}, y_1y_2^{-1}) = \min\{ A^+(x_1x_2^{-1}), B^-(y_1y_2^{-1}) \} \geq \min\{ \min\{A^+(x_1), A^+(x_2)\}, \min\{B^-(y_1), B^-(y_2)\}\} = \min\{ \min\{A^+(x_1), B^-(y_1)\}, \min\{A^+(x_2), B^-(y_2)\}\} = \min\{A^+(x_1), A^+(x_2)\}, \min\{B^-(y_1), B^-(y_2)\}\} = \min\{A^-(x_1x_2^{\cdot-1}), B^-(y_1y_2^{\cdot-1})\} \leq \max\{\max\{A^-(x_1), A^-(x_2)\}, \max\{B^-(y_1), B^-(y_2)\}\} = \max\{\max\{A^-(x_1), B^-(y_1)\}, \max\{A^-(x_2), B^-(y_2)\}\} = \max\{A^-(x_1), B^-(y_1)\}, \max\{A^-(x_2), B^-(y_2)\}\} = \max\{A^-((x_1, y_1)), (A \times B)^+((x_1, y_2))\}.$ Therefore, $(A \times B)^-[(x_1, y_1)(x_2, y_2)] = (A \times B)^-(x_1x_2^{-1}, y_1y_2^{-1}) \leq \max\{A^-(x_1x_2^{\cdot-1}), B^-(y_1y_2^{\cdot-1})\} = \max\{A^-((x_1, y_1)), (A \times B)^-(x_1, y_2)\}. Hence $A \times B$ is a bipolar-valued fuzzy subring of $G_1 \times G_2$.

2.12 Theorem:

Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be any two bipolar-valued fuzzy subsets of the rings $G$ and $H$ respectively. Suppose that $e$ and $e$' are the identity elements of $G$ and $H$ respectively. If $A \times B$ is a bipolar-valued fuzzy subring of $G \times H$, then at least one of the following two statements must hold.

(i) $B^+(e') \geq A^+(x)$, for all $x$ in $G$ and $B^-(e') \leq A^-(x)$, for all $x$ in $G$,
(ii) $A^+(e) \geq B^+(y)$, for all $y$ in $H$ and $A^-(e) \leq B^-(y)$, for all $y$ in $H$.

Proof:

Let $A \times B$ be a bipolar-valued fuzzy subring of $G \times H$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find $a$ in $G$ and $b$ in $H$ such that $A^+(a) > B^+(e')$, $A^-(-a) < B^-(e')$ and $B^+(b) > A^+(e)$, $B^-(b) < A^-(e)$. We have, $(A \times B)^+ (a, b) = \min\{ A^+(a), B^+(b)\} > \min\{ A^-(a), B^-(b)\}$. Also, $(A \times B)^- (a, b) = \max\{ A^-(a), B^-(b)\} < \max\{ A^+(a), B^+(b)\}$. Thus $A \times B$ is not a bipolar-valued fuzzy subring of $G \times H$. Hence either $B^+(e') \geq A^+(x)$, for all $x$ in $G$ and $B^+(e') \leq A^-(x)$, for all $x$ in $G$ or $A^+(e) \geq B^+(y)$, for all $y$ in $H$ and $A^-(e) \leq B^-(y)$, for all $y$ in $H$.

2.13 Theorem:

Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be any two bipolar-valued fuzzy subsets of the rings $G$ and $H$, respectively and $A \times B$ is a bipolar-valued fuzzy subring of $G \times H$. Then the following are true:
(i) if $A^+(x) \leq B^+(e^i)$, for all $x$ in $G$ and $A^-(x) \geq B^-(e^i)$, for all $x$ in $G$, then $A$ is a bipolar-valued fuzzy subring of $G$, where $e^i$ is identity element of $H$.

(ii) if $B^+(x) \leq A^+(e)$ for all $x$ in $H$ and $B^-(x) \geq A^-(e)$, for all $x$ in $H$, then $B$ is a bipolar-valued fuzzy subring of $H$, where $e$ is identity element of $G$.

(iii) either $A$ is a bipolar-valued fuzzy subring of $G$ or $B$ is a bipolar-valued fuzzy subring of $H$, where $e$ and $e^i$ are the identity elements of $G$ and $H$ respectively.

**Proof:**

Let $A \times B$ be a bipolar-valued fuzzy subring of $G \times H$ and $x$ and $y$ in $G$. Then $(x, e^i)$ and $(y, e^i)$ are in $G \times H$. Now, using the property if $A^+(x) \leq B^+(e^i)$, for all $x$ in $G$ and $A^+(x) \geq B^+(e^i)$, for all $x$ in $G$, where $e^i$ is identity element of $H$, we get, $A^+(xy^1) = \min \{ A^+(xy^1), B^+(e^i) \} = (A \times B)^+ (xy^1), (e^i) \} = $ 

\begin{align*}
&\min \{ (A \times B)^+(x, e), (A \times B)^+(y, e) \} = \min \{ \min \{ A^+(x), B^+(e^i) \}, \min \{ A^+(y), B^+(e^i) \} \} = \min \{ A^+(x), A^+(y) \} \geq \min \{ A^+(x), A^+(y) \}.
\end{align*}

Therefore, $A^+(xy^1) \geq \min \{ A^+(x), A^+(y) \}$, for all $x$ and $y$ in $G$. Also, $A^+(xy^1) = \max \{ A^+(xy^1), B^-(e^i) \} = (A \times B)^-(xy^1), (e^i) \} = \max \{ A^-(x), B^-(e^i) \} = \max \{ A^-(x), A^-(y) \}$. Therefore, $A^+(xy^1) \leq \max \{ A^-(x), A^-(y) \}$, for all $x$ and $y$ in $G$. Hence $A$ is a bipolar-valued fuzzy subring of $G$. Thus (i) is proved. Now, using the property $B^+(x) \leq A^+(e)$ for all $x$ in $H$ and $B^-(x) \geq A^-(e)$, for all $x$ in $H$, we get, $B^+(xy^1) = \min \{ B^+(xy^1), A^+(e) \} = (A \times B)^+ (xy^1, (e) \} = \min \{ \min \{ B^+(x), A^+(y) \}, \min \{ B^+(x), A^+(y) \} \} \geq \min \{ (A \times B)^+(x, e), (A \times B)^+(y, e) \} = \min \{ \min \{ B^+(x), B^+(y) \} \} = \min \{ B^+(x), B^+(y) \}.$

Therefore, $B^+(xy^1) \leq \max \{ B^-(x), B^-(y) \}$, for all $x$ and $y$ in $H$. Hence $B$ is a bipolar-valued fuzzy subring of $H$. Thus (ii) is proved. Hence (iii) is clear.

**2.14 Theorem:**

Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subset of a ring $(G, .)$ and $V = \langle V^+, V^- \rangle$ be the strongest bipolar-valued fuzzy relation of $G$. Then $A$ is a bipolar-valued fuzzy subring of $G$ if and only if $V$ is a bipolar-valued fuzzy subring of $G \times G$.

**Proof:**

Suppose that $A$ is a bipolar-valued fuzzy subring of $G$. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $G \times G$. We have, $V^+(xy^1) = V^+[ (x_1, x_2)(y_1, y_2) ] = V^+(x_1y_1^{-1}, x_2y_2^{-1}) = \min \{ A^+(x_1y_1^{-1}), A^+(x_2y_2^{-1}) \} \geq \min \{ \min \{ A^+(x_1), A^+(x_2) \}, \min \{ A^+(y_1), A^+(y_2) \} \} = \min \{ V^+(x_1), V^+(x_2), V^+(y_1), V^+(y_2) \} = \min \{ V^+(x), V^+(y) \}.$

Therefore, $V^+(xy^1) \geq \min \{ V^+(x), V^+(y) \}$, for all $x$ and $y$ in $G \times G$. Also we have, $V^-(xy^1) = V^-[(x_1, x_2)(y_1, y_2)] = V^-[(x_1y_1^{-1}, x_2y_2^{-1})] = \max \{ A^-(x_1y_1^{-1}), A^-(x_2y_2^{-1}) \} = \max \{ \max \{ A^-((x_1), A^-((x_2), A^-(y_1), A^-(y_2)) \} = \max \{ V^-(x_1), V^-(x_2), V^-(y_1), V^-(y_2) \} = \max \{ V^-(x), V^-(y) \}.$

Therefore, $V^-(xy^1) \leq \max \{ V^-(x), V^-(y) \}$.
The bipolar valued fuzzy subring of a ring has been studied extensively due to its applications in various fields such as decision-making, information fusion, and pattern recognition. The characterization theorems of bipolar valued fuzzy subrings of a ring are obtained, which are essential in understanding the structure and properties of these subrings.

**CONCLUSION:**
In the study of the structure of bipolar valued fuzzy algebraic system, we notice that bipolar valued fuzzy subring of a ring with special properties always play an important role. In this paper, we define bipolar valued fuzzy subring of a ring and investigate the relationship among these bipolar valued fuzzy subrings of a ring. Some characterization theorems of bipolar valued fuzzy subrings of a ring are obtained. We hope that the research along this direction can be continued, and in fact, this work would serve as a foundation for further study of the theory of ring, it will be necessary to carry out more theoretical research to establish a general framework for the practical application.

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