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Classes of Nonparametric Regression Tests

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Abstract

Two classes of tests based on integrated squared error (ISE) of generalized varying kernel regression estimators and density estimates for testing parametric regression against nonparametric regression are proposed. The generalized kernel regression estimator has local bandwidth factor as function of some statistical measure. The density estimates, viz. pilot density estimates and varying kernel density estimates are used to obtain conditional expectation of regression function that is $E(Y|X = x)$. Empirical powers of some members of the proposed classes of tests are obtained using wild bootstrap method and are compared with existing tests in the literature.

Keywords: Density estimates, Empirical power, ISE, Kernel regression estimator, Nonparametric regression, Wild bootstrap.

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1 INTRODUCTION

Nonparametric regression is one among the flexible data analytic techniques. It does not assume a predetermined form for the function of the predictor. Suppose $(X_1, Y_1), \dots, (X_n, Y_n)$ is a random sample of size n from a bivariate population having density $f(x, y)$ whose relation is established by the model

$$Y_i = m(X_i) + \varepsilon_i, i = 1,2, \dots, n \tag{1}$$

where $m(\cdot)$ is an unknown regression function, $m(x) = E(Y|X = x)$ and ε_i are independent and identically distributed (iid) symmetric random errors with finite variance σ^2. For instance, in a study of soil, it is of interest to model the effect of different chemical compounds present in the soil on the fertility of the soil. The effect of temperature on atmospheric air pressure is studied through regression models.

Kernel smoothing technique is one of the prominent approaches to estimate $m(x)$. Nadaraya (1964) and Watson (1964) developed a kernel smoother based on fixed bandwidth to estimate $m(x)$ and this estimator is popularly known as Nadaraya-Watson (NW) estimator.Silverman (1986) developed adaptive kernel estimator which is a varying density estimator for estimating density. Bhat and Deshpande (2019a) constructed a generalized class of varying kernel estimators which includes estimators due to Demir and Toktamiş (2010), Aljuhani and Al Turk (2014), Joshi and Deshpande (2016) and Deshpande and Bhat (2019) as its particular members.These estimators are analysed taking 0.5 as the value of sensitivity parameter of the local bandwidth factor λ since Abramson (1982) established that the varying kernel estimators with sensitivity parameter value as 0.5 out perform NW estimator.

Testing the hypothesis about nonparametric regression function is important as it has many applications in reality. It tests whether $m(x)$ has a specific form which is parametric in nature against some other form which is not parametric. A detailed discussion on various tests for nonparametric regression is given in González-Manteiga and Crujeiras (2013). The Ttest due to Härdle and Mammen (1993) is ISE of $\widehat{m}_h(\cdot)$ and $\hat{l}_h(\cdot)$ and is defined by

$$T = nh^{\frac{1}{2}} \int \left(\widehat{m}_h - \hat{l}_h\right)^2 \pi(x) f(x) dx, \tag{2}$$

whose alternative form is given by

$$T = h^{\frac{1}{2}} \sum_{i=1}^{n} \left(\widehat{m}_h(x_i) - \hat{l}_h(x_i)\right)^2 \pi(x), \tag{3}$$

where $\widehat{m}_h(x) = \frac{\sum_{i=1}^{n} Y_i K_h(x - X_i)}{\sum_{i=1}^{n} K_h(x - X_i)}$, (4)

$K_h(x - X_i) = \frac{1}{h} K\left(\frac{x - X_i}{h}\right)$ is a kernel function with fixed bandwidth h,

$$\hat{l}_h(x) = \frac{\sum_{i=1}^{n} m_{\hat{\theta}}(x) K_h(x - X_i)}{n \tilde{p}(x)}, \tag{5}$$

where $\tilde{p}(x) = \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i)$ (6)

is pilot density estimate and $\hat{l}_h(x)$ is conditional expectation of $\widehat{m}_h(x)$. $\pi(x)$ is a weight function, $f(x)$ is true density at $x$and $m_{\hat{\theta}}(x)$ is parametric estimate at x with $\hat{\theta}$ being least square estimate. For mathematical simplicity, $\pi(x)$ is taken to be unity by Härdle and Mammen (1993) while obtaining properties of T.

The classes of tests due to Bhat and Deshpande (2019b) and Bhat and Deshpande (2019c) respectively are ISE of $\widehat{m}_{h_v}(\cdot)$,$\hat{l}_h(\cdot)$and ISE of $\widehat{m}_{h_v}(\cdot)$, $\hat{l}_{h_v}(\cdot)$ and are given by

$$B = n(h_v)^{1/2} \int \left(\widehat{m}_{h_v} - \hat{l}_h\right)^2 \pi(x) f(x) dx, \tag{7}$$

$$A = n(h_v)^{1/2} \int \left(\widehat{m}_{h_v} - \hat{l}_{h_v}\right)^2 \pi(x) f(x) dx, \tag{8}$$

where $\widehat{m}_{h_v}(x) = \dfrac{\frac{1}{n}\sum_{i=1}^{n} Y_i K_{h_v}(x-X_i)}{\frac{1}{n}\sum_{i=1}^{n} K_{h_v}(x-X_i)}$, (9)

$K_{h_v}(\cdot)$ is kernel function with varying bandwidth h_v,

$h_v = h\lambda$ with $\lambda = \left(\dfrac{\tilde{p}(x)}{g(\tilde{p}(x))}\right)^{-0.5}$, (10)

and $\hat{l}_{h_v}(x) = \dfrac{\sum_{i=1}^{n} m_{\hat{\theta}}(x) K_{h_v}(x-X_i)}{n\tilde{p}_v(x)}$. (11)

where $n\tilde{p}_v(x) = \sum_{i=1}^{n} K_{h_v}(x - X_i)$ (12)

is varying density estimate and $\hat{l}_{h_v}(x)$ is conditional expectation of $\widehat{m}_{h_v}(x)$. Both the classes of tests are modified T tests. The alternative forms of B and A are similar to that of (3) and $\pi(x)$ is taken as unity in these tests for mathematical simplicity.

In the present study, we propose two classes of tests whose local bandwidth factor is function of some statistical measures. Each class consists of some subclasses with extensive number of members with various λ. One of the classes is based on pilot density and the other is based on varying density estimates.

In section 2, we propose two classes of tests for testing parametric regression against nonparametric regression along with discussion on their distributions. We furnishempirical powers of the classes of tests in section 3,results and discussions in section 4 and conclusions in section 5.

2 CLASSES OF NONPARAMETRIC REGRESSION TESTS

The generalized class of adaptive kernel regression estimators based on some statistical function due to Bhat and Deshpande (2019a) is given by

$$\widehat{m}_{f_{(\eta)}}(x) = \frac{\sum_{i=1}^{n} Y_i K_{h_{f_{(\eta)}}}(x-X_i)}{\sum_{i=1}^{n} K_{h_{f_{(\eta)}}}(x-X_i)} \tag{13}$$

where$f$= Function of a statistical measure of $\tilde{p}(x)$,

$$h_{f_{(\eta)}} = h\lambda_{f_{(\eta)}}, \lambda_{f_{(\eta)}} = \left(\frac{\tilde{p}(x)}{g_{f_{(\eta)}}(\tilde{p}(x))}\right)^{-0.5}, \quad (14)$$

$g_{f_{(\eta)}}(\tilde{p}(x)) = \frac{f}{\eta}$, $f = R, MR, \widetilde{MD}$, $\eta = k^l$, $l \geq 0$, $k > 0$,

$R = \tilde{p}(x_{(n)}) - \tilde{p}(x_{(1)})$, $MR = \frac{\{\tilde{p}(x_{(n)}) + \tilde{p}(x_{(1)})\}}{2}$, $\widetilde{MD} = \frac{\sum_{i=1}^{n} |\tilde{p}(x_i) - \tilde{\tilde{p}}(x)|}{n}$, $\tilde{\tilde{p}}(x)$ is median of $\tilde{p}(x)$, $\tilde{p}(x_{(i)})$ is i^{th} order statistic of $\tilde{p}(.)$ and $|.|$ is absolute value. Various members of $\widehat{m}_{f_{(\eta)}}(x)$ emerge for different values of k and l.

Various values of k,l and η.

$\boldsymbol{k}$	1	2	3	2	2	2	2	2	3	3	n	n	n	n	n
$\boldsymbol{l}$	1	1	1	2	3	4	5	6	2	3	1	2	3	4	5
$\boldsymbol{\eta}$	1	2	3	4	8	16	32	64	9	81	n	n^2	n^3	n^4	n^5

We propose classes of tests $S_{1f_{(\eta)}}$ and $S_{2f_{(\eta)}}$ for testing

$$H_0: m(x) = m_{\theta_0}(x) \text{ against } H_1: m(x) = m_\theta(x) > m_{\theta_0}(x), \quad (15)$$

where $m_{\theta_0}(x)$ is parametric regression and $m_\theta(x)$ is nonparametric regression.

The proposed classes of tests $S_{1f_{(\eta)}}$ based on pilot density estimates and $S_{2f_{(\eta)}}$ based on varying density estimates are respectively given by

$$S_{1f_{(\eta)}} = n(h_{f_{(\eta)}})^{1/2} \int \left(\widehat{m}_{h_{f_{(\eta)}}} - \hat{l}_h\right)^2 \pi(x) f(x) dx, \quad (16)$$

$$S_{2f_{(\eta)}} = n(h_{f_{(\eta)}})^{1/2} \int \left(\widehat{m}_{h_{f_{(\eta)}}} - \hat{l}_{h_{f_{(\eta)}}}\right)^2 \pi(x) f(x) dx, \quad (17)$$

and their alternative forms are given by

$$S_{1f_{(\eta)}} = (h_{f_{(\eta)}})^{\frac{1}{2}} \sum_{i=1}^{n} \left(\widehat{m}_{h_{f_{(\eta)}}}(x_i) - \hat{l}_h(x_i)\right)^2 \pi(x) \quad (18)$$

and $S_{2f_{(\eta)}} = (h_{f_{(\eta)}})^{1/2} \sum_{i=1}^{n} \left(\widehat{m}_{h_{f_{(\eta)}}}(x_i) - \hat{l}_{h_{f_{(\eta)}}}(x_i)\right)^2 \pi(x)$. (19)

For $k = 2, n, l = 1, \eta = 2, n$ and $f = R$, the tests discussed in Bhat and Deshpande (2019b) become particular cases of $S_{1f_{(\eta)}}$. For $f = MR, \widetilde{MD}$, some of the tests discussed in Bhat and Deshpande (2019c) happen to be other members of $S_{1f_{(\eta)}}$. For $f = R, MR, \widetilde{MD}$, some members of the class of tests A due to Bhat and Deshpande (2019c) are particular members of $S_{2f_{(\eta)}}$. For $f = R, \eta = 1$, the tests are based on estimator due to Aljuhani and Al Turk (2014).

From Bhat and Deshpande (2019a), it is found that the varying kernel estimators whose local bandwidth factors are function of R, MR and $\widetilde{MD}$ of $\tilde{p}(x)$ outperform the estimators based on other statistical measures. Hence the preliminary tests based on these functions viz. for $\eta = 2$ and n were discussed in Bhat and Deshpande (2019b and 2019c). Proceeding on similar lines, in the present study, we consider general classes of tests based on R, MR and $\widetilde{MD}$ for various values of η.

From Bhat and Deshpande (2019b and 2019c), it follows that$S_{if_{(\eta)}}, i = 1,2$ is consistent and asymptotically follows $N(\mu_{if_{(\eta)}}, \tau^2_{if_{(\eta)}})$ with

$$\mu_{if_{(\eta)}} = E\left(S_{if_{(\eta)}}\right) = b_{ih_{f_{(\eta)}}} + o_p(1),\ \tau^2_{if_{(\eta)}} = Var\left(S_{if_{(\eta)}}\right),$$

where, $b_{1h_{f_{(\eta)}}} = \frac{(h_{f_{(\eta)}})^{1/2}}{n^2} K_h^{(2)}(0) K^{(2)}_{h_{f_{(\eta)}}}(0) \int_0^1 \frac{\sigma^2(x)\pi(x)}{f(x)} dx,$

$b_{2h_{f_{(\eta)}}} = h^{1/2}_{f_{(\eta)}} K^{(2)}_{h_{f_{(\eta)}}}(0) \int_0^1 \frac{\sigma^2(x)\pi(x)}{f(x)} dx,$

$\tau^2_{1f_{(\eta)}} = \frac{2h_{f_{(\eta)}}}{n} \left(K_h^{(2)}(0) K^{(2)}_{h_{f_{(\eta)}}}(0)\right)^2 \int_0^1 \frac{\sigma^4(x)\pi^2(x)}{f^2(x)} dx$

and $\tau^2_{2f_{(\eta)}} = \frac{2h_{vj}}{n} K^{(4)}_{h_{f_{(\eta)}}}(0) \int_0^1 \frac{\sigma^4(x)\pi^2(x)}{f^2(x)} dx.$

The classes of tests$S_{if_{(\eta)}}, i = 1,2$ reject the null hypothesis for their large values.

3 EMPIRICAL POWER OF THE TESTS

In this section, we study the performance of the proposed classes of tests in terms of their empirical power. We consider the regression function, $y = 2x - x^2 + c(x - 1/4)(x - 1/2)(x - 3/4) + \varepsilon$ with $\varepsilon \sim N(0,0.01)$ for testing $H_0: c = 0$ versus $H_1: c > 0$ since the same function is considered by Härdle and Mammen (1993) to establish their test.To obtain empirical power of $S_{if_{(\eta)}}, i = 1,2$ by simulation, we apply wild bootstrap method due to Wu (1986) since it yields $E(Y_i|(X_i, Y_i)_{i=1,\dots,n}) = m_{\hat{\theta}}(X_i)$. The wild bootstrap method involves resampling of the errors obtained from parametric fit under H_0. Proceeding on similir lines of Härdle et al. (2004), we present the following algorithm to construct the test$S_{if_{(\eta)}}, i = 1,2$ and obtain its empirical power.

(i) We generate n random numbers $X_i, i = 1, \dots, n$ from Uniform $(0,1)$ and obtain $m_{\hat{\theta}}(X_i)$ under H_0.

(ii) We compute the errors, $\tilde{\varepsilon}_i = Y_i - m_{\hat{\theta}}(X_i), i = 1, \dots, n.$

(iii) By wild bootstrapping $\tilde{\varepsilon}_i$ from two-point distribution, we obtain ε_i^*, taking $\varepsilon_i^* \sim \hat{F}_i$ with $E_{\hat{F}_i}(Z) = 0,\quad E_{\hat{F}_i}(Z^k) = \tilde{\varepsilon}_i^{\,k}, k = 2$ and 3.

(iv) We compute $Y_i^* = m_{\hat{\theta}}(X_i) + \varepsilon_i^*$.

The steps (ii) to (iv) are repeated 1000 times to approximate the distributions of $S_{if_{(\eta)}}, i = 1,2$, empirically and are used to obtain critical values of the tests under H_0. These are termed as bootstrap repetitions. To obtain the empirical power of $S_{1f_{(\eta)}}, S_{2f_{(\eta)}}$, 10000 replications are performed for various values of c, h and n under quartic kernel, $\frac{15}{16}(1-u^2)^2 I_{\{|u|\leq 1\}}$. For T test, $u = \frac{(x-X_i)}{h}$ and for proposed classes of tests, $u = \frac{(x-X_i)}{h_{f_{(\eta)}}}$.

We compare the performances of members of $S_{if_{(\eta)}}, i = 1,2$ and T test. For $h = 0.01, 0.1, 0.25, 0.3, 0.4$, the empirical powers of proposed classes of tests and T at 5% level of significance $(\alpha = 0.05)$ are computed with $n = 50, 100, c = 1, 2, 3, 4$ and $n = 200, c = 1, 2, 3$. The values of empirical power of $T, S_{iR_{(\eta)}}$, $S_{iMR_{(\eta)}}$ and $S_{i\widetilde{MD}_{(\eta)}}, i = 1, 2$ are respectively furnished in tables 1, 2, 3 and 4.

4. RESULTS AND DISCUSSIONS

From tables 1, 2, 3 and 4, we observe that, for λ based on all the statistical functions considered, the empirical power of $S_{if_{(\eta)}}, i = 1,2$ increase as c increases for all the values of n, h and η. $S_{1f_{(\eta)}}$ has higher power than T for increasing n, c and η for η value other than n^3. As c increases, $S_{2f_{(\eta)}}$ outperforms T.

From tables 2, 3 and 4,we find that, for $i = 1, 2, S_{i\widetilde{MD}_{(\eta)}}$ possess higher power than $S_{iR_{(\eta)}}$ and $S_{iMR_{(\eta)}}$ for all the values of η for increasing values of c and n.

$S_{1f_{(\eta)}}$ outperforms $S_{2f_{(\eta)}}$ for increasing valuesof c, for $\eta = 3, 8$ and 32. $S_{2f_{(\eta)}}$ has higher power than $S_{1f_{(\eta)}}$ for lower values of bandwidth and sample sizes.

Both the classes of tests have zero as their empirical power for lower values of h and $\eta = n^3$ which leads to oversmoothing of $m(x)$ by $S_{1f_{(\eta)}}$ and $S_{2f_{(\eta)}}$. The traces of this fact is exhibited when $\eta = n^2$.

5. CONCLUSIONS

Two classes of tests proposed are based on ISE of varying kernel estimators and regression function obtained from pilot density estimates and varying density estimates. We record our conclusions as follows.

- The proposed classes of tests outperform Härdle and Mammen test for moderate sample sizes and values of η.
- The empirical power of $S_{1f_{(\eta)}}$ and $S_{2f_{(\eta)}}$ increase for increasing values of h and decrease for $h > 0.25$.
- For increasing n, the empirical power of $S_{1f_{(\eta)}}$ and $S_{2f_{(\eta)}}$ increase for all values of c.
- The proposed classes of tests lead to oversmoothing of regression function for $\eta = n^3$ when bandwidth values are smaller.

APPENDIX

Table 1: Empirical power of T for various values of c, h and n.

n	h	c			
		1	2	3	4
50	0.10	0.1075	0.4573	0.8271	0.9711
	0.25	0.2547	0.6002	0.8680	0.9706
	0.30	0.2012	0.4884	0.7901	0.9349
	0.40	0.1438	0.3371	0.5717	0.7840
100	0.10	0.4478	0.9235	0.9984	1.0000
	0.25	0.4129	0.8737	0.9918	0.9998
	0.30	0.3364	0.7858	0.9772	0.9992
	0.40	0.2155	0.5505	0.8505	0.9695
n	h	c			
		1	2	3	
200	0.10	0.7187	0.9980	1.0000	
	0.25	0.6385	0.9922	1.0000	
	0.30	0.5406	0.9729	1.0000	
	0.40	0.3499	0.8243	0.9888	

Table 2: Empirical power of $S_{1R_{(\eta)}}, S_{2R_{(\eta)}}$ for various values of $\eta, h, c$and n.

n	η	h	c							
			1		2		3		4	
			$S_{1R_{(\eta)}}$	$S_{2R_{(\eta)}}$	$S_{1R_{(\eta)}}$	$S_{2R_{(\eta)}}$	$S_{1R_{(\eta)}}$	$S_{2R_{(\eta)}}$	$S_{1R_{(\eta)}}$	$S_{2R_{(\eta)}}$
50	3	0.10	0.3870	0.1544	0.7353	0.4924	0.9425	0.8424	0.9939	0.9718
		0.25	0.4464	0.1942	0.7987	0.5654	0.9547	0.8630	0.9946	0.9730
		0.30	0.4128	0.1705	0.7468	0.5102	0.9385	0.8315	0.9922	0.9623
		0.40	0.3201	0.1275	0.6648	0.4265	0.8984	0.7638	0.9788	0.9342
	8	0.10	0.6235	0.0343	0.8705	0.2050	0.9748	0.5757	0.9978	0.8791
		0.25	0.3491	0.1421	0.7064	0.4733	0.9309	0.8229	0.9909	0.9705
		0.30	0.6029	0.2903	0.8755	0.6641	0.9803	0.9183	0.9987	0.9902
		0.40	0.5080	0.2249	0.8207	0.5889	0.9679	0.8891	0.9967	0.9820
	32	0.10	0.9815	0.4488	0.9972	0.7173	0.9996	0.9234	0.9998	0.9879
		0.25	0.9984	0.6233	0.9997	0.8571	0.9999	0.9773	0.9999	0.9968
		0.30	0.9969	0.5635	0.9994	0.8278	1.0000	0.9691	1.0000	0.9967
		0.40	0.9885	0.4525	0.999	0.785	0.9998	0.9579	1.0000	0.9941
	n^2	0.10	0.0594	0.6392	0.1841	0.8195	0.4809	0.9504	0.7942	0.9927
		0.25	0.9352	0.4252	0.9773	0.6628	0.9964	0.8819	0.9996	0.9779
		0.30	0.9999	0.1045	1.0000	0.2828	1.0000	0.5991	1.0000	0.8701
		0.40	1.0000	0.1939	1.0000	0.4116	1.0000	0.7229	1.0000	0.9285

	n^3	0.10	0.0000	0.2490	0.0000	0.4625	0.0000	0.7591	0.0000	0.9340
		0.25	0.0000	0.8312	0.0000	0.9260	0.0047	0.9831	0.0493	0.9976
		0.30	0.0001	0.9019	0.0002	0.9647	0.0107	0.9924	0.0894	0.9987
		0.40	0.0001	0.0808	0.0024	0.2205	0.0335	0.5144	0.1964	0.8186
100	3	0.10	0.6290	0.3339	0.9520	0.8544	0.9990	0.9940	1.0000	0.9999
		0.25	0.6624	0.3652	0.9608	0.8697	0.9988	0.9925	1.0000	0.9999
		0.30	0.6049	0.3281	0.9511	0.8409	0.9985	0.9909	1.0000	0.9998
		0.40	0.5102	0.2529	0.9137	0.7600	0.9950	0.9785	0.9998	0.9990
	8	0.10	0.5365	0.0972	0.9225	0.5817	0.9970	0.9542	1.0000	0.9986
		0.25	0.5244	0.2712	0.9387	0.8246	0.9985	0.9928	0.9999	0.9997
		0.30	0.4760	0.2502	0.9214	0.8056	0.9973	0.9881	1.0000	1.0000
		0.40	0.3972	0.3972	0.8811	0.8811	0.9949	0.9949	1.0000	1.0000
	32	0.10	0.9837	0.2527	0.9987	0.7106	1.0000	0.9749	1.0000	0.9992
		0.25	0.9802	0.5111	0.9991	0.9172	1.0000	0.9981	1.0000	1.0000
		0.30	0.9584	0.4401	0.9985	0.8989	1.0000	0.9968	1.0000	1.0000
		0.40	0.9146	0.3604	0.9950	0.8712	1.0000	0.9950	1.0000	1.0000
	n^2	0.10	0.5291	0.5291	0.8116	0.8116	0.9716	0.9716	0.9988	0.9988
		0.25	0.9999	0.5681	1.0000	0.8393	1.0000	0.9823	1.0000	0.9995
		0.30	1.0000	0.6962	1.0000	0.9043	1.0000	0.9908	1.0000	0.9999
		0.40	1.0000	0.8320	1.0000	0.9605	1.0000	0.9982	1.0000	0.9999
	n^3	0.10	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0016	1.0000
		0.25	0.0202	0.9920	0.1215	0.9992	0.5021	1.0000	0.8891	1.0000
		0.30	0.0699	0.9989	0.2752	1.0000	0.6912	1.0000	0.9552	1.0000
		0.40	0.2645	0.2645	0.5849	0.5849	0.8963	0.8963	0.9930	0.9930

	η	h	c					
			1		2		3	
			$S_{1MR(\eta)}$	$S_{2MR(\eta)}$	$S_{1MR(\eta)}$	$S_{2MR(\eta)}$	$S_{1MR(\eta)}$	$S_{2MR(\eta)}$
200	3	0.10	0.8509	0.6240	0.9985	0.9931	1.0000	1.0000
		0.25	0.8626	0.6320	0.9993	0.9946	1.0000	1.0000
		0.30	0.8210	0.5796	0.9990	0.9905	1.0000	0.9999
		0.40	0.7327	0.7327	0.9950	0.9950	1.0000	1.0000
	8	0.10	0.9528	0.5212	0.9996	0.9862	1.0000	1.0000
		0.25	0.9485	0.7795	1.0000	0.9981	1.0000	1.0000
		0.30	0.9221	0.7319	0.9997	0.9972	1.0000	1.0000
		0.40	0.8745	0.6540	0.9995	0.9947	1.0000	1.0000
	32	0.10	0.9989	0.5707	1.0000	0.9797	1.0000	1.0000
		0.25	0.9964	0.7527	1.0000	0.9977	1.0000	1.0000
		0.30	0.9917	0.6919	1.0000	0.9961	1.0000	1.0000
		0.40	0.9776	0.6099	1.0000	0.9923	1.0000	1.0000
	n^2	0.10	1.0000	0.9979	1.0000	1.0000	1.0000	1.0000

		0.25	1.0000	0.9991	1.0000	1.0000	1.0000	1.0000
		0.30	1.0000	0.9599	1.0000	0.9975	1.0000	1.0000
		0.40	1.0000	0.9913	1.0000	0.9998	1.0000	1.0000
200	n^3	0.10	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
		0.25	0.1162	1.0000	0.4771	1.0000	0.9303	1.0000
		0.30	0.3505	1.0000	0.7620	1.0000	0.9858	1.0000
		0.40	0.7954	0.7954	0.9702	0.9702	0.9997	0.9997

Table 3: Empirical power of $S_{1MR_{(\eta)}}$, $S_{2MR_{(\eta)}}$ for various values of η, h, c and n.

n	η	h	c							
			1		2		3		4	
			$S_{1MR_{(\eta)}}$	$S_{2MR_{(\eta)}}$	$S_{1MR_{(\eta)}}$	$S_{2MR_{(\eta)}}$	$S_{1MR_{(\eta)}}$	$S_{2MR_{(\eta)}}$	$S_{1MR_{(\eta)}}$	$S_{2MR_{(\eta)}}$
50	3	0.10	0.1733	0.1733	0.515	0.515	0.8596	0.8596	0.9796	0.9796
		0.25	0.1832	0.1832	0.5442	0.5442	0.8531	0.8531	0.9716	0.9716
		0.30	0.1491	0.1491	0.4772	0.4772	0.8174	0.8174	0.9543	0.9543
		0.40	0.113	0.113	0.3852	0.3852	0.7215	0.7215	0.9185	0.9185
	8	0.10	0.3193	0.0323	0.6536	0.2017	0.9087	0.5883	0.9858	0.8917
		0.25	0.6526	0.1247	0.8959	0.4685	0.9832	0.8283	0.9991	0.9702
		0.30	0.5699	0.2627	0.862	0.6415	0.9761	0.9066	0.9975	0.9866
		0.40	0.4655	0.2	0.8006	0.5716	0.9619	0.8634	0.9945	0.9762
	32	0.10	0.8153	0.1682	0.9346	0.4178	0.9895	0.7632	0.9984	0.9468
		0.25	0.9031	0.2793	0.9756	0.6287	0.9983	0.8927	0.9998	0.9867
		0.30	0.856	0.2325	0.9632	0.5883	0.9943	0.8889	0.9997	0.9839
		0.40	0.7577	0.4133	0.9327	0.7556	0.9934	0.9516	0.999	0.9938
	n^2	0.10	0.0007	0.0397	0.0056	0.1389	0.0609	0.4224	0.2858	0.7579
		0.25	0.9483	0.0776	0.9829	0.2312	0.9971	0.5520	0.9999	0.8494
		0.30	0.9653	0.0158	0.9873	0.0703	0.9988	0.3000	0.9999	0.6504
		0.40	0.9817	0.0385	0.9934	0.1537	0.9991	0.4502	1.0000	0.7902
	n^3	0.10	0.0000	0.1858	0.0000	0.3865	0.0000	0.7013	0.0000	0.9180
		0.25	0.0000	0.0096	0.0001	0.0539	0.0042	0.2405	0.0579	0.5938
		0.30	0.0303	0.0303	0.1204	0.1204	0.3767	0.3767	0.7226	0.7226
		0.40	0.1276	0.1276	0.3020	0.3020	0.6178	0.6178	0.8764	0.8764
100	3	0.10	0.6387	0.3439	0.9614	0.8602	0.9989	0.9953	1.0000	1.0000
		0.25	0.6328	0.3400	0.9595	0.8564	0.9991	0.9913	1.0000	0.9999
		0.30	0.5562	0.2794	0.9416	0.8233	0.9977	0.9864	0.9999	0.9998
		0.40	0.4502	0.4502	0.8876	0.8876	0.9939	0.9939	0.9997	0.9997
	8	0.10	0.9848	0.8447	0.9994	0.9890	1.0000	0.9998	1.0000	1.0000
		0.25	0.9867	0.8062	0.9999	0.9862	1.0000	0.9998	1.0000	1.0000
		0.30	0.9704	0.7432	0.9994	0.9824	0.9998	0.9995	1.0000	1.0000
		0.40	0.9347	0.9347	0.9975	0.9975	1.0000	1.0000	1.0000	1.0000

	32	0.10	0.9865	0.2550	0.9997	0.7143	0.9999	0.9737	1.0000	0.9996
		0.25	0.9713	0.2184	0.9987	0.7555	1.0000	0.9861	1.0000	1.0000
		0.30	0.9416	0.4045	0.9983	0.8901	1.0000	0.9967	1.0000	1.0000
		0.40	0.8756	0.3033	0.9934	0.8458	1.0000	0.9934	1.0000	1.0000
	n^2	0.10	0.0219	0.5098	0.1467	0.7988	0.5585	0.9731	0.9177	0.9989
		0.25	0.9821	0.1430	0.9979	0.4445	0.9999	0.8415	1.0000	0.9879
		0.30	0.9908	0.0318	0.9991	0.1853	0.9999	0.6245	1.0000	0.9436
		0.40	1.0000	0.0976	1.0000	0.3780	1.0000	0.8119	1.0000	0.9826
	n^3	0.10	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0007	1.0000
		0.25	0.0337	0.9965	0.1857	1.0000	0.6009	1.0000	0.9333	1.0000
		0.30	0.1278	0.9992	0.3856	1.0000	0.8098	1.0000	0.9807	1.0000
		0.40	0.4170	1.0000	0.7287	1.0000	0.9540	1.0000	0.9978	1.0000

n	η	h	c					
			1		2		3	
			$S_{1MR_{(\eta)}}$	$S_{2MR_{(\eta)}}$	$S_{1MR_{(\eta)}}$	$S_{2MR_{(\eta)}}$	$S_{1MR_{(\eta)}}$	$S_{2MR_{(\eta)}}$
200	3	0.10	0.9757	0.8476	1.0000	0.9991	1.0000	1.0000
		0.25	0.9819	0.8407	1.0000	0.9991	1.0000	1.0000
		0.30	0.9685	0.7906	1.0000	0.9978	1.0000	1.0000
		0.40	0.9277	0.6850	0.9999	0.9921	1.0000	1.0000
	8	0.10	0.9502	0.5061	0.9997	0.9834	1.0000	1.0000
		0.25	0.9359	0.5095	1.0000	0.9884	1.0000	1.0000
		0.30	0.7047	0.4673	0.9957	0.9829	1.0000	1.0000
		0.40	0.8460	0.6118	0.9987	0.9925	1.0000	1.0000
	32	0.10	0.9747	0.1034	0.9999	0.7863	1.0000	0.9984
		0.25	0.9241	0.4563	0.9998	0.9832	1.0000	1.0000
		0.30	0.8772	0.3987	0.9993	0.9789	1.0000	1.0000
		0.40	0.8143	0.5757	0.9991	0.9934	1.0000	1.0000
	n^2	0.10	1.0000	0.7077	1.0000	0.9518	1.0000	0.9990
		0.25	1.0000	0.5185	1.0000	0.8939	1.0000	0.9959
		0.30	1.0000	0.7056	1.0000	0.9634	1.0000	0.9994
		0.40	1.0000	0.4835	1.0000	0.8832	1.0000	0.9970
	n^3	0.10	0.9991	0.7106	1.0000	0.9495	1.0000	0.9989
		0.25	1.0000	0.5197	1.0000	0.8928	1.0000	0.9981
		0.30	1.0000	0.7210	1.0000	0.9566	1.0000	0.9997
		0.40	1.0000	0.4805	1.0000	0.8849	1.0000	0.9965

Table 4: Empirical power of $S_{1\widetilde{MD}(\eta)}$, $S_{2\widetilde{MD}(\eta)}$ for various values of η, h, c and n.

n	η	h	c							
			1		2		3		4	
			$S_{1\widetilde{MD}(\eta)}$	$S_{2\widetilde{MD}(\eta)}$	$S_{1\widetilde{MD}(\eta)}$	$S_{2\widetilde{MD}(\eta)}$	$S_{1\widetilde{MD}(\eta)}$	$S_{2\widetilde{MD}(\eta)}$	$S_{1\widetilde{MD}(\eta)}$	$S_{2\widetilde{MD}(\eta)}$
50	3	0.10	0.3789	0.0409	0.6851	0.2099	0.9179	0.5864	0.9868	0.8884
		0.25	0.7886	0.1899	0.9430	0.5463	0.9907	0.8736	0.9996	0.9804
		0.30	0.7307	0.3811	0.9235	0.7375	0.9893	0.9420	0.9992	0.9928
		0.40	0.6235	0.3032	0.8803	0.6832	0.9834	0.9202	0.9980	0.9891
50	8	0.10	0.4560	0.0075	0.7127	0.0628	0.9200	0.3063	0.9896	0.6945
		0.25	0.6383	0.0350	0.8701	0.2053	0.9778	0.5963	0.9980	0.8887
		0.30	0.8908	0.0316	0.9739	0.1983	0.9972	0.5819	0.9998	0.8882
		0.40	0.8157	0.0754	0.9490	0.3430	0.9955	0.7440	0.9996	0.9472
	32	0.10	0.7590	0.3193	0.8960	0.5621	0.9779	0.8335	0.9968	0.9649
		0.25	0.9797	0.4108	0.9947	0.6951	0.9996	0.9139	0.9999	0.9889
		0.30	0.9738	0.4158	0.9951	0.7088	0.9991	0.9177	1.0000	0.9876
		0.40	0.9583	0.3777	0.9893	0.6799	0.9994	0.9155	0.9999	0.9862
	n^2	0.10	0.0004	0.9696	0.0034	0.9894	0.0489	0.9984	0.2324	1.0000
		0.25	0.6077	0.6077	0.7906	0.7906	0.9340	0.9340	0.9891	0.9891
		0.30	0.9997	0.7095	0.9996	0.8649	0.9999	0.9620	1.0000	0.9950
		0.40	0.9999	0.2258	1.0000	0.4399	1.0000	0.7315	1.0000	0.9263
	n^3	0.10	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
		0.25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0015	1.0000
		0.30	0.0000	1.0000	0.0000	1.0000	0.0001	1.0000	0.0027	1.0000
		0.40	0.0000	0.5819	0.0000	0.7679	0.0002	0.9254	0.0096	0.9866
100	3	0.10	0.9324	0.3750	0.9946	0.8376	0.9999	0.9922	1.0000	0.9999
		0.25	0.9170	0.3709	0.9962	0.8756	1.0000	0.9955	1.0000	1.0000
		0.30	0.8804	0.5986	0.9935	0.9533	0.9998	0.9990	1.0000	1.0000
		0.40	0.7905	0.4861	0.9840	0.9232	0.9996	0.9969	1.0000	1.0000
	8	0.10	0.9886	0.2745	0.9995	0.7193	1.0000	0.9761	1.0000	0.9994
		0.25	0.9864	0.5754	0.9993	0.9344	1.0000	0.9986	1.0000	1.0000
		0.30	0.9705	0.4874	0.9992	0.9142	0.9999	0.9973	1.0000	1.0000
		0.40	0.9254	0.6686	0.9972	0.9694	1.0000	0.9997	1.0000	1.0000
	32	0.10	1.0000	0.7070	1.0000	0.9287	1.0000	0.9970	1.0000	1.0000
		0.25	0.9995	0.8579	0.9999	0.9834	1.0000	0.9996	1.0000	1.0000
		0.30	1.0000	0.7953	1.0000	0.9771	1.0000	0.9993	1.0000	1.0000
		0.40	1.0000	0.6739	1.0000	0.9525	1.0000	0.9991	1.0000	1.0000
	n^2	0.10	0.0000	0.6146	0.0006	0.8522	0.0155	0.9805	0.1951	0.9996
		0.25	0.9895	0.0275	0.9986	0.1505	0.9999	0.5262	1.0000	0.8878
		0.30	0.9974	0.0759	0.9995	0.2802	1.0000	0.6949	1.0000	0.9517
		0.40	1.0000	0.0180	0.9999	0.1198	1.0000	0.4941	1.0000	0.8844
	n^3	0.10	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
		0.25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0008	1.0000

		0.30	0.0000	0.6872	0.0009	0.8873	0.0295	0.9854	0.2526	0.9987
		0.40	0.0008	0.9237	0.0132	0.9862	0.1547	0.9992	0.5729	1.0000

n	η	h	c					
			1		2		3	
			$S_{1\widetilde{MD}(\eta)}$	$S_{2\widetilde{MD}(\eta)}$	$S_{1\widetilde{MD}(\eta)}$	$S_{2\widetilde{MD}(\eta)}$	$S_{1\widetilde{MD}(\eta)}$	$S_{2\widetilde{MD}(\eta)}$
200	3	0.10	0.9895	0.4095	1.0000	0.9622	1.0000	1.0000
		0.25	0.9830	0.6411	1.0000	0.9952	1.0000	1.0000
		0.30	0.9699	0.5919	0.9999	0.9918	1.0000	1.0000
		0.40	0.9220	0.4879	0.9995	0.9858	1.0000	1.0000
200	8	0.10	0.9998	0.6819	1.0000	0.9877	1.0000	1.0000
		0.25	0.9981	0.8107	1.0000	0.9979	1.0000	1.0000
		0.30	0.9951	0.7529	1.0000	0.9970	1.0000	1.0000
		0.40	0.9804	0.6409	1.0000	0.9929	1.0000	1.0000
	32	0.10	1.0000	0.0389	1.0000	0.4669	1.0000	0.9675
		0.25	0.9989	0.1202	1.0000	0.7980	1.0000	0.9989
		0.30	1.0000	0.0917	1.0000	0.7676	1.0000	0.9978
		0.40	0.9997	0.1580	1.0000	0.8794	1.0000	0.9991
	n^2	0.10	0.2760	0.9972	0.6798	0.9999	0.9672	1.0000
		0.25	1.0000	0.9954	1.0000	0.9998	1.0000	1.0000
		0.30	1.0000	0.9995	1.0000	1.0000	1.0000	1.0000
		0.40	1.0000	0.9803	1.0000	0.9987	1.0000	1.0000
	n^3	0.10	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
		0.25	0.8865	1.0000	0.9824	1.0000	1.0000	1.0000
		0.30	0.9795	0.9795	0.9986	0.9986	0.9999	0.9999
		0.40	0.9997	0.9997	1.0000	1.0000	1.0000	1.0000

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