Triple Connected Domination Number of Some Specific Classes of Complementary Prism

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Abstract

Let $G = (V(G), E(G))$ be a graph and $\overline{G}$ be the complement of $G$. The complementary prism of $G$, denoted by $G\overline{G}$, is the graph obtained from the disjoint union of $G$ and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of $G$ and $\overline{G}$. A subset $S$ of $V$ of a nontrivial connected graph $G$ is said to be a triple connected domination set if $S$ is a dominating set and the induced subgraph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number of $G$. The complementary prism of a graph $G$, denoted as $G\overline{G}$, is obtained from the graph $G\cup \overline{G}$ by adding a perfect matching between the corresponding vertices of $G$ and $\overline{G}$. In this paper, we study the triple connected domination number of complementary prisms. The triple connected domination number of a complementary prism $G\overline{G}$, denoted by $\gamma_{tc}(G\overline{G})$, is the cardinality of a minimum triple connected dominating set of $G\overline{G}$. We determine the triple connected domination number of $G\overline{G}$ for specific graphs $G$.

Keywords: Domination number, triple connected domination number, Cartesian product, complementary product, complementary prism, triple connected domination number of complementary prism.
1. INTRODUCTION

We consider a finite, simple connected and undirected graph \(G(V,E)\), where \(V\) is called the vertex set and \(E\) is called the edge set. Degree of a vertex \(v\) is denoted by \(\text{deg}(v)\), the maximum degree of a graph \(G\) is denoted by \(\Delta(G)\) and the minimum degree of a graph is denoted by \(\delta(G)\). We denote a cycle on \(n\) vertices by \(C_n\), a path on \(n\) vertices by \(P_n\) and a complete graph on \(n\) vertices by \(K_n\). A graph \(G\) is connected if any two vertices of \(G\) are connected by a path. A maximal connected sub graph of a graph \(G\) is called a component of \(G\). The number of components of \(G\) is denote by \(\chi(G)\). The complement \(\overline{G}\) of \(G\) is the graph with vertex set \(V\) in which two vertices are adjacent if and only if they are not adjacent in \(G\). A tree is a connected acyclic graph. A leaf of a graph is a vertex of degree one and the vertex adjacent to leaf is called a support vertex . A bipartite graph or bigraph is a graph whose vertex set can be divided into two disjoint sets \(V_1\) and \(V_2\) such that every edge has one end in \(V_1\) and another end in \(V_2\). A complete bipartite graph is a bipartite graph where every vertex of \(V_1\) is adjacent to every vertex in \(V_2\). The complete bipartite graph with partitions of order \(|V_1|=m\) and \(|V_2|=n\) is denoted by \(K_{m,n}\). A star, denoted by \(K_{1,n-1}\) is a tree with one root vertex and \(n-1\) pendant vertices. A bistar, denoted by \(B(m,n)\) is the graph obtained by joining the root vertices of the stars \(K_{1,m}\) and \(K_{1,n}\). The friendship graph, denoted by \(F_n\) can be constructed by identifying \(n\) copies of the cycle \(C_3\) at a common vertex. A wheel graph, denoted by \(W_n\) is a graph with \(n\) vertices, formed by connecting a single vertex to all vertices of \(C_{n-1}\). A helm graph, denoted by \(H_n\) is a graph obtained from the wheel \(W_n\) by attaching a pendant vertex to each vertex in the outer cycle of \(W_n\). Corona of two graphs \(G_1\) and \(G_2\), denoted by \(G_1 \circ G_2\) is the graph obtained by taking one copy of \(G_1\) and \(|V_1|\) copies of \(G_2\) (\(|V_1|\) is the number of vertices in \(G_1\)) in which \(i^{th}\) vertex of \(G_1\) is joined to every vertex in the \(i^{th}\) copy of \(G_2\). If \(S\) is a subset of \(V\), then \(\langle S \rangle\) denotes the vertex induced sub graph of \(G\) induced by \(S\). A crown graph, denoted by \(C_{n}^+\) is a graph obtained from any cycle \(C_n\) with a pendant edge attached at each vertex. A Hoffman tree, denoted by \(P_n^+\) is a graph obtained from any path \(P_n\) with a pendant edge attached at each vertex. A graph \(G\) is said to be planar if \(G\) can be drawn on a plane without intersecting edges. Bull graph is a planar undirected graph with 5 vertices and edges in the form of a triangle with two disjoint pendant edges. Diamond graph is a planar undirected graph with 4 vertices and 5 edges. It consists of a complete graph \(K_4\) minus one edge.

A subset \(S\) of \(V\) is called a dominating set of \(G\) if every vertex in \(V-S\) is adjacent to at least one vertex in \(S\). The domination number \(\gamma(G)\) of \(G\) is the minimum
cardinality taken over all dominating sets in $G$. A dominating set $S$ of a connected graph $G$ is said to be connected dominating set of $G$ if the induced sub graph $\langle S \rangle$ is connected. The minimum cardinality taken over all connected dominating sets is the connected domination number and is denoted by $\gamma_c(G)$.

Recently, the concept of triple connected graphs has been introduced by Paulraj Joseph.J.et.al [2] by considering the existence of a path containing any three vertices of $G$. They have studied the properties of triple connected graphs and established many results on them. A graph $G$ is said to be triple connected if any three vertices lie on a path in $G$. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs. In [3] Mahadevan.G.et.al introduced triple connected domination number of a graph and found many results on them. A subset $S$ of $V$ of a nontrivial connected graph $G$ is said to be triple connected dominating set, if $S$ is a dominating set and the induced sub graph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number of $G$ and is denoted by $\gamma_{tc}(G)$.

In [6] Haynes et.al. introduced the complementary prism of two graphs. The cartesian product of two graphs $G$ and $H$, $G \circ H$ has a vertex set of $V(G) \times V(H)$. The edges of $G \circ H$ are formed by replacing each vertex in $G$ with a copy of $H$ and replacing each vertex of $H$ with a copy of $G$. Let $A \subseteq V(G)$ and $B \subseteq V(H)$, the complementary product, denoted by $G(A) \circ G(B)$ has the vertex set $V(G) \times V(H)$. The edge set is defined as follows. There is an edge between the vertices $(g_i,h_j)$ and $(g_k,h_l)$ if one of the following holds:

(i) $i=k$, $g_i$ is in $A$ and there is an edge between $h_j$ and $h_l$.

(ii) or if $i=k$, $g_i$ is not in $A$ and there is no edge between $h_j$ and $h_l$.

(iii) if $j=l$, $h_j$ is in $B$ and there is an edge between $g_i$ and $g_k$.

(iv) or if $j=l$, $h_j$ is not in $B$ and there is no edge between $g_i$ and $g_k$.

That is, the complementary product is a graph on $V(G) \times V(H)$ vertices and for a vertex in $G$, we replace that vertex with a copy of $H$ if it is in $A$ and a copy of $\overline{H}$ if it is not in $A$. For a vertex in $H$, we replace that vertex with a copy of $G$ if it is in $B$ and a copy of $\overline{G}$ if it is not in $B$. 
The complementary prism, a special case of the complementary product $G \circ K_2(S)$ where $|S| = 1$. The complementary prism of a graph $G$, denoted as $GG$, is obtained from the graph $G \cup \overline{G}$ by adding a perfect matching between the corresponding vertices of $G$ and $\overline{G}$.

**Figure 1**: Complementary Product

**Figure 2**: Complementary prism $C_6 \overline{C_6}$
In this paper, we use this idea to develop the concept of triple connected dominating set and triple connected domination number of complementary prism.

2. TRIPLE CONNECTED DOMINATION NUMBER OF COMPLEMENTARY PRISM.

2.1 Definition:
A subset $S$ of $V$ of a complementary prism $\overline{G}$ is said to be a triple connected dominating set of $\overline{G}$ if $S$ is a dominating set and $\langle S \rangle$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number of $\overline{G}$ and is denoted by $\gamma_{tc}(\overline{G})$. Any triple connected dominating set with $\gamma_{tc}$ vertices is called a $\gamma_{tc}$-set of $\overline{G}$.

2.2 Example

![Diagram]

Figure 3:

Theorem: 2.3
If $G = K_n$ then $\gamma_{tc}(\overline{G}) = n$, where $n \geq 3$

Proof:
Let $G = K_n$

Since $G$ is a complete graph, $\overline{G}$ will be a set of isolated vertices. It makes all $n$ vertices of $\overline{G}$ leaves in $\overline{G}$.

Therefore we need at least $n$ vertices to dominate $\overline{G}$.
Hence $\gamma_n(G\overline{G}) = n$.

**Theorem: 2.4**

If $G$ is a wheel of order $n$, where $n \geq 4$, then $\gamma_n(G\overline{G}) = 4$.

**Proof:**

Let $G$ be a wheel of order $n$, where $n \geq 4$.

Since the centre vertex $v$ of $G$ is adjacent to every other vertex of $G$, it is an isolate vertex in $\overline{G}$ and it is a leaf in $G\overline{G}$.

The dominating set must contain the vertex $v$ since all the $n-1$ vertices of $G$ and $\overline{v}$ in $\overline{G}$ are dominated by $v$.

In $G\overline{G}$, 2 vertices of $\overline{G}$ have degree $n-2$ and $n-3$ vertices of $\overline{G}$ have degree $n-3$.

To get the triple connected path, select $v$, the adjacent vertex $v_i$, the corresponding vertex $v_i$ in $G\overline{G}$ and a vertex $v_j$ or $\overline{v}_j$ where $j \neq i$.

Hence $\gamma_n(G\overline{G}) = 4$ when $G = W_n$ with $n \geq 4$.

**Proposition: 2.5**

1. If $G = K_1,n$ then $\gamma_n(G\overline{G}) = 3$ where $n \geq 2$.
2. If $G = F_n$ then $\gamma_n(G\overline{G}) = 4$ where $n \geq 2$.
3. If $G = B(m,n)$ then $\gamma_n(G\overline{G}) = 4$ where $m,n \geq 1$.
4. If $G = K_{m,n}$ then $\gamma_n(G\overline{G}) = 4$ where $m,n \geq 2$.
5. If $G$ is a cubic graph with $n \geq 4$, then $\gamma_n(G\overline{G}) \leq \frac{n}{2} + 2$.
6. If $G = F_{1,n}$ then $\gamma_n(G\overline{G}) = 4$ where $n \geq 3$.
7. If $G = C_3$ then $\gamma_n(G\overline{G}) = n + 1$ where $n \geq 3$.
8. If $G = L_n$ then $\gamma_n(G\overline{G}) = n$ where $n \geq 3$.
9. If $G = C_n^*$ then $\gamma_n(G\overline{G}) = n + 2 = n + \delta$ where $n \geq 3$.
10. If $G = P_n^*$ then $\gamma_n(G\overline{G}) = n + 2 = n + \delta$ where $n \geq 2$.
11. If $G = H_n$ then $\gamma_n(G\overline{G}) = n + 1$ where $n \geq 3$.

**Theorem: 2.6**

If $G \neq K_n$ is a connected graph with $n \geq 7$ then the triple connected dominating set of $G\overline{G}$ must contain at least a vertex in $\overline{G}$.

**Proof:**

Let $G$ be a connected graph with $n \geq 7$ vertices except $K_n$. 


We prove by contradiction.

\( \gamma_{\kappa} \)-dominating set does not contain at least a vertex in \( \overline{G} \).

That is, all the vertices of \( \gamma_{\kappa} \)-dominating set belongs to \( G \).

By the definition of complementary prism, we know that each \( v_i \) is adjacent to the corresponding \( \overline{v}_i \) where \( 1 \leq i \leq n \).

Since more number of higher degree vertices belong to \( \overline{G} \) select the vertices from \( \overline{G} \) as much as possible. If we omit the vertices in \( \overline{G} \), we have to select more number of vertices in \( G \).

Which is a contradiction to the fact that minimum cardinality of the domination number.

Hence the triple connected dominating set of \( G \overline{G} \) must contain an element in \( G \).

**Theorem:** 2.7

If \( G = C_n \) and \( n \geq 3 \) then,

\[
\gamma_{\kappa}(G\overline{G}) = \begin{cases} 
\frac{2n}{3} + 1 & \text{if } n = 3k \text{ for } k \geq 1 \\
\left\lfloor \frac{2n}{3} \right\rfloor + 2 & \text{if } n = 3k + i \text{ for } k \geq 1 \text{ and } i = 1, 2
\end{cases}
\]

Proof:

We prove this theorem by induction method. Put \( k = 1 \)

Case(i): If \( n = 3 \) or \( k = 1 \), assume that \( G = C_3 \) with the vertices \( v_1, v_2, v_3 \). To get a triple connected path we need at least 3 vertices. Then the \( \gamma_{\kappa}(G\overline{G}) \) set is \( \{v_1, v_2, v_3\} \), so that \( \gamma_{\kappa}(G\overline{G}) = 3 \).

Case(ii): If \( n = 4 \) (or) \( k = 1, i = 1 \) assume that \( G = C_4 \) with the vertices \( v_1, v_2, v_3, v_4 \). To show that at least 4 vertices are necessary for triple connected domination of \( G\overline{G} \) we assume to the contrary that \( \gamma_{\kappa}(G\overline{G}) \leq 3 \).

Let \( D \) be a \( \gamma_{\kappa}(G\overline{G}) \) set.

If \( D \subseteq V(G) \) (respectively, \( D \subseteq V(G) \)), then atmost three vertices are dominated in \( \overline{G} \) (respectively in \( G \)), a contradiction.

Hence \( \gamma_{\kappa}(G\overline{G}) \geq 4 \).

Therefore \( \gamma_{\kappa}(G\overline{G}) = 4 \).
3. EXACT VALUE FOR SOME SPECIAL GRAPHS:

3.1 The bull graph is a planar undirected graph with 5 vertices and 5 edges in the form of a triangle with two disjoint pendent edges. For the bull graph $G$, $\gamma_n(G\overline{G}) = 4$

![Figure 4](image1)

3.2 The diamond graph is a planar undirected graph with 4 vertices and 5 edges. It consists of a complete graph $K_4$ minus one edge. For the diamond graph $G$, $\gamma_n(G\overline{G}) = 4$.

![Figure 5](image2)
REFERENCES


