On Edge Regular Square Fuzzy Graphs

K. Radha ¹ and N. Kumaravel ²

¹ P.G. Department of Mathematics, Periyar E.V.R. College, Tiruchirappalli – 620 023, Tamil Nadu, India. E-mail: radhagac@yahoo.com
² Department of Mathematics, K S R Institute for Engineering and Technology, Namakkal – 637 215, Tamil Nadu, India. E-mail: kumaramaths@gmail.com

Abstract

In this paper, degree of an edge in square fuzzy graph is obtained. Square fuzzy graph of an edge regular fuzzy graph need not be edge regular. Conditions under which it is edge regular are provided. A necessary and sufficient condition for square fuzzy graph to be edge regular is determined.

Keywords: Strong fuzzy graph, complete fuzzy graph, edge regular fuzzy graph, totally edge regular fuzzy graph, square fuzzy graph.

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1. INTRODUCTION

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975. Though it is very young, it has been growing fast and has numerous applications in various fields. During the same time Yeh and bang have also introduced various connectedness concepts in fuzzy graphs. A.Nagoorgani and J.Malarvizhi discussed the concept of square fuzzy graph and its properties. K.Radha and N.Kumaravel (2014) introduced the concept of edge regular fuzzy graphs. In this paper, we study about edge regular property of square fuzzy graphs. First we go through some basic definitions in the next section from [1], [2], [3], [4], [5] and [7].

2. BASIC CONCEPTS

Let \( V \) be a non-empty finite set and \( E \subseteq V \times V \). A fuzzy graph \( G:(\sigma,\mu) \) is a pair of functions \( \sigma:V \to [0,1] \) and \( \mu:E \to [0,1] \) such that \( \mu(x,y) \leq \sigma(x) \land \sigma(y) \) for all
The order and size of a fuzzy graph $G: (\sigma, \mu)$ are defined by $O(G) = \sum_{x \in V} \sigma(x)$ and $S(G) = \sum_{xy \in E} \mu(xy)$. A fuzzy Graph $G: (\sigma, \mu)$ is strong, if

\[ \mu(xy) = \sigma(x) \land \sigma(y) \]  

for all $xy \in E$. A fuzzy Graph $G: (\sigma, \mu)$ is complete, if

\[ \mu(xy) = \sigma(x) \land \sigma(y) \]  

for all $x, y \in V$.

Let $G: (\sigma, \mu)$ be a fuzzy graph on $G^*: (V, E)$. The degree of a vertex $x$ is

\[ d_G(x) = \sum_{xy \in E} \mu(xy). \]

If each vertex in $G$ has same degree $k$, then $G$ is said to be a regular fuzzy graph or $k$ – regular fuzzy graph.

Let $G^* : (V, E)$ be a graph and let $e = uv$ be an edge in $G^*$. Then the degree of an edge $e = uv \in E$ is defined by

\[ d_{G^*}(uv) = d_{G^*}(u) + d_{G^*}(v) - 2. \]

If each edge in $G^*$ has same degree, then $G^*$ is said to be edge regular. The degree of an edge $xy \in E$ is

\[ d_G(xy) = \sum_{x \in x} \mu(xz) + \sum_{y \in y} \mu(zy) - 2\mu(xy). \]

If each edge in $G$ has same degree $k$, then $G$ is said to be an edge regular fuzzy graph or $k$ – edge regular fuzzy graph.

If $\mu(x, y) > 0$ then $x$ and $y$ are called neighbours, $x$ and $y$ are said to lie on the edge $e = xy$. A path $\rho$ in a fuzzy graph $G: (\sigma, \mu)$ is a sequence of distinct nodes $v_0, v_1, v_2, ..., v_n$ such that $\mu(v_i, v_{i-1}) > 0, 1 \leq i \leq n$. Here ‘$n$’ is called the length of the path. The consecutive pairs $(v_i, v_{i-1})$ are called arcs of the path.

If $u, v$ are nodes in $G: (\sigma, \mu)$ and if they are connected by means of a path then the strength of that path is defined as

\[ \bigwedge_{i=1}^{m} \mu(v_{i-1}, v_i) \]  
i.e., it is the strength of the weakest arc.

If $u, v$ are connected by means of paths of length ‘$k$’ then $\mu^k(u, v)$ is defined as

\[ \mu^k(u, v) = \sup \{ \mu(u, v_1) \land \mu(v_1, v_2) \land \mu(v_2, v_3) \land ... \land \mu(v_{k-1}, v)/u, v_1, v_2, ..., v_{k-1}, v \in V \}. \]

If $u, v \in V$ the strength of connectedness between $u$ and $v$ is,

\[ \mu^\circ(u, v) = \sup \{ \mu^k(u, v)/k = 1, 2, 3, ...\}. \]

A fuzzy graph $G: (\sigma, \mu)$ is connected if $\mu^\circ(u, v) > 0$ for all $u, v \in V$. An arc $uv$ is said to be a strong arc if $\mu(u, v) \geq \mu^\circ(u, v)$. A node $u$ is said to be an isolated node if $\mu(u, v) = 0, \forall u \neq v$.

The $\mu$ – distance $\delta(u, v)$ is the smallest $\mu$ – length of any $u - v$ path, where the $\mu$ – length of a path $\rho: u_0, u_1, u_2, ..., u_n$ is

\[ \ell(\rho) = \sum_{i=1}^{n} 1/\mu(u_{i-1}, u_i). \]

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\( v \) is defined as \( e(v) = \max_p(\delta(u, v)) \). The diameter \( \text{diam}(G) = \forall \{e(v) \mid v \in V\} \), radius \( r(G) = \land \{e(v) \mid v \in V\} \). A node whose eccentricity is minimum in a connected fuzzy graph is called a central node. A connected fuzzy graph is called self-centered if each node is a central node.

Let \( G : (\sigma, \mu) \) be a fuzzy graph with \( G^* : (V, E) \). A square fuzzy graph \( S^2(G) : (\sigma, \mu_{sq}) \) is defined as follows:

The fuzzy relation \( \mu_{sq} \) is defined as,

\[
\mu_{sq}(uv) = \mu(uv), \text{if } uv \in E; \text{ else }
\]

\[
\mu_{sq}(uv) = \begin{cases} 
\text{strength of the strongest path of length 2} & \text{if } uv \in E \\
0, & \text{otherwise}
\end{cases}
\]

2.1. Theorem [9]:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on a \( k \) – regular graph \( G^* : (V, E) \). Then \( \mu \) is a constant function if and only if \( G \) is both regular and edge regular.

2.2. Theorem [2]:
If \( G : (\sigma, \mu) \) is a fuzzy graph with the underlying crisp graph \( G^* : (V, E) \) being complete, then \( G \) and \( S^2(G) \) are same.

2.3. Theorem [10]:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on an odd cycle \( G^* : (V, E) \). Then \( G \) is edge regular if and only if \( \mu \) is a constant function.

2.4. Theorem [10]:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on an even cycle \( G^* : (V, E) \) with \( n \) vertices and let \( n \not\equiv 0 \pmod{4} \). Then \( G \) is an edge regular fuzzy graph if and only if \( \mu \) is a constant function.

3. DEGREE OF AN EDGE IN SQUARE FUZZY GRAPH

3.1. Theorem:
Let \( G : (\sigma, \mu) \) be a fuzzy graph. Then the degree of an edge in its square fuzzy graph is given by

\[
d_{S^2(G)}(uv) = d_G(uv) + \sum_{ux \in E} \left[ \sup(\mu(uw) \land \mu(wx)) \right] + \sum_{vy \in E} \left[ \sup(\mu(vw) \land \mu(wy)) \right]
\]

for \( uv \in E \) and
\[ d_{S^2(G)}(uv) = d_G(u) + d_G(v) + \sum_{uv \in E_x, x \neq v} \left[ \sup(\mu(uw) \land \mu(wx)) \right] + \sum_{vy \in E, y \neq u} \left[ \sup(\mu(vw) \land \mu(zy)) \right] \]
for \( uv \in E_{sq} \) with \( uv \notin E \).

**Proof:**

By definition, \( d_{S^2(G)}(uv) = \sum_{uv \in E_{sq}} \mu_{sq}(uw) + \sum_{vy \in E_{sq}} \mu_{sq}(vw). \)

When \( uv \in E \), \( d_{S^2(G)}(uv) = \sum_{uv \in E} \mu(uw) + \sum_{vy \in E} \left[ \sup(\mu(uw) \land \mu(wx)) \right] \]
\[ + \sum_{vy \in E} \mu(vw) + \sum_{vy \in E} \left[ \sup(\mu(vw) \land \mu(zy)) \right] \]
\[ = d_G(u) - \mu(uv) + \sum_{uv \in E} \left[ \sup(\mu(uw) \land \mu(wx)) \right] \]
\[ + d_G(v) - \mu(uv) + \sum_{vy \in E} \left[ \sup(\mu(vw) \land \mu(zy)) \right] \]
\[ = d_G(u) + \sum_{uv \in E} \left[ \sup(\mu(uw) \land \mu(wx)) \right] + \sum_{vy \in E} \left[ \sup(\mu(vw) \land \mu(zy)) \right] \]
When \( uv \notin E \) with \( uv \in E_{sq} \), \( d_{S^2(G)}(uv) = \sum_{uv \in E} \mu(uw) + \sum_{vy \in E_{sq}} \left[ \sup(\mu(uw) \land \mu(wx)) \right] \]
\[ + \sum_{vy \in E} \mu(vw) + \sum_{vy \in E} \left[ \sup(\mu(vw) \land \mu(zy)) \right] \]
\[ = d_G(u) + d_G(v) + \sum_{uv \in E} \left[ \sup(\mu(uw) \land \mu(wx)) \right] + \sum_{vy \in E} \left[ \sup(\mu(vw) \land \mu(zy)) \right] \].

4. EDGE REGULAR PROPERTY OF SQUARE FUZZY GRAPH

4.1. Remark:

If \( G : (\sigma, \mu) \) is an edge regular fuzzy graph, then \( S^2(G) : (\sigma, \mu_{sq}) \) need not be edge regular fuzzy graph. For example, consider \( G^* : (V, E) \), where \( V = \{a, b, c, d\} \) and \( E = \{ab, ad, bc, bd, cd\} \). Define \( G : (\sigma, \mu) \) by \( \sigma(a) = 0.5, \sigma(b) = 1, \sigma(c) = 0.6, \sigma(d) = 0.9 \) and \( \mu(ab) = 0.4, \mu(ad) = 0.3, \mu(bc) = 0.6, \mu(bd) = 0.9, \mu(cd) = 0.5 \). Then \( S^2(G) : (\sigma, \mu_{sq}) \) is defined by \( \mu_{sq}(ab) = 0.4, \mu_{sq}(ad) = 0.3, \mu_{sq}(bc) = 0.6, \mu_{sq}(bd) = 0.9, \mu_{sq}(cd) = 0.5 \) and \( \mu_{sq}(ac) = 0.4 \). Therefore \( G : (\sigma, \mu) \) is 1.8 – edge regular fuzzy graph, but \( S^2(G) : (\sigma, \mu_{sq}) \) is not an edge regular fuzzy graph.
4.2. Remark:
If $S^2(G): (\sigma, \mu_{sq})$ is an edge regular fuzzy graph, then $G: (\sigma, \mu)$ need not be edge regular fuzzy graph. For example, consider $G^* : (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{ab, ad, bc, bd, cd\}$. Define $G: (\sigma, \mu)$ by $\sigma(a) = 0.5$, $\sigma(b) = 0.7$, $\sigma(c) = 0.6$, $\sigma(d) = 0.9$ and $\mu(ac) = 0.5$, $\mu(ad) = 0.5$, $\mu(bd) = 0.4$, $\mu(cd) = 0.5$. Then $S^2(G): (\sigma, \mu_{sq})$ is defined by $\mu_{sq}(ac) = 0.5$, $\mu_{sq}(ad) = 0.5$, $\mu_{sq}(bd) = 0.4$, $\mu_{sq}(cd) = 0.5$, $\mu_{sq}(ab) = 0.4$ and $\mu_{sq}(bc) = 0.4$. Therefore $S^2(G): (\sigma, \mu_{sq})$ is 1.8 – edge regular fuzzy graph. But $G: (\sigma, \mu)$ is not an edge regular fuzzy graph.

4.3. Theorem:
Let $G: (\sigma, \mu)$ be a complete fuzzy graph. Then $G: (\sigma, \mu)$ is an edge regular fuzzy graph if and only if $S^2(G): (\sigma, \mu_{sq})$ is an edge regular fuzzy graph.

Proof:
By theorem 2.2, $G$ and $S^2(G)$ are same. It follows that, $G: (\sigma, \mu)$ is an edge regular fuzzy graph if and only if $S^2(G): (\sigma, \mu_{sq})$ is an edge regular fuzzy graph.

4.4. Theorem:
Let $G: (\sigma, \mu)$ be a fuzzy graph on a complete bipartite graph $G^*$ such that $\mu$ is a constant function. Then its square fuzzy graph is edge regular.

Proof:
Let $\mu(e) = c$, $\forall \ e \in E$, where $c$ is a constant.
Let $(V_1, V_2)$ be the partition of $G^*$, where $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$.
Then $u_i, v_j \in E$, $\forall i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.
Therefore there are $n$ paths of length two between any two vertices $u_i$ and $u_j$, namely $u_i v_j u_j$, $j = 1, 2, \ldots, n$. Therefore $u_i u_j \in E_{sq}, \forall r \neq s$.

Since $\mu$ is a constant function, $\mu_{sq}(u_i u_j) = \sup\left[\mu(u_i v_j) \wedge \mu(v_j u_j)\right] = c$, $\forall r \neq s$.

Similarly, $v_i v_j \in E_{sq}, \mu_{sq}(v_i v_j) = c$, $\forall i \neq j$.

Now, $d_{S^2(G)} (u_i v_j) = d_G(u_i v_j) + \sum_{r=1 \atop r \neq i}^{m} \sup\left[\mu(u_i v_k) \wedge \mu(v_k u_j)\right] + \sum_{s=1 \atop s \neq j}^{n} \sup\left[\mu(v_l u_i) \wedge \mu(u_i v_j)\right] = d_G(u_i) + d_G(v_j) - 2 \mu(u_i v_j) + \sum_{r=1 \atop r \neq i}^{m} c + \sum_{s=1 \atop s \neq j}^{n} c = nc + mc - 2c + (m-1)c + (n-1)c = 2(m+n-2)c$. 

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For \( i \neq j \), \( d_{S^2(G)}(u_iu_j) = d_G(u_i) + d_G(u_j) + \sum_{r=1}^{m} \sup_{1 \leq k \leq n} \left[ \mu(u_i v_k) \wedge \mu(v_k u_j) \right] + \sum_{s=1}^{m} \sup_{1 \leq k \leq n} \left[ \mu(u_j v_s) \wedge \mu(v_s u_i) \right] \).

\[
= nc + nc + (m - 2)c + (m - 2)c = 2(m + n - 2)c.
\]

Similarly, for \( i \neq j \), \( d_{S^2(G)}(v_iu_j) = 2(m + n - 2)c \).

Hence \( G \) is edge regular.

4.5. Theorem:
Let \( G: (\sigma, \mu) \) be a fuzzy graph on a cycle with \( n \) vertices such that \( \mu \) is a constant function. Then its square fuzzy graph \( S^2(G) \) is edge regular.

Proof:
Since \( \mu \) is a constant function, \( G \) is edge regular.
Let \( \mu(e) = c, \forall e \in E \), where \( c \) is a constant.
When \( n = 3 \), \( G \) is complete. Therefore \( S^2(G) \) is \( G \) itself. Therefore it is edge regular.
When \( n = 4 \), each pair of nonadjacent vertices are connected by two distinct paths of length 2.
Therefore, in \( S^2(G) \), each vertex is adjacent to only one more vertex. (Hence apart from the edges in \( G \), two more edges are incident at each edge in \( S^2(G) \).
Hence \( S^2(G^*) \) is 3–regular and hence it is complete. Therefore \( S^2(G) \) is edge regular.

Let \( n \geq 5 \) and let \( u \) be any vertex. Exactly two vertices are at distance 2 from \( u \).
Therefore \( u \) is adjacent to exactly two more vertices in \( S^2(G) \). Hence \( S^2(G^*) \) is 4–regular. Since \( \mu \) is a constant function, \( S^2(G) \) is 6c – edge regular.

4.6. Theorem:
Let \( G: (\sigma, \mu) \) be a fuzzy graph on a cycle with \( n \) vertices such that \( n \equiv 0 \) (mod 4). If \( G \) is edge regular, then its square fuzzy graph \( S^2(G) \) is edge regular.

Proof:
If \( G \) is a fuzzy graph on a cycle with \( n \equiv 0 \) (mod 4), then by theorems 2.3 and 2.4, \( \mu \) is a constant function.
Hence the result follows from theorem 4.5.

4.7. Remark:
If \( G: (\sigma, \mu) \) is an edge regular fuzzy graph on an even cycle \( G^* : (V, E) \) with \( n \equiv 0 \) (mod 4), then \( S^2(G): (\sigma, \mu_{eq}) \) need not be edge regular. For example, consider...
$G^2 := (V, E)$, where $V = \{a, b, c, d, e, f, g, h\}$ and $E = \{ab, bc, cd, de, ef, fg, gh, ha\}$.

Define $G : (\sigma, \mu)$ by $\sigma(a) = 0.7$, $\sigma(b) = 0.6$, $\sigma(c) = 1$, $\sigma(d) = 0.8$, $\sigma(e) = 0.6$, $\sigma(f) = 0.6$, $\sigma(g) = 0.9$, $\sigma(h) = 0.8$ and $\mu(ab) = 0.6$, $\mu(bc) = 0.4$, $\mu(cd) = 0.5$, $\mu(de) = 0.7$, $\mu(ef) = 0.6$, $\mu(fg) = 0.4$, $\mu(gh) = 0.5$, $\mu(ha) = 0.7$. Then $S^2(G) : (\sigma, \mu_{sq})$ is defined by $\mu_{sq}(ab) = 0.6$, $\mu_{sq}(bc) = 0.4$, $\mu_{sq}(cd) = 0.5$, $\mu_{sq}(de) = 0.7$, $\mu_{sq}(ef) = 0.6$, $\mu_{sq}(fg) = 0.4$, $\mu_{sq}(gh) = 0.5$, $\mu_{sq}(ha) = 0.7$, $\mu_{sq}(ac) = 0.4$, $\mu_{sq}(ce) = 0.5$, $\mu_{sq}(eg) = 0.4$, $\mu_{sq}(ga) = 0.5$, $\mu_{sq}(bd) = 0.4$, $\mu_{sq}(df) = 0.6$, $\mu_{sq}(fh) = 0.4$, $\mu_{sq}(hb) = 0.6$. Therefore $G : (\sigma, \mu)$ is 1.1 – edge regular fuzzy graph. But, $S^2(G)$ is not an edge regular fuzzy graph.

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