Generlized Hyers-Ulam-Rassias Stability of new Leibniz type of n-dimensional Quartic functional equation in Quasi-Beta Normed Space: Direct and fixed Point methods

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Abstract

In this paper, the authors has proved the generalized Hyers-Ulam-Rassias stability of new Leibnitz type of n-dimensional quartic functional equation

\[ f \left( \sum_{i=1}^{n} x_i \right) + \sum_{i=1}^{n} f \left( -x_j + \sum_{i \neq j}^{n} x_i \right) = \sum_{i=1}^{n} f \left( x_i + x_j \right) + 12 \sum_{1 \leq i < j \leq k \leq n} f \left( \frac{4}{x_i} x_j x_k \right) + 24 \sum_{1 \leq i < j \leq k \leq n} f \left( \frac{4}{x_i} x_j x_k x_l \right) - (n+14) \sum_{i=1}^{n} f \left( x_i \right) + \sum_{j=1}^{n} f \left( 2x_j \right) \]

in Quasi-Beta normed spaces, where \( n \) is a positive integer with \( n \geq 4 \) using direct and fixed point methods.

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1. Introduction

The study of stability problems for functional equations is relate to a question of Ulam[31] concerning the stability of group homomorphism was affirmatively
answered for Banach Spaces by Hyers[13]. It was further generalized via excellent results obtained by a number of authors[1, 2, 3, 4, 8, 19].

Over the last six or seven decades[29, 30, 31], the above Ulam[31] problems was tackled by numerous authors who provided solutions in various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations involving only these types of functional equations were[10, 11, 12, 16, 17] discussed. We refer[17, 21] the interested readers for more information on such problems to the monographs.

In 1941, D. H. Hyers[13] gave an affirmative answer to the question of S. M. Ulam[31] for banach spaces. In 1950 T. Aoki[2] was the second author to treat this problem for additive Mppings. In 1978 TH. M. Rassias[23] succeed in exending Hyers theorem by weakening the condition for the Cauchy difference cotrolled by \( \left( \|x\|^p + \|y\|^p \right), p \in (0,) \), to be unbounded.

In 1982, J. M. Rassias[21, 22]replaced the factor \( \left( \|x\|^p + \|y\|^p \right), p \in (0,1) \) by \( \left( \|x\|^p \|y\|^q \right), p \in (0,1) \) for p, q \( \in \mathbb{R} \). A generalization of all the above inequality results was obtained by Gavruta[12] in 1994 by replacing the unbounded Cauchy difference by a general control function \( \varphi(x, y) \).

In 2008, a special case of Gavruta’s theorem for the unbounded Cauchy difference was obtained by Ravi et. al.. [25] by considering the summation of both sum and product of p-two norms. The stability problem of several functional equations have been extensively investigated by number of authors and there are many interesting results concering this problem (see [6], [7], [8], [24]) and reference cited there in.

The solution and stability of following mixed type additive quadratic functional equations

\[
\begin{align*}
&f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x) \\
&f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x) \\
&f(x+y)+f(x-y)=2f(x)+f(y)+f(-y) \\
&f(x-t)+f(y-t)+f(z-t)=3f\left(\frac{x+y+z}{3}-t\right) \\
&+f\left(\frac{2x-y-z}{3}\right)+f\left(\frac{-x+2y-z}{3}\right)+f\left(\frac{-x-y+2z}{3}\right) \\
&f(x+2y+3z)+f(x-2y+3z)+f(x+2y-3z)+f(-x+2y+3z) \\
&=4f(x)+8[f(y)+f(-y)]+18[f(z)+f(-z)] \\
&f(3x+2y+z)+f(3x-2y+z)+f(3x+2y-z)+f(3x-2y-z) \\
&=12f(x)+12[f(x)+f(-x)]+8[f(y)+f(-y)]+2[f(z)+f(-z)]
\end{align*}
\]

Were investigated by A. Najati, M. B. Moghimi, M. E. Gordji et. al., [10], M. Rassias et. al., [20], M. Arunkumar, J. M. Rassias[21], and M. Arunkumar, P. Agilan[4, 5].

In 2006, K. W. Jun and H. M. Kim[15] introduced the following generalized additive and quadratic type functional equation
Generlized Hyers-Ulam-Rassias Stability of new Leibniz type

\[ f\left(\sum_{i=1}^{n} x_i\right) + \left(n-2\right) \sum_{i=1}^{n} f\left(x_i\right) = \sum_{1 \leq i < j \leq n} f\left(x_i + x_j\right) \]  

(1. 7)

in the class of function between real vector spaces. For \( n=3 \), P1. Kannappan[] proved that a function \( f \) satisfies the functional equation (1. 1) if and only if there exits a symmetric bi-additive function \( A \) and additive function \( B \) such that \( f(x)=B(x, x)+A(x) \) for all \( x \). The Hyers-Ulam-stability for the equation when \( n=3 \) was proved by S. M. Jung[16]. The Hyers-Ulam-stability[31] for the equation (1. 1) When \( n=4 \) was also investigated by I. S. Chang et al[8].

In 2014, Yeol Je Cho, Madjid Eshaghi Gordji, Seong Sik Kim, and Yougoh Yang, we point out the generalized Hyers-Ulam stability results controlled by approximately mappings by the radical quadratic and radical quartic functional equations are

\[ f\left(\sqrt{ax^2 + bx^2}\right) = af(x) + bf(x) \quad \text{and} \]

\[ f\left(\sqrt{ax^2 + bx^2}\right) + f\left(\sqrt{ax^2 - bx^2}\right) = 2a^2 f(x) + 2b^2 f(x) \]  

(1. 8)


\[ f(\Gamma^l_u + \Gamma^m_v + n^{-1} w) + f(\Gamma^l_u - \Gamma^m_v + n^{-1} w) + f(\Gamma^l_u + \Gamma^m_v - n^{-1} w) \]

\[ + f(\Gamma^l_u - \Gamma^m_v - n^{-1} w) = 4f(\Gamma^l_u) + 2f(\Gamma^m_v) + f(-n^{-1} w) + 2f(\Gamma^l_u) + f(\Gamma^m_v) \]  

(1. 9)

Where \( l \neq m \neq n \) are positive numbers with \( l = m = n \neq 0 \) in Quasi-Beta normed spaces.

In this paper, the authors has proved the generalized Hyers-Ulam stability of new Leibnitz type of n-dimensional quartic functional equation

\[ f\left(\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} f\left(x_i + x_i\right) = \sum_{i=1}^{n} f\left(x_i + x_i\right) + 12 \sum_{1 \leq i < j \leq n} f\left(\sqrt{x_i x_j}\right) \]

\[ + 24 \sum_{1 \leq i < j \leq k \leq n} f\left(\sqrt{x_i x_j x_k}\right) - (n+14) \sum_{i=1}^{n} f\left(x_i\right) + \sum_{j=1}^{n} f\left(2x_j\right) \]  

(1. 10)

in Quasi-Beta normed space with \( n \geq 4 \), using direct and fixed point methods.

In section 2, the Basic definition of n-dimensional quartic functional equation(1. 10) is given. In section 3, 4, the generalized Hyers-Ulam-Rassias stability of new Leibnitz type of n-dimensional quartic functional equation using direct and fixed point methods.

2. Definitions and notations of then-dimensional quartic functional equation in Quasi-Beta normed Spaces

**Definition 2.1**

Let \( X \) be a lineal space over \( k \). A quasi-\( \beta \)-norm \( \|\cdot\|\) is a real valued function on satisfying the following:

(1) \( \|x\| \geq 0 \) for all \( x \in X \) and \( \|x\| = 0 \) if and only if \( x = 0 \)

(2) \( \|\lambda x\| = |\lambda|^\beta \|x\| \) for all \( \lambda \in k \) and all \( x \in X \).
There is a constant $k \geq 1$ such that $\|x + y\| \leq k(\|x\| + \|y\|)$ for all $x \in X$.

The pair $(X, \|\cdot\|)$ is called quasi-$\beta$-normed space if $\|\cdot\|$ is a quasi-$\beta$-norm on $X$.

The smallest possible $k$ is called the modulus of concavity of $\|\cdot\|$.

**Definition 2.2**

A quasi-$\beta$-Banach space is a complete quasi-$\beta$-normed space.

**Definition 2.3**

A quasi-$\beta$-norm $\|\cdot\|$ is called a $(\beta, p)$-norm ($0 < p \leq 1$) if

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

for every $x, y \in X$. In this case, a quasi-$\beta$-Banach space is called a $(\beta, p)$-Banach space.

For more information, one can refer [6, 220] for the concept of quasi-normed spaces and $p$-Banach space.

### 3. Stability of then-dimensional quartic functional equation (1.10): Direct method

In this section, we obtain the generalized Hyers-Ulam-Rassias stability of new type of Leibniz type $n$-dimensional quartic functional equation in quasi-Beta normed space. Throughout this section, let us take $X$ is a linear space over $k$ and $Y$ is a $(\beta, p)$-Banach space with $p$-norm $\|\cdot\|_p$. Let $k$ be the modulus of concavity of $\|\cdot\|_p$.

For notational convenience, we denote for a given mapping $f : X \to Y$ and define the difference operator $Df : X \to Y$ by

$$Df(x_1, x_2, x_3, \ldots, x_n) = f\left(\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} f\left(-x_i + \sum_{j \neq i} x_j\right) - \sum_{i=1}^{n} f(x_i + x_j) - 12 \sum_{1 \leq i < j \leq k \leq n} f\left(\sqrt[4]{x_i x_j x_k x_l}\right) - 24 \sum_{1 \leq i < j \leq k \leq n} f\left(\sqrt[4]{x_i x_j x_k x_l}\right) + (n+14) \sum_{i=1}^{n} f(x_i) - \frac{n}{2} \sum_{j=1}^{n} f(2x_j)$$

for all $x_1, x_2, \ldots, x_n \in X$.

**Theorem 3.1**

Let $j = \pm 1$. Let $f : X^n \to Y$ be a mapping for which there exits a function $\alpha : X^n \to [0, \infty)$ with the condition

$$\lim_{n \to \infty} \frac{1}{n^j} \alpha\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i, \ldots, \sum_{i=1}^{n} x_i\right) = 0 \quad (3.1)$$

Such that the functional inequality

$$\|Df(x_1, x_2, \ldots, x_n)\|_Y \leq \alpha(x_1, x_2, \ldots, x_n) \quad (3.2)$$

for all $x_1, x_2, \ldots, x_n \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ which satisfies (1.10) and the inequality...
\[ \left\| f(x) - Q(x) \right\|_Y^p \leq \frac{K^p}{16^p \beta} \sum_{k=0}^{\infty} \alpha \left( 2^k x, 0, \ldots, 0 \right)^p \]  

(3.3)

for all \( x \in X \).

**Proof:**
Replacing \((x_1, x_2, \ldots, x_n)\) by \((x, 0, \ldots, 0)\) in the functional inequality (3.1), we get
\[ \left\| f(2x) - 15(x) - f(-x) \right\|_Y \leq \alpha(x, 0, \ldots, 0) \]  

(3.4)

for all \( x \in X \). Using evenness of \( f \) in (3.4), we obtain
\[ \left\| f(2x) - 16f(x) \right\|_Y \leq \alpha(x, 0, \ldots, 0) \]  

(3.5)

for all \( x \in X \). It follows from (3.5) that
\[ \left\| \frac{f(2x)}{16} - f(x) \right\|_Y \leq \frac{1}{16^\beta} \alpha(x, 0, \ldots, 0) \]  

(3.6)

for all \( x \in X \). Replacing \( x \) by \( 2x \) and dividing by 16 in (3.6), we get
\[ \left\| \frac{f(2^2x)}{16^2} - f(2x) \right\|_Y \leq \frac{1}{16^\beta} \alpha(2^2x, 0, \ldots, 0) \]  

(3.7)

for all \( x \in X \). From (3.6) and (3.7), we have
\[ \left\| \frac{f(2^2x)}{16^2} - f(x) \right\|_Y \leq K \left[ \left\| \frac{f(2x)}{16} - f(x) \right\|_Y + \left\| \frac{f(2^2x)}{16^2} - f(2x) \right\|_Y \right] \]
\[ \leq \frac{K}{16^\beta} \left[ \alpha(2x, 0, \ldots, 0) + \frac{\alpha(2^2x, 0, \ldots, 0)}{16} \right] \]  

(3.8)

for all \( x \in U \). Proceeding further and using induction on a positive integer \( n \), we get
\[ \left\{ f \left( \frac{2^n x}{16^n} \right) \right\} . \]

Replacing, \( x \) by \( 2^n x \) and dividing by \( 16^n \) in (3.8), for any \( m, n > 0 \), we deduce
\[ \left\| \frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m} \right\|_Y = \frac{1}{16^m} \left\| \frac{f(2^n.2^m x)}{16^n} - f(2^m x) \right\|_Y \]
\[ \leq \frac{K}{16^\beta} \sum_{k=0}^{n-1} \frac{\alpha(2^{k+m}x, 0, \ldots, 0)}{16^{k+m}} \]
\[
\leq \frac{K}{16^\beta} \sum_{\kappa=0}^{\infty} \frac{\alpha(2^{\kappa+m}x,0,...,0)}{16^{\kappa+m}}
\]

\[
\rightarrow 0 \quad \text{as} \quad m \to \infty
\]

for all \(x \in X\). Thus it follows that a sequence
\[
\left\{ \frac{f(2^n x)}{16^n} \right\}
\]

is a Cauchy in \(Y\) and so it converges. Therefore we see that a mapping \(Q: X \to Y\) defined by
\[
Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{16^n}
\]
is well defined for all \(x \in X\). In quartic it is clear from (3.1) that the following inequality
\[
\left\| DQ(x_1, x_2, \ldots, x_n) \right\|_Y = \lim_{n \to \infty} \frac{1}{16^p n} \left\| Df(2^n x, 0, \ldots, 0) \right\|_Y^p \leq \lim_{n \to \infty} \frac{1}{16^p n} \alpha(2^n x, 0, \ldots, 0)^p
\]

\[
\rightarrow 0 \quad \text{as} \quad m \to \infty
\]
holds for all \(x, x_1, \ldots, x_n \in X\) and so the mapping \(Q\) is quartic. Letting \(n \to \infty\) in (3.9) and using the definition of \(Q(x)\) we see that (3.3) holds for all \(x \in X\). To prove uniqueness, we assume now that there is another quartic function \(Q': X \to Y\) which satisfies (1.10) and the inequality (3.3), then it follows that
\[
Q(2x) = 16Q(x), \quad Q'(2x) = 16Q'(x) \quad \text{for all} \ x \in X \text{ and all} \ n \in \mathbb{N}. \text{ Thus}
\]
\[
\left\| Q(x) - Q'(x) \right\|_Y^p = \frac{1}{16^p n} \left\| Q(2^n x) - Q'(2^n x) \right\|_Y^p
\]
\[
= \frac{k^p}{16^p n} \left\{ \left\| Q(2^n x) - f(2^n x) \right\|_Y^p + \left\| f(2^n x) - Q'(2^n x) \right\|_Y^p \right\}
\]
\[
\leq \frac{K^p}{16^p n} \left( \sum_{\kappa=0}^{\infty} \frac{\alpha(2^{\kappa+n}x, 0, \ldots, 0)^p}{16^{p(\kappa+n)}} \right)
\]
\[
\rightarrow 0 \quad \text{as} \quad m \to \infty
\]

For all \(x \in X\). Hence \(Q\) is unique.

For \(j = -1\), we can prove a similar stability result. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.10).

**Corollary 3.2.**

Let \(f: X^n \to Y\) be an quartic mapping and there exits real numbers \(\lambda\) and \(s\) such that
Generalized Hyers-Ulam-Rassias Stability of new Leibniz type

\[ \|Df (x_1, x_2, \ldots, x_n)\| \]

\[
\leq \lambda \left\{ \|x_1\|^s + \|x_2\|^s + \cdots + \|x_n\|^s \right\}, s < 4 \quad \text{or} \quad s > 4; \\
\leq \lambda \left\{ \|x_1\|^s + \|x_2\|^s + \cdots + \|x_n\|^s + \left\{ \|x_1\|^{n_s} + \|x_2\|^{n_s} + \cdots + \|x_n\|^{n_s} \right\} \right\}, s < \frac{4}{n} \quad \text{or} \quad s > \frac{4}{n};
\]

for all \( x_1, x_2, \ldots, x_n \in X \), then there exists a unique quartic function \( Q : X \to Y \) such that

\[
\|f(x) - Q(x)\|_Y \leq \left( \frac{16 \lambda K}{15 (16^B)} \right)^p \left( \frac{16 \lambda K \|x\|^s}{16^B |16 - 2^s|} \right)^p \left( \frac{16 \lambda K \|x\|^{n_s}}{16^B |16 - 2^{n_s}|} \right)^p
\]

for all \( x \in X \).

4. Stability of the n-dimensional quartic functional equation (1.10): Fixed point method

In this section, the generalized Hyers-Ulam-Rassias stability of the Leibniz Quartic-functional equation (1.10) is given by the Fixed point method.

For notational convenience, we denote for a given mapping \( f : X \to Y \) and define the difference operator \( Df : X \to Y \) by

\[
Df(x_1, x_2, x_3, \ldots, x_n) = f\left( \sum_{i=1}^n x_i \right) + \sum_{i=1}^n f\left( -x_j + \sum_{i=1, i \neq j}^n x_i \right) - \sum_{i=1}^n f\left( x_i + x_j \right) \\
-12 \sum_{l \leq i < j, k \leq n} f\left( \frac{4x_i^2 x_j x_k}{x_l} \right) - 24 \sum_{l \leq i < j, k \leq n} f\left( \frac{4x_i x_j x_k x_l}{x_l} \right) + (n+14) \sum_{i=1}^n f(x_i) - \sum_{j=1}^n f(2x_j)
\]

for all \( x_1, x_2, \ldots, x_n \in X \).

Now we will recall the fundamental results in fixed point theory.

**Theorem 4.1. (Banach’s contraction principle)**

Let \( (X, d) \) be a complete metric space and consider a mapping \( T : X \to X \) which is strictly contractive mapping, that is
(A1) \[ d(Tx,Ty) \leq Ld(x,y) \]
for some (Lipschitz constant) \( L < 1 \). Then,

(1) The mapping \( T \) has one and only fixed point theory \( x^* = T(x^*) \)

(2) The fixed point for each given element \( x^* \) is globally attractive, that is

\[ \lim_{n \to \infty} T^n x = x^* , \text{ for any starting point } x \in X ; \]

(3) one has the following estimation inequalities:

\[ d\left(T^n x, x^*\right) \leq \frac{1}{1-L} d\left(T^n x, T^{n+1} x\right), \forall n \geq 0, \forall x \in X ; \]

\[ d\left(x, x^*\right) \leq \frac{1}{1-L} d\left(x, x^*\right), \forall x \in X . \]

**Theorem 4.2. (The alternative fixed point)**

Suppose that for a complete generalized metric space \( (X,d) \) and a strictly contractive mapping \( T : X \to X \) with lipschitz constant \( L \). Then, for each given element \( x \in X \), either

(B1) \[ d\left(T^n x, T^{n+1} x\right) = \infty \quad \forall n \geq 0, \]

Or

(B2) there exists a natural number \( n_0 \) such that:

(1) \[ d\left(T^n x, T^{n+1} x\right) < \infty \quad \text{for all} \quad n \geq n_0 ; \]

(2) The sequence \( \left(T^n x\right) \) is convergent to a fixed point \( y^* \) of \( T \)

(3) \( y^* \) is the unique fixed point of \( T \) in the set \( Y = \left\{ y \in X : d\left(T^{n_0} x, y\right) < \infty \right\} ; \)

(4) \[ d\left(y, y^*\right) \leq \frac{1}{1-L} d\left(y, Ty\right) \quad \text{for all} \quad y \in Y . \]

In the section, let us assume \( V \) be a vector space and \( B \) banace space respectively.

**Theorem 4.3**

Let \( f : V^n \to B \) be a mapping for which there exists a functional \( \alpha : V^n \to [0, \infty) \)
with the condition

\[ \lim_{n \to \infty} \alpha\left(\mu_i^n x_1, \mu_i^n x_2, \ldots, \mu_i^n x_n\right) = 0 \quad (4.1) \]

Where \( \mu_1 = 2 \) if \( i = 0 \) and \( \mu_1 = \frac{1}{2} \) if \( i = 1 \) such that the functional inequality with

\[ \|Df\left(x_1, x_2, \ldots, x_n\right)\|_Y \leq \alpha\left(x_1, x_2, \ldots, x_n\right) \quad (4.2) \]
for all $x_1, x_2, \ldots, x_n \in V$. If there exists $L = L(i)$ such that the function $x \to \gamma(x) = \alpha\left(\frac{x}{2}, 0, \ldots, 0\right)$.

Has the property

$$
\gamma(x) \leq L\mu_i^4 \gamma\left(\frac{x}{\mu_i}\right)
$$

(4.3)

for all $x \in V$. Then there exists unique quartic function $A : V \to B$ satisfying the functional equation (1.5) and

$$
\|f(x) - Q(x)\|^p_Y \leq \left(\frac{L^{1-i}}{1-L}\right)^p \gamma(x)^p
$$

(4.4)

holds for all $x \in V$.

**Proof:**

consider the set $\Omega = \left\{ g : V \to B, g(0) = 0 \right\}$ and introduce the generalized metric on $\Omega$,

$$
d(g, h) = \left\{ M \in (0, \infty) : \|g(x) - h(x)\|^p_Y \leq M \gamma(x), x \in V \right\}.
$$

It is easy to see that $(\Omega, d)$ is complete. Define $T : \Omega \to \Omega$ by

$$
Tg(x) = \frac{1}{\mu_i^4} g\left(\mu_i x\right), \text{ for all } x \in V.
$$

Now $g, h \in \Omega$

$$
d(g, h) \leq M \Rightarrow \|g(x) - h(x)\|^p_Y \leq M \gamma(x), x \in V.
$$

$$
\Rightarrow \left\|\frac{1}{\mu_i^4} g(\mu_i x) - \frac{1}{\mu_i^4} h(\mu_i x)\right\|^p_Y \leq \frac{1}{\mu_i^4} M \gamma(\mu_i x)
$$

$$
\Rightarrow \left\|\frac{1}{\mu_i^4} g(\mu_i x) - \frac{1}{\mu_i^4} h(\mu_i x)\right\|^p_Y \leq LM \gamma(x), x \in V,
$$

$$
\Rightarrow \|Tg(x) - Th(x)\|^p_Y \leq LM \gamma(y), x \in V,
$$

$$
\Rightarrow d(Tg, Th) \leq LM.
$$

This implies $d(Tg, Th) \leq Ld(g, h)$, for all $g, h \in \Omega$, i.e., $T$ strictly contractive mapping on $\Omega$ with Lipschitz constant $L$.

It follows (4.2) that,
\[
\left\| \frac{f(2x)}{16} - f(x) \right\| \leq \frac{1}{16^B} \alpha(x,0,\ldots,0) \tag{4.5}
\]
for all \(x \in V\). Using (4.3) for the case \(i = 0\), it reduces to

\[
\left\| \frac{f(2x)}{16} - f(x) \right\| \leq L\gamma(x)
\]
for all \(x \in V\).

\[d(f_0, T0f) \leq L = \frac{1}{16^B} \Rightarrow d(f_0, T0f) \leq L = L^1 < \infty \]

Again replacing \(x = \frac{x}{2}\) in (4.5), we get

\[
\left\| f(x) - 16f(\frac{x}{2}) \right\| \leq \alpha\left(\frac{x}{2}, 0, \ldots, 0\right) \tag{4.6}
\]
for all \(x \in V\). using (4.3) for the case \(i = 1\), it reduces to

\[
\left\| f(x) - 16f(\frac{x}{2}) \right\| \leq \gamma(x)
\]
for all \(x \in V\).

\[d(f, Tf) \leq 1 \Rightarrow d(f, Tf) \leq 1 = L^0 < \infty \]

In both cases, we have Therefore \((B1(i))\) holds.

By \((B1(ii))\), it follows that there exists a fixed \(A\) of \(T\) in \(\Omega\) such that

\[
Q(x) = \lim_{n \to \infty} \frac{f(\mu^n_i x)}{\mu^{An}_i} \forall x \in V \tag{4.7}
\]

In order to prove \(Q: V \to B\) is quartic. Replacing \((x_1, x_2, \ldots, x_n)\) by \((\mu^n_i x_1, \mu^n_i x_2, \ldots, \mu^n_i x_n)\) in (4.2) and dividing by \(\mu^{An}_i\), it follows from (4.1) and (4.7) for all \(x_1, x_2, \ldots, x_n \in V\).

By \((B1(iii))\), \(Q\) is the unique fixed point of \(T\) in the set

\[\Delta = \{f \in X : d(f, Q) < \infty\}, \text{ such that} \]

\[
\left\| f(x) - Q(x) \right\|_Y \leq M\beta(x)
\]
For all \(x \in V\) and \(M > 0\). Finally, by \((B1(iV))\), we obtain

\[d(f, Q) \leq \frac{1}{1-L} d(f, Tf)\]
this implies
\[
\| f(x) - Q(x) \|_Y^p \leq \left( \frac{L^{-i}}{1-L} \right)^p \gamma(x)^p
\]

For all \( x \in V \). This completes the proof of the theorem.

From Theorem 4.3, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability for the functional equation (1.10).

**Corollary 4.4**

Let \( f : X^n \to V \) be a mapping and there exits real numbers \( \lambda \) and \( s \) such that

\[
\| Df(x_1, x_2, \ldots, x_n) \|_Y \\
\leq \lambda \left\{ \left\| x_1 \right\|^s + \left\| x_2 \right\|^s + \ldots + \left\| x_n \right\|^s \right\}; \quad \text{s < 1 or s > 1};
\]

\[
\left\{ \lambda \left\{ \left\| x_1 \right\|^s \left\| x_2 \right\|^s \ldots \left\| x_n \right\|^s + \left\| x_1 \right\|^{ns} + \left\| x_2 \right\|^{ns} + \ldots + \left\| x_n \right\|^{ns} \right\} \right\}; \quad \text{s < 1/4 or s > 1/4};
\]

For all \( x_1, x_2, \ldots, x_n \in V \) then there exits a unique quartic function \( Q : X \to Y \) such that

\[
\| f(x) - Q(x) \|_Y^p \leq \left( \frac{\lambda \left\| x \right\|^s}{16^\beta \left\| 16 - 2^s \right\|} \right)^p,
\]

For all \( x \in V \).

**Proof:**

setting

\[
\alpha(x_1, x_2, \ldots, x_n) \leq \lambda \left\{ \left\| x_1 \right\|^s + \left\| x_2 \right\|^s + \ldots + \left\| x_n \right\|^s \right\}; \quad \text{s < 1 or s > 1};
\]

\[
\left\{ \lambda \left\{ \left\| x_1 \right\|^s \left\| x_2 \right\|^s \ldots \left\| x_n \right\|^s + \left\| x_1 \right\|^{ns} + \left\| x_2 \right\|^{ns} + \ldots + \left\| x_n \right\|^{ns} \right\} \right\}; \quad \text{s < 1/4 or s > 1/4};
\]

For all \( x_1, x_2, \ldots, x_n \in V \)

Then, for \( s < 1 \) if \( i = 0 \) and for \( s > 1 \) if \( i = 1 \), we get

\[
\frac{\alpha(x_1^n, x_2^n, \ldots, x_n^n)}{\mu_i^{4n}}
\]
Thus, (4.1) is holds.

But we have $\gamma(x) = \alpha \left( \frac{x}{2}, 0, \ldots, 0 \right)$ has the property $\gamma(x) \leq L \mu_i(\mu, x)$ for all $x \in X$.

Hence

$$\gamma(x) = \frac{1}{16^B} \left( \frac{x}{2}, 0, \ldots, 0 \right) = \left[ \frac{\lambda}{16^B} \left( \frac{\|x\|^S}{2} \right), \frac{\lambda}{16^B} \left( \frac{\|x\|^nS}{2} \right) \right].$$

Now,

$$\frac{1}{\mu_i} \gamma(\mu_i x) = \left[ \frac{\lambda}{16^B} \mu_i^4 \left\{ \left( \frac{\|x\|^S}{2} \right) \right\}, \frac{\lambda}{16^B} \mu_i^4 \left\{ \left( \frac{\|x\|^nS}{2} \right) \right\} \right].$$
Hence the inequality (4. 3) holds either, $L = 2^{s-1}$ for $s < 4$ if $i = 0$ and $L = \frac{1}{2^{s-1}}$ for $s > 4$ if $i = 1$.

Now from (4. 4), we prove the following cases for condition $(i)$.

**Case 1**

$L = 2^{s-1}$ for $s < 4$ if $i = 0$

$$\| f(x) - Q(s) \|_Y \leq \left( \frac{1}{2^{s-1}} \right) \left( \frac{1}{2^s} \right) \frac{\lambda}{16^B} \|x\|^s$$

$$\leq \frac{2^s}{16 - 2^s} \left( 1 - \frac{1}{2^s} \right) \frac{\lambda}{16^B} \|x\|^s$$

$$\leq \frac{\lambda \|x\|^s}{16^B (16 - 2^s)}$$

**Case 2**

$L = \frac{1}{2^{s-1}}$ for $s > 4$ if $i = 1$.

$$\| f(x) - Q(s) \|_Y \leq \left( \frac{1}{2^{s-1}} \right) \left( \frac{1}{2^s} \right) \frac{\lambda}{16^B} \|x\|^s$$

$$\leq \frac{2^s}{16^s - 2^s} \left( 1 - \frac{1}{2^s} \right) \frac{\lambda}{16^B} \|x\|^s$$

$$\leq \frac{\lambda \|x\|^s}{16^B (2^s - 16)}$$

Again, the inequality (4. 3) holds either, $L = 2^{ns-1}$ for $s < \frac{4}{n}$ if $i = 0$ and $L = \frac{1}{2^{ns-1}}$ for $s > \frac{4}{n}$ if $i = 1$.

Now from (4. 4), we prove the following cases for condition $(ii)$.
Case 1

\[ L = 2^{ns-1} \text{ for } s < \frac{4}{n} \text{ if } i = 0 \]

\[
\| f(x) - Q(x) \|_Y \leq \frac{\left( \frac{1}{2^{[ns-1]}} \right)^{1-0}}{1 - \frac{1}{2^{[ns-1]}}} \left( \frac{1}{2^{ns}} \right) \frac{\lambda}{16B} \|x\|^{ns} 
\]

\[
\leq \frac{2^{ns}}{16 - 2^{ns}} \left( \frac{1}{2^{ns}} \right) \frac{\lambda}{16B} \|x\|^{ns} 
\]

\[
\leq \frac{\lambda \|x\|^{ns}}{16B (16 - 2^{ns})} 
\]

Case 2

\[ L = \frac{1}{2^{ns-1}} \text{ for } s > \frac{4}{n} \text{ if } i = 1. \]

\[
\| f(x) - Q(x) \|_Y \leq \frac{\left( \frac{1}{2^{[ns-1]}} \right)^{1-1}}{1 - \frac{1}{2^{[ns-1]}}} \left( \frac{1}{2^{ns}} \right) \frac{\lambda}{16B} \|x\|^{ns} 
\]

\[
\leq \frac{2^{ns}}{2^{ns} - 16} \frac{\lambda}{16B} \|x\|^{ns} 
\]

\[
\leq \frac{\lambda \|x\|^{ns}}{16B (2^{ns} - 16)} 
\]

Hence the proof is complete.

REFERENCES


