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Dominator Local Colourings of Graphs

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Abstract

The local colouring of graphs was introduced by Gary Chartrand et al., [5] in 2003. A local colouring of a graph G of order $n \geq 2$ is a function $c: V(G) \to \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exists two vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_s$, where m_s is the size of the induced subgraph $\langle S \rangle$. The value of a local colouring c is the maximum colour it assigns to a vertex of G and is denoted by $\chi_l(c)$. A graph has a dominator colouring if it has a standard colouring in which each vertex of the graph dominates every vertex of some colour class. The dominator chromatic number is the minimum number of colour classes in a dominator colouring of a graph G. In this paper, dominator local colouring of graphs and dominator local chromatic number of graphs are defined. A bound for a dominator local chromatic number of graphs is obtained. The dominator local chromatic number of paths, cycles and wheels are determined.

Keywords: local colouring, local chromatic number, dominator local colouring, dominator local chromatic number

AMS Subject Classification: 05C15, 05C69

1. Introduction

Graphs considered here are undirected, connected and simple graphs on n vertices. For standard notation and terminology we follow Harary[7]. Graph colouring and domination are two major areas in graph theory that have been well studied.

A *standard colouring*, or simply a *colouring* of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. The *chromatic number* $\chi(G)$ is defined as the minimum number of colours used in any colouring of G. A *k-colouring* of G uses k colours.The *value of a colouring c* of G is defined by $\chi(c) = \max\ \{c(v) : v \in V(G)\}$ .Then $\chi(G) = \min\ \{\chi(c) : c$ is a colouring of $G\}$. The generalisations of graph colouring have been introduced and the variations are developed in a tremendous way. The area of research in graph colouring is branching out in many directions. The standard colouring of a graph G may also be defined as a function $c : V(G) \to \mathbb{N}$ such that for every 2-element set $S = \{u, v\}$ of vertices of G, $|c(u) - c(v)| \geq m_s$, where m_s is the size of the induced subgraph $\langle S \rangle$ of G. Defining the standard colouring of a graph in this way, Gary Chartrand et al., have introduced the study of local colourings of graphs and local chromatic number[5, 6].

A *local colouring* of a graph G of order $n \geq 2$ is a function $c : V \to \mathbb{N}$ such that for each subset $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exists two distinct vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_s$, where m_s is the size of the induced subgraph $\langle S \rangle$ of G. The *value of a local colouring c* is the maximum colour it assigns to a vertex of G and is denoted by $\chi_l(c)$. The *local chromatic number* of G is the minimum value of any local colouring of the graph G and is denoted by $\chi_l(G)$. If H is a subgraph of G, then $\chi_l(H) \leq \chi_l(G)$. Behnaz Omoomi et al., have proved results based on the local colouring of Kneser graphs, and locally rainbow graphs[1, 2].

In a graph G, the *open neighbourhood of* $v \in V(G)$ is denoted by $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighbourhood of* $v \in V(G)$ is denoted by $N[v] = \{u \in V(G) : uv \in E(G)\} \cup \{v\}$. A subset S of $V(G)$ is said to be an *independent set* if no two vertices in S are adjacent. The largest number of vertices in such a set is called the *point independence number of G* and is denoted by $\beta_0(G)$ or β_0. In a graph G a subset S of $V(G)$ is called a *dominating set* if every vertex in $V(G) - S$ is adjacent to atleast one vertex in S. A *colour class* is the set of all vertices having the same colour. The colour class corresponding to a colour i is denoted by V_i and every such colour class is an independent set. A *dominator colouring* of a graph G is a standard colouring in which every vertex of G dominates every vertex of atleast one colour class. The convention is that if $\{v\}$ is a colour class, then v dominates the colour class $\{v\}$ [11]. The *dominator chromatic number* $\chi_d(G)$ is the minimum number of colours required for a dominator colouring of G. The concepts of dominator partition and dominator colouring of a graph are introduced by Hedetniemi et al.[8, 9] and motivated by Cockayne et al.[10]

2. Dominator Local Colouring and Dominator Local Chromatic Number

In this section, we define dominator local colouring, dominator local chromatic number and obtain some bounds for dominator local chromatic number.

Definition 2.1 A graph has a *dominator local colouring* if it has a local colouring in which each vertex of the graph dominates every vertex of some colour class.

Note 2.1[11] The convention is that if $\{v\}$ is a colour class, then v dominates the colour class $\{v\}$.

Definition 2.2 The *dominator local chromatic number* of G denoted as $\chi_{dl}(G)$ is the minimum value of any dominator local colouring of G.
We now determine the dominator local chromatic number of some graphs.
The local chromatic number of the complete graph G of order n is $\left\lfloor\frac{3n-1}{2}\right\rfloor$[4].

Observation 2.1 The dominator local chromatic number of the complete graph K_n is $\left\lfloor\frac{3n-1}{2}\right\rfloor$.

Observation 2.2 For the star graph, $K_{1,n}$, $\chi_{dl}(K_{1,n}) = 3$.

Double Star A double star $S_{a,b}$, $(a,b \geq 2)$ is a tree of diameter 3.

Observation 2.3 The dominator local chromatic number of the double star $S_{a,b}, (a,b \geq 2)$, is 4.

Proof Let $S_{a,b}$ be the double star with central vertices u and v with deg $u = a \geq 2$ and deg $v = b \geq 2$. Let $X = \{x_1, x_2, \cdots, x_{a-1}\}$ be the set of vertices adjacent to u and $Y = \{y_1, y_2, \cdots, y_{b-1}\}$ be the set of vertices adjacent to v. Let $V_1 = X$, $V_2 = \{v\}, V_3 = \{u\}$ and $V_4 = Y$. Let V_1, V_2, V_3, V_4 be the colour classes of the colours 1,2,3,4 respectively. Hence the local colouring condition is satisfied. Therefore $\chi_l(S_{a,b}) \leq 4$. This local coluring is minimum. So, $\chi_l(S_{a,b}) = 4$. Each vertex in the set X dominates the colour class V_3 and each vertex in the set Y dominates the colour class V_2. The vertices u, v dominates the colour classes V_1 and V_4 respectively. Therefore this is a dominator local colouring and it is minimum. Hence $\chi_{dl}(S_{a,b}) = 4$.

Observation 2.4 For any graph G, $\chi_l(G) \leq \chi_{dl}(G)$.
When $G = K_n$, equality occurs, for $\chi_l(K_n) = \left\lfloor\frac{3n-1}{2}\right\rfloor = \chi_{dl}(K_n)$.
When $G = P_5$, strict inequality occurs, for $\chi_l(P_5) = 3$, but $\chi_{dl}(P_5) = 4$. Hence $\chi_l(P_5) < \chi_{dl}(P_5)$.

Observation 2.4 For any connected graph G of order $n \geq 3$ we have $3 \leq \chi_{dl}(G) \leq \left\lfloor \frac{3n-1}{2} \right\rfloor$ and these bounds are sharp.
$\chi_{dl}(G) = 3$ when $G = P_3$.
$\chi_{dl}(G) = \left\lfloor \frac{3n-1}{2} \right\rfloor$ when $G = K_n$.

Theorem 2.1 Let G be a graph of order n with local chromatic number r. Then we can construct a graph G' with G as subgraph and with $\chi_{dl}(G') \leq r+2$.

ProofLet G be a graph of order n with the local chromatic number r. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. Add a vertex v ' to G and an edge from v ' to every vertex in G. Then $V(G') = v' \cup V(G)$ and $E(G') = E(G) \cup \{vv' : v \in V(G)\}$. Since $\chi_l(G) = r$, there exists a colouring $c : V(G) \to \mathbb{N}$ such that c is a minimum local colouring of G. Let $c(v') = r+2$.

Claim: $c : V(G') \to \mathbb{N}$ is a local colouring of G'.
Suppose u_i, u_j are not adjacent and coloured by the same colour, (say) s, where $s \leq r$ and if the three vertices u_i, u_j, v' are considered as the set S, then the colour difference is $r+2-s \geq r+2-r = 2 = m_s$. Suppose if there are two adjacent vertices u_i, u_j coloured with the colours s_1, s_2 where $s_1 < s_2 < r$, then also the colour condition is satisfied. Therefore c is a local colouring of G' and hence, $\chi_l(G') \leq r+2$.
Now each vertex of G dominates the colour class of v', as all the vertices are adjacent to it. $v' \in N[u]$ for every $u \in V(G')$, therefore v' dominates all the colour classes of G. Hence c is a dominator local colouring. The value of the dominator local colouring c, is $r+2$. Therefore we have constructed the graph G' with G as subgraph with dominator local colouring c. Hence $\chi_{dl}(G') \leq r+2$.

Theorem 2.2 For a given integer $r < n$, a graph G of order $n \geq 3$ with dominator local chromatic number $\left\lfloor \frac{3r-1}{2} \right\rfloor$ can be constructed.

Proof Let K_r be a complete graph on $r(< n)$ vertices $v_1, v_2, \cdots, v_r$. Let G be a graph on n vertices with the vertex set as $\{v_1, v_2, \cdots, v_r, u_1, u_2, \cdots, u_{n-r}$ and edge set as $E(G) = E(K_r) \cup \{v_1 u_i : 1 \leq i \leq n-r\}$.
Define $c : V(G) \to \mathbb{N}$ as $c(v_1) = 1$; $c(v_2) = 2$ $c(v_i) = c(v_{i-2}) + 3$, for $3 \leq i \leq r$.
$c(u_i) = 3$, for $1 \leq i \leq n-r$.

The vertex v_r is coloured by $\left\lfloor\frac{3r-1}{2}\right\rfloor$. Hence the value of the local colouring of G is $\left\lfloor\frac{3r-1}{2}\right\rfloor$. We can easily check that the local colour condition is satisfied. Therefore$\chi_l(G) \leq \left\lfloor\frac{3r-1}{2}\right\rfloor$.

As K_r is the subgraph of G, $\chi_l(K_r) \leq \chi_l(G)$. And$\chi_l(K_r) = \left\lfloor\frac{3r-1}{2}\right\rfloor$[5]. Therefore, $\chi_l(G) \geq \left\lfloor\frac{3r-1}{2}\right\rfloor$. Hence $\chi_l(G) = \left\lfloor\frac{3r-1}{2}\right\rfloor$. Each vertex $v_i, 1 \leq i \leq r$ is given a unique colour and hence each vertex $v_i, 1 \leq i \leq r$ dominates its own colour class. Each vertex $u_i, 1 \leq i \leq n-r$ dominates the colour class V_1 of the vertex v_1 as they are adjacent to v_1. Hence this is a dominator local colouring and is minimum. Therefore, $\chi_{dl}(G) = \left\lfloor\frac{3r-1}{2}\right\rfloor$.

Theorem 2.3 For a connected graph G of order $n \geq 3$, and with the point independence number $t = \beta_0(G)$, $\chi_{dl}(G) \leq r+2$ where$r = \left\lfloor\frac{3(n-t)-1}{2}\right\rfloor$ and the bound is sharp.

Proof Let G be a connected graph of order $n \geq 3$. Let I be a maximum independent set of G with t vertices and let $V(G) - I = \{v_1, v_2, \cdots, v_{n-t}\}$. Define a local colouring $c: V(G) \to \mathbb{N}$ as $c(v_1) = 1$; $c(v_2) = 2$; $c(v_i) = c(v_{i-2}) + 3$, for $3 \leq i \leq n-t$; $c(v_i) = r+2$, for $v_i \in I$ and $r = \left\lfloor\frac{3(n-t)-1}{2}\right\rfloor$. It can be easily checked that the local colour condition is satisfied. Hence c is a local colouring for G. Therefore $\chi_l(G) \leq r+2$.

The vertices $v_i, (1 \leq i \leq n-t)$ have unique colours and hence they dominate their own respective colour classes. Since G is connected and I is independent, it follows that, every vertex in I, is adjacent to atleast one vertex $v_j, 1 \leq j \leq n-t$. So each vertex of I dominates the colour class of the vertex to which it is adjacent. Hence this colouring is a dominator local colouring. So $\chi_{dl}(G) \leq r+2$, where $r = \left\lfloor\frac{3(n-t)-1}{2}\right\rfloor$ and $t = \beta_0(G)$.

The bound is sharp.

Consider the star graph $K_{1,n}$. $\beta_0(K_{1,n}) = n = t$ and $\chi_{dl}(K_{1,n}) = 3$, $r = \left\lfloor\frac{3(n-t)-1}{2}\right\rfloor = \left\lfloor\frac{3(n+1-n)-1}{2}\right\rfloor = 1$.Therefore $r+2 = 3 = \chi_{dl}(K_{1,n})$.

Hence $\chi_{dl}(K_{1,n}) = r+2 = 3$.

3. Dominator Local Chromatic Number of Paths, Cycles and Wheels

In this section we determine the dominator local chromatic number of paths, cycles and wheels.

Theorem A[12]The path P_n of order $n \geq 2$ has $\chi_d(P_n) = 1 + \left\lceil \frac{n}{3} \right\rceil$, for $n = 2,3,4,5,7$ and is, $2 + \left\lceil \frac{n}{3} \right\rceil$ otherwise.

Theorem 3.1 The path P_n of order $n \geq 3$ has $\chi_{dl}(P_n) = 1 + \left\lceil \frac{n}{3} \right\rceil$, for $n = 4,7$ and is $2 + \left\lceil \frac{n}{3} \right\rceil$, otherwise.

Proof For $3 \leq n \leq 7$, the result can be verified directly. Let $n > 7$. Consider a path P_n on n vertices $v_1, v_2, \cdots, v_n$. Define $c : V \to \mathbb{N}$ as,

1, if $i \equiv 1 \pmod 6$ and $i \neq n$
$c(v_i) = 1$, if $i \equiv 3 \pmod 6$
2, if $i \equiv 4 \pmod 6$ and $i \neq n$
2, if $i \equiv 0 \pmod 6$
$c(v_{3i-1}) = i+2$, if $1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor$
$c(v_n) = 2 + \left\lceil \frac{n}{3} \right\rceil$ if $n \not\equiv 0 \pmod 3$

The adjacent vertices coloured by 1 and 2 are adjacent to a vertex of colour $\geq$ 3. Hence if $S \subseteq V(P_n)$ is taken with any three consecutive vertices, then $m_s = 2$. Then there exist two vertices of S whose colour difference is ≥ 2. Suppose the vertices are not consecutive, $m_s \leq 1$. As consecutive vertices have different colours, we can easily prove that the colour condition is satisfied. Therefore c is a local colouring. $\therefore$ $\chi_l(P_n) \leq 2 + \left\lceil \frac{n}{3} \right\rceil$ for $n > 7$.
Each vertex which are coloured as 1 or 2 dominates some uniquely coloured neighbour and each vertex coloured as k, for $3 \leq k \leq 2 + \left\lceil \frac{n}{3} \right\rceil$ dominates its own colour class. Therefore c is a dominator local colouring of P_n $\therefore$ $\chi_{dl}(P_n) \leq 2 + \left\lceil \frac{n}{3} \right\rceil$ for $n > 7$. By theorem[A], $\chi_d(P_n) = 2 + \left\lceil \frac{n}{3} \right\rceil$ for $n > 7$. We know that $\chi_d(P_n) \leq \chi_{dl}(P_n)$. $\therefore$ $\chi_{dl}(P_n) \geq 2 + \left\lceil \frac{n}{3} \right\rceil$ for $n > 7$. Hence $\therefore$ $\chi_{dl}(P_n) = 2 + \left\lceil \frac{n}{3} \right\rceil$ for $n > 7$.
Hence $\chi_{dl}(P_n) = 1 + \left\lceil \frac{n}{3} \right\rceil$, for $n = 4,7$ and is $2 + \left\lceil \frac{n}{3} \right\rceil$, otherwise.

Theorem 3.2 The dominator local chromatic number of the cycle C_n, $n > 7$ is $2 + \left\lceil \frac{n}{3} \right\rceil$.

Proof Let C_n be a cycle on n vertices $u_1, u_2, \cdots, u_n$ where $n > 7$ and consider the subgraph $P_n : u_1, u_2, \cdots, u_n$. By the theorem 3.1 $\therefore$ $\chi_{dl}(P_n) = 2 + \left\lceil \frac{n}{3} \right\rceil$ for $n > 7$. Hence $\chi_{dl}(C_n) \geq \chi_{dl}(P_n) = 2 + \left\lceil \frac{n}{3} \right\rceil$ for $n > 7$. Define the colouring of the vertices

$u_1, u_2, \cdots, u_n$ as defined in theorem 3.1 except when $n \equiv 3 \ (mod\ 6)$. In that case, define $c(v_n) = 2$ and the other vertices are coloured as in theorem 3.1. This is a dominator local colouring of the path $u_1, u_2, \cdots, u_n$ with $2 + \left\lceil \frac{n}{3} \right\rceil$ colours, such that vertices u_1 and u_n are dominated by themselves or by u_2 and u_{n-1} respectively. The cycle C_n is obtained from the path P_n, by adding the edge $u_1 u_n$ and hence each vertex of C_n is dominated as it was in the path P_n. Hence the colouring is a dominator local colouring and is minimum therefore $\chi_{dl}(C_n) = 2 + \left\lceil \frac{n}{3} \right\rceil$ for $n > 7$.

Note 3.1 It can easily be verified that, $\chi_{dl}(C_3) = \chi_{dl}(C_6) = 4$, $\chi_{dl}(C_4) = \chi_{dl}(C_5) = 3$ and $\chi_{dl}(C_7) = 5$.

Wheel Graph[7] The wheel graph W_n is defined to be the graph $K_1 + C_{n-1}$.

Theorem 3.3 The dominator local chromatic number of the wheel W_n, $n \geq 4$, is 5.

Proof Let W_n be a wheel graph with v_0 as the central vertex and $v_1, v_2, \cdots, v_{n-1}$ as the other vertices on the cycle of order $n-1$. Define $c: V \to \mathbb{N}$, when n is even as, $c(v_0) = 1$; $c(v_{2i-1}) = 3$ for $1 \leq i \leq \frac{n-2}{2}$; $c(v_{2i}) = 5$ for $2 \leq i \leq \frac{n-2}{2}$; and $c(v_{n-1}) = 4$. If n is odd, $c(v_0) = 1$; $c(v_{2i-1}) = 3$ for $1 \leq i \leq \frac{n-1}{2}$; and $c(v_{2i}) = 5$ for $2 \leq i \leq \frac{n-1}{2}$. Clearly this colouring is a local colouring and value of this local colouring is 5 and it is easy to check that 5 is the minimum such value. Hence $\chi_l(W_n) = 5$.
Each vertex on the rim of the wheel dominates the colour class of the central vertex v_0 and the vertex v_0 dominates the colour classes of the other vertices. Therefore this colouring is a dominator local colouring. It can be easily proved that this dominator local colouring is minimum. Hence the dominator local chromatic number of the wheel graph is 5.

4. Conclusion

Dominator local colouring and dominator local chromatic number are defined in this article. Theorems related to the bounds of dominator local chromatic numbers are proved. Also the dominator local chromatic number of the paths, cycles and wheels are determined.

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