Few more results on Geometric Mean Labeling of graphs

Ujwala Deshmukh
Department of Mathematics, Mithibai College, Vile Parle (West)
Mumbai-400056, Maharastra, India.

Vahida Y. Shaikh
Department of Mathematics, Maharashtra College of Arts, Science & Commerce,
Mumbai-400008, Maharastra, India.

Abstract
A Graph with $p$ vertices and $q$ edges is said to be Geometric Mean Graph if it is
possible to label vertices $x \in V$ with distinct elements $f(x)$ from $1, 2, \ldots, q + 1$ in
such a way that when edge $e = uv$ is labelled with $f^*(uv) = \left\lceil \sqrt{(f(u)f(v))} \right\rceil$ or
$f^*(uv) = \left\lfloor \sqrt{(f(u)f(v))} \right\rfloor$ then the resulting edge labels are distinct. In this case $f$ is
called Geometric Mean Labeling of $G$. In this paper, we prove the following graphs
$A(T_n) \circ K_1$, $T(T_n) \circ K_1$, $Q_n \circ \bar{K}_2$, $D(Q_n) \circ \bar{K}_2$, $C_n \circ P_2$ are Geometric
Mean Graphs.

AMS subject classification: 05C78.
Keywords: Geometric mean labeling, Geometric Mean graphs.

1. Introduction

All graphs considered here are simple, finite, connected and undirected. Given a graph
$G$, the symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of the graph $G$
respectively. Let $G(p, q)$ be a graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges.
A graph labeling is an assignment of integers to the vertices or edges or both, subject
to certain conditions. There are various types of graph labeling and a detailed survey
is available in [1]. The concept of Geometric mean labeling was introduced by S.
Somasundaram, P. Vidhyarani and R. Ponraj in [2]. We use the following definitions in
the subsequent sections.
Definition 1.1. An Alternate Triangular snake $A(T_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ alternatively to a new vertex $v_i$ for $1 \leq i \leq n - 1$.

Definition 1.2. A Triple Triangular snake $T(T_n)$ consists of three triangular snakes that have a common path. That is, a Triple Triangular snake is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$ for $i = 1, 2, \ldots, n - 1$ and also to new vertex $w_i$ for $i = 1, 2, \ldots, n - 1$ and also to a new vertex $z_i$ for $i = 1, 2, \ldots, n - 1$.

Definition 1.3. A Quadrilateral snake $Q_n$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ to two new vertices $v_i$ and $w_i$ for $1 \leq i \leq n - 1$ respectively and then joining $v_i$ and $w_i$.

Definition 1.4. A Double Quadrilateral snake $D(Q_n)$ consists of two quadrilateral snakes that have a common path.

Definition 1.5. The Corona $G = G_1 \odot G_2$ of two graphs $G_1$ and $G_2$ is defined as a graph obtained by taking one copy of $G_1$ (with $p$ vertices) and $p$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ to every vertex of $i^{th}$ copy of $G_2$.

2. Results

Theorem 2.1. $A(T_n) \odot K_1$ is a geometric mean graph.

Proof. Let $G = A(T_n) \odot K_1$ be the graph. We consider the following two cases.

Case 1: $n$ is even

Subcase 1.1: The first triangle starts from the first vertex of the path

Let,

$V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$

$E(G) = \{u_iw_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n - 1\}$

$\cup \{u_{2i-1}v_{2i-1} : 1 \leq i \leq \frac{n}{2}\} \cup \{u_{2i}v_{2i} : 1 \leq i \leq \frac{n}{2}\} \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq \frac{n}{2}\}$

Then,

$|V(G)| = p = 3n$

$|E(G)| = q = \frac{7n - 2}{2}$

Define a function

$f : V(G) \rightarrow \{1, 2, \ldots, q + 1\}$
as follows:

\[
\begin{align*}
  f(u_{2i-1}) &= 7i - 5 & 1 \leq i \leq \frac{n}{2} \\
  f(u_{2i}) &= 7i - 1 & 1 \leq i \leq \frac{n}{2} \\
  f(w_{2i-1}) &= 7i - 6 & 1 \leq i \leq \frac{n}{2} \\
  f(w_{2i}) &= 7i & 1 \leq i \leq \frac{n}{2} \\
  f(v_{2i}) &= 7i - 2 & 1 \leq i \leq \frac{n}{2} \\
  f(v_{2i-1}) &= 7i - 4 & 1 \leq i \leq \frac{n}{2}
\end{align*}
\]

Then \( f \) induces a bijective function \( f^* : E(G) \to \{1, 2, \ldots, q\} \) as follows:

\[
\begin{align*}
  f^*(u_{2i-1}u_{2i}) &= 7i - 3 & 1 \leq i \leq \frac{n}{2} \\
  f^*(u_{2i}u_{2i+1}) &= 7i & 1 \leq i \leq \frac{n-2}{2} \\
  f^*(u_{2i-1}w_{2i-1}) &= 7i - 6 & 1 \leq i \leq \frac{n}{2} \\
  f^*(u_{2i}w_{2i}) &= 7i - 1 & 1 \leq i \leq \frac{n}{2} \\
  f^*(u_{2i-1}v_{2i-1}) &= 7i - 5 & 1 \leq i \leq \frac{n}{2} \\
  f^*(v_{2i-1}v_{2i}) &= 7i - 4 & 1 \leq i \leq \frac{n}{2} \\
  f^*(v_{2i-1}u_{2i}) &= 7i - 2 & 1 \leq i \leq \frac{n}{2}
\end{align*}
\]

We observe that all the edge labels are distinct. Also,

\[
\begin{align*}
  |V(G)| &= p = \frac{3n}{7n-2} \\
  |E(G)| &= q = \frac{2}{7n-2}
\end{align*}
\]

Hence, \( f \) is Geometric Mean Labeling of \( G \).

**Subcase 1.2**: The first triangle starts from the second vertex of the path.

Let,

\[
V(G) = \{u_i, w_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n - 2\}
\]

\[
E(G) = \{u_iw_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_{2i}v_{2i-1} : 1 \leq i \leq \frac{n-2}{2}\}
\]

\[
\cup \{u_{2i+1}v_{2i-1} : 1 \leq i \leq \frac{n-2}{2}\} \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq \frac{n-2}{2}\}
\]

Then,

\[
|V(G)| = p = 3n - 2
\]
Define a function \( f : V(G) \to \{1, 2, \ldots, q + 1\} \) as follows:

\[
\begin{align*}
\quad f(u_{2i}) & = 7i - 4 & 1 \leq i \leq \frac{n}{2} \\
\quad f(u_{2i - 1}) & = 7i - 5 & 1 \leq i \leq \frac{n}{2} \\
\quad f(w_{2i}) & = 7i - 3 & 1 \leq i \leq \frac{n}{2} \\
\quad f(w_{2i - 1}) & = 7i - 6 & 1 \leq i \leq \frac{n}{2} \\
\quad f(v_{2i}) & = 7i - 1 & 1 \leq i \leq \frac{n - 2}{2} \\
\quad f(v_{2i - 1}) & = 7i & 1 \leq i \leq \frac{n - 2}{2}
\end{align*}
\]

Then \( f \) induces a bijective function \( f^* : E(G) \to \{1, 2, \ldots, q\} \) as follows:

\[
\begin{align*}
\quad f^*(u_{2i - 1}u_{2i}) & = 7i - 5 & 1 \leq i \leq \frac{n}{2} \\
\quad f^*(u_{2i}w_{2i}) & = 7i - 4 & 1 \leq i \leq \frac{n}{2} \\
\quad f^*(u_{2i - 1}w_{2i - 1}) & = 7i - 6 & 1 \leq i \leq \frac{n}{2} \\
\quad f^*(u_{2i}u_{2i + 1}) & = 7i - 2 & 1 \leq i \leq \frac{n - 2}{2} \\
\quad f^*(u_{2i}v_{2i - 1}) & = 7i - 3 & 1 \leq i \leq \frac{n - 2}{2} \\
\quad f^*(u_{2i + 1}v_{2i - 1}) & = 7i & 1 \leq i \leq \frac{n - 2}{2} \\
\quad f^*(v_{2i - 1}v_{2i}) & = 7i - 1 & 1 \leq i \leq \frac{n - 2}{2}
\end{align*}
\]

We observe that, all the edge labels are distinct. Also,

\[
|V(G)| = p = 3n - 2 \\
|E(G)| = q = \frac{7n - 8}{2}
\]

Hence \( f \) is a Geometric Mean Labeling of \( G \).

**Case 2:** \( n \) is odd

**Subcase 2.1:** The first triangle starts from the first vertex of the path. Let,

\[
\begin{align*}
V(G) & = \{u_i, w_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n - 1\} \\
E(G) & = \{u_iw_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n - 1\}
\end{align*}
\]
Few more results on Geometric Mean Labeling of graphs

\[ \cup \left\{ u_{2i-1}v_{2i-1}, u_{2i}v_{2i-1}, v_{2i-1}v_{2i} : 1 \leq i \leq \frac{n-1}{2} \right\} \]

Then,
|V(G)| = p = 3n - 1
|E(G)| = q = \frac{7n - 5}{2}

Define a function \( f : V(G) \to \{1, 2, \ldots, q+1\} \) as follows:

\[
\begin{align*}
f(u_{2i-1}) &= 7i - 5 & 1 \leq i \leq \frac{n+1}{2} \\
f(w_{2i-1}) &= 7i - 6 & 1 \leq i \leq \frac{n+1}{2} \\
f(u_{2i}) &= 7i - 1 & 1 \leq i \leq \frac{n-1}{2} \\
f(w_{2i}) &= 7i & 1 \leq i \leq \frac{n-1}{2} \\
f(v_{2i}) &= 7i - 2 & 1 \leq i \leq \frac{n-1}{2} \\
f(v_{2i-1}) &= 7i - 4 & 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

Then \( f \) induces a bijective function \( f^* : E(G) \to \{1, 2, \ldots, q\} \) as follows:

\[
\begin{align*}
f^*(u_{2i-1}w_{2i-1}) &= 7i - 6 & 1 \leq i \leq \frac{n+1}{2} \\
f^*(u_{2i-1}u_{2i}) &= 7i - 4 & 1 \leq i \leq \frac{n-1}{2} \\
f^*(u_{2i}u_{2i+1}) &= 7i & 1 \leq i \leq \frac{n-1}{2} \\
f^*(u_{2i}w_{2i}) &= 7i - 1 & 1 \leq i \leq \frac{n-1}{2} \\
f^*(u_{2i-1}v_{2i-1}) &= 7i - 5 & 1 \leq i \leq \frac{n-1}{2} \\
f^*(u_{2i}v_{2i-1}) &= 7i - 2 & 1 \leq i \leq \frac{n-1}{2} \\
f^*(v_{2i}v_{2i-1}) &= 7i - 3 & 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

We observe that all the edge labels are distinct. Also,
|V(G)| = p = 3n - 1
|E(G)| = q = \frac{7n - 5}{2}

Hence, \( f \) is Geometric Mean Labeling of G.

**Subcase 2.2:** The first triangle starts from the second vertex of the path

Let,

\[ V(G) = \{u_i, w_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n - 1\} \]
\[ E(G) = \{ u_iw_i : 1 \leq i \leq n \} \cup \{ u_iu_{i+1} : 1 \leq i \leq n-1 \} \]
\[ \cup \{ u_{2i}v_{2i-1}, u_{2i+1}v_{2i-1}, v_{2i-1}v_{2i} : 1 \leq i \leq \frac{n-1}{2} \} \]

Then,
\[ |V(G)| = p = 3n - 1 \]
\[ |E(G)| = q = \frac{7n - 5}{2} \]

Define a function \( f : V(G) \to \{ 1, 2, \ldots, q + 1 \} \) as follows:

\[
\begin{align*}
f(u_{2i-1}) &= 7i - 5 \quad 1 \leq i \leq \frac{n+1}{2} \\
f(w_{2i-1}) &= 7i - 6 \quad 1 \leq i \leq \frac{n+1}{2} \\
f(u_{2i}) &= 7i - 4 \quad 1 \leq i \leq \frac{n-1}{2} \\
f(v_{2i-1}) &= 7i \quad 1 \leq i \leq \frac{n-1}{2} \\
f(v_{2i}) &= 7i - 1 \quad 1 \leq i \leq \frac{n-1}{2} \\
f(w_{2i}) &= 7i - 3 \quad 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

Then \( f \) induces a bijective function \( f^* : E(G) \to \{ 1, 2, \ldots, q \} \) as follows:

\[
\begin{align*}
f^*(u_{2i-1}w_{2i-1}) &= 7i - 6 \quad 1 \leq i \leq \frac{n+1}{2} \\
f^*(u_{2i-1}u_{2i}) &= 7i - 5 \quad 1 \leq i \leq \frac{n-1}{2} \\
f^*(u_{2i}u_{2i+1}) &= 7i - 2 \quad 1 \leq i \leq \frac{n-1}{2} \\
f^*(u_{2i}w_{2i}) &= 7i - 4 \quad 1 \leq i \leq \frac{n-1}{2} \\
f^*(u_{2i+1}v_{2i-1}) &= 7i \quad 1 \leq i \leq \frac{n-1}{2} \\
f^*(v_{2i-1}v_{2i}) &= 7i - 1 \quad 1 \leq i \leq \frac{n-1}{2} \\
f^*(u_{2i}v_{2i-1}) &= 7i - 3 \quad 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

We observe that all the edge labels are distinct. Also,
\[ |V(G)| = p = 3n - 1 \]
\[ |E(G)| = q = \frac{7n - 5}{2} \]

Hence, \( f \) is Geometric Mean Labeling of \( G \).  

\[ \square \]
Example 2.2. Geometric Mean Labeling of $A(T_6) \circ K_1$ where the first triangle starts from first vertex is shown in Figure 1.

Example 2.3. Geometric Mean Labeling of $A(T_6) \circ K_1$ where the first triangle starts from the second vertex is shown in Figure 2.

Theorem 2.4. $T(T_n) \circ K_1$ is a geometric mean graph.

Proof. Let, $G = T(T_n) \circ K_1$ be the graph.
Let,
\[ V(G) = \{u_i, u_{i+1} : 1 \leq i \leq n \} \cup \{v_i, v_{i+1}, z_i, z_{i+1}, t_i, t_{i+1} : 1 \leq i \leq n - 1 \} \]
\[ E(G) = \{u_i u_{i+1} : 1 \leq i \leq n \} \]
\[ \cup \{u_i v_i, v_i v_{i+1}, z_i z_{i+1}, t_i t_{i+1}, u_i v_i, u_i t_i, u_i z_i, u_{i+1} v_i, u_{i+1} t_i, u_{i+1} z_i : 1 \leq i \leq n - 1 \} \]
Then,
\[ |V(G)| = p = 8n - 6 \]
\[ |E(G)| = q = 11n - 10 \]
Define a function \( f : V(G) \to \{1, 2, \ldots, q + 1\} \) as follows:

\[
\begin{align*}
  f(u_i) &= 11i - 9 & 1 \leq i \leq n \\
  f(u_{i1}) &= 11i - 10 & 1 \leq i \leq n \\
  f(v_i) &= 11i - 3 & 1 \leq i \leq n - 1 \\
  f(v_{i1}) &= 11i - 4 & 1 \leq i \leq n - 1 \\
  f(z_i) &= 11i - 8 & 1 \leq i \leq n - 1 \\
  f(z_{i1}) &= 11i - 7 & 1 \leq i \leq n - 1 \\
  f(t_i) &= 11i - 1 & 1 \leq i \leq n - 1 \\
  f(t_{i1}) &= 11i - 2 & 1 \leq i \leq n - 1
\end{align*}
\]

Then \( f \) induces a bijective function \( f^* : E(G) \to \{1, 2, \ldots, q\} \) as follows:

\[
\begin{align*}
  f^*(u_iu_{i1}) &= 11i - 10 & 1 \leq i \leq n \\
  f^*(u_iv_i) &= 11i - 7 & 1 \leq i \leq n - 1 \\
  f^*(u_iu_{i+1}) &= 11i - 5 & 1 \leq i \leq n - 1 \\
  f^*(u_it_i) &= 11i - 6 & 1 \leq i \leq n - 1 \\
  f^*(v_iv_{i1}) &= 11i - 3 & 1 \leq i \leq n - 1 \\
  f^*(z_iz_{i1}) &= 11i - 8 & 1 \leq i \leq n - 1 \\
  f^*(t_it_{i1}) &= 11i - 2 & 1 \leq i \leq n - 1 \\
  f^*(u_{i+1}v_i) &= 11i - 1 & 1 \leq i \leq n - 1 \\
  f^*(u_{i+1}t_i) &= 11i & 1 \leq i \leq n - 1 \\
  f^*(u_{i+1}z_i) &= 11i - 9 & 1 \leq i \leq n - 1 \\
  f^*(u_{i+1}z_{i1}) &= 11i - 4 & 1 \leq i \leq n - 1
\end{align*}
\]

We observe that, all the edge labels are distinct.

Also,

\[
|V(G)| = p = 8n - 6 \\
|E(G)| = q = 11n - 10
\]

Hence, \( f \) is Geometric Mean Labeling of \( G \).

\[\square\]

**Example 2.5.** Geometric Mean Labeling of \( T(T_3) \bigodot K_1 \) is shown in figure 3.

**Theorem 2.6.** \( Q_n \bigodot \bar{K}_2 \) is a geometric mean graph.

**Proof.** Let, \( G = Q_n \bigodot \bar{K}_2 \).

Let,

\[
V(G) = \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{v_{ij}, w_{ij} : 1 \leq i \leq n - 1; 1 \leq j \leq 3\}
\]
Few more results on Geometric Mean Labeling of graphs

Figure 3:

\[ E(G) = \{u_{i1}u_{i+1,1} : 1 \leq i \leq n - 1\} \cup \{u_{i1}u_{ij} : 1 \leq i \leq n, 2 \leq j \leq 3\} \]

\[ \cup \{v_{i1}v_{ij}, w_{i1}w_{ij} : 1 \leq i \leq n - 1, 2 \leq j \leq 3\} \]

\[ \cup \{u_{i1}v_{i1}, v_{i1}w_{i1}, u_{i+1,1}w_{i1} : 1 \leq i \leq n - 1\} \]

Then,

\[ |V(G)| = p = 9n - 6 \]

\[ |E(G)| = q = 10n - 8 \]

Define a function \( f : V(G) \to \{1, 2, \ldots, q + 1\} \) as follows:

\[
\begin{align*}
  f(u_{i1}) &= 10i - 7 & 1 \leq i \leq n \\
  f(u_{i2}) &= 10i - 8 & 1 \leq i \leq n \\
  f(u_{i3}) &= 10i - 9 & 1 \leq i \leq n \\
  f(v_{i2}) &= 10i - 4 & 1 \leq i \leq n - 1 \\
  f(v_{i1}) &= 10i - 5 & 1 \leq i \leq n - 1 \\
  f(v_{i3}) &= 10i - 6 & 1 \leq i \leq n - 1 \\
  f(w_{i2}) &= 10i & 1 \leq i \leq n - 1 \\
  f(w_{i1}) &= 10i - 1 & 1 \leq i \leq n - 1 \\
  f(w_{i3}) &= 10i - 2 & 1 \leq i \leq n - 1 
\end{align*}
\]
Then $f$ induces a bijective function $f^*: E(G) \to \{1, 2, \ldots, q\}$ as follows:

\[
\begin{align*}
  f^*(u_{i1}u_{i3}) &= 10i - 9 & 1 \leq i \leq n \\
  f^*(u_{i1}u_{i2}) &= 10i - 8 & 1 \leq i \leq n \\
  f^*(v_{i1}v_{i3}) &= 10i - 6 & 1 \leq i \leq n - 1 \\
  f^*(v_{i1}v_{i2}) &= 10i - 5 & 1 \leq i \leq n - 1 \\
  f^*(w_{i1}w_{i3}) &= 10i - 2 & 1 \leq i \leq n - 1 \\
  f^*(w_{i1}w_{i2}) &= 10i - 1 & 1 \leq i \leq n - 1 \\
  f^*(u_{i1}u_{i+1,1}) &= 10i - 3 & 1 \leq i \leq n - 1 \\
  f^*(u_{i1}v_{i1}) &= 10i - 7 & 1 \leq i \leq n - 1 \\
  f^*(v_{i1}w_{i1}) &= 10i - 4 & 1 \leq i \leq n - 1 \\
  f^*(w_{i1}u_{i+1,1}) &= 10i & 1 \leq i \leq n - 1
\end{align*}
\]

We observe that, all the edge labels are distinct.

Also,

\[|V(G)| = p = 9n - 6\]
\[|E(G)| = q = 10n - 8\]

Hence, $f$ is Geometric Mean Labeling of $G$. \hfill \blacksquare

**Example 2.7.** Geometric Mean Labeling of $Q_3 \odot \bar{K}_2$ is shown in figure 4.

![Figure 4](image)

**Theorem 2.8.** $D(Q_n) \odot \bar{K}_2$ is a geometric mean graph.

**Proof.** Let, $G = D(Q_n) \odot \bar{K}_2$ be the graph.

Let,

\[V(G) = \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{v_{ij}, w_{ij}, t_{ij}, p_{ij} : 1 \leq i \leq n - 1, 1 \leq j \leq 3\}\]
Few more results on Geometric Mean Labeling of graphs

\[ E(G) = \{u_{i1}u_{i+1,1} : 1 \leq i \leq n - 1\} \cup \{u_{i1}u_{ij} : 1 \leq i \leq n, 2 \leq j \leq 3\} \]

\[ \cup \{v_{i1}v_{ij}, w_{i1}w_{ij}, t_{i1}t_{ij}, p_{i1}p_{ij} : 1 \leq i \leq n - 1, 2 \leq j \leq 3; \} \]

\[ \cup \{u_{i1}v_{i1}, u_{i1}t_{i1}, v_{i1}w_{i1}, t_{i1}p_{i1}, u_{i+1,1}w_{i1}, u_{i+1,1}p_{i1} : 1 \leq i \leq n - 1\} \]

Then,

\[ |V(G)| = p = 15n - 12; \]
\[ |E(G)| = q = 17n - 15 \]

Define a function \( f : V(G) \rightarrow \{1, 2, \ldots, q + 1\} \) as follows:

\[
\begin{align*}
  f(u_1) &= 17i - 14 & 1 \leq i \leq n \\
f(u_2) &= 17i - 15 & 1 \leq i \leq n \\
f(u_3) &= 17i - 16 & 1 \leq i \leq n \\
f(v_1) &= 17i - 12 & 1 \leq i \leq n - 1 \\
f(v_2) &= 17i - 11 & 1 \leq i \leq n - 1 \\
f(v_3) &= 17i - 13 & 1 \leq i \leq n - 1 \\
f(w_1) &= 17i - 8 & 1 \leq i \leq n - 1 \\
f(w_2) &= 17i - 7 & 1 \leq i \leq n - 1 \\
f(w_3) &= 17i - 9 & 1 \leq i \leq n - 1 \\
f(t_1) &= 17i - 5 & 1 \leq i \leq n - 1 \\
f(t_2) &= 17i - 6 & 1 \leq i \leq n - 1 \\
f(t_3) &= 17i - 4 & 1 \leq i \leq n - 1 \\
f(p_1) &= 17i - 1 & 1 \leq i \leq n - 1 \\
f(p_2) &= 17i - 2 & 1 \leq i \leq n - 1 \\
f(p_3) &= 17i & 1 \leq i \leq n - 1
\end{align*}
\]

Then \( f \) induces a bijective function \( f^* : E(G) \rightarrow \{1, 2, \ldots, q\} \) as follows:

\[
\begin{align*}
  f^*(u_{i1}u_{21}) &= 8 \\
f^*(w_{i1}w_{12}) &= 10 \\
f^*(w_{i1}w_{i3}) &= 9 \\
f^*(v_{i1}w_{11}) &= 7 \\
f^*(u_{i1}t_{i1}) &= 6 \\
f^*(u_{i1}u_{i2}) &= 17i - 15 & 1 \leq i \leq n \\
f^*(u_{i1}u_{i3}) &= 17i - 16 & 1 \leq i \leq n \\
f^*(v_{i1}v_{i2}) &= 17i - 12 & 1 \leq i \leq n - 1 \\
f^*(v_{i1}v_{i3}) &= 17i - 13 & 1 \leq i \leq n - 1
\end{align*}
\]
Theorem 2.10. We observe that, all the edge labels are distinct.

Example 2.9. Geometric Mean Labeling of \( D(Q_3) \circ \tilde{K}_2 \) is shown in figure 5.

Theorem 2.10. \( C_n \circ P_2 \) is a geometric mean graph.

Proof. Let, \( G = C_n \circ P_2 \).

Let, 
\[ s = 4n, \quad t = \lceil \sqrt{12n - 3} \rceil \]
\[ V(G) = \{ v_i : 1 \leq i \leq 4n / v_{4j} = v_{4j-3}, 1 \leq j \leq n \} \]
\[ E(G) = \{ v_iv_{i+1} : 1 \leq i \leq s - 1 \} \cup \{ v_s v_1 \} \]

Then, 
\[ |V(G)| = p = 3n \]
\[ |E(G)| = q = 4n \]

Define a function \( f : V(G) \rightarrow \{1, 2, \ldots, q + 1\} \) as follows:
\[
\begin{align*}
    f(v_{4i-3}) &= 4i - 1 & 1 \leq i \leq n \\
    f(v_{4i-2}) &= 4i - 3 & 1 \leq i \leq n \\
    f(v_{4i-1}) &= 4i & 1 \leq i \leq n 
\end{align*}
\]

Then \( f \) induces a bijective function \( f^* : E(G) \rightarrow \{1, 2, \ldots, q\} \) as follows:
\[
\begin{align*}
    f^*(v_iv_{i+1}) &= i & 1 \leq i \leq t - 1 \\
    f^*(v_{i}v_{i+1}) &= i + 1 & t \leq i \leq 4n - 1 \\
    f^*(v_s v_1) &= t 
\end{align*}
\]

We observe that, all the edge labels are distinct.

Also,
\[
\begin{align*}
    f^*(t_1t_2) &= 17i - 6 & 1 \leq i \leq n - 1 \\
    f^*(t_1t_3) &= 17i - 5 & 1 \leq i \leq n - 1 \\
    f^*(p_1p_2) &= 17i - 2 & 1 \leq i \leq n - 1 \\
    f^*(p_1p_3) &= 17i - 1 & 1 \leq i \leq n - 1 \\
    f^*(t_1p_1) &= 17i - 4 & 1 \leq i \leq n - 1 \\
    f^*(u_1v_1) &= 17i - 14 & 1 \leq i \leq n - 1 \\
    f^*(w_1u_{i+1,1}) &= 17i - 3 & 1 \leq i \leq n - 1 \\
    f^*(p_1u_{i+1,1}) &= 17i & 1 \leq i \leq n - 1 \\
    f^*(u_1u_{i+1,1}) &= 17i - 10 & 2 \leq i \leq n - 1 \\
    f^*(w_1w_{i+1}) &= 17i - 7 & 2 \leq i \leq n - 1 \\
    f^*(w_1w_{i+2}) &= 17i - 8 & 2 \leq i \leq n - 1 \\
    f^*(w_1w_{i+3}) &= 17i - 9 & 2 \leq i \leq n - 1 \\
    f^*(v_1w_{i+1}) &= 17i - 11 & 2 \leq i \leq n - 1 
\end{align*}
\]
Few more results on Geometric Mean Labeling of graphs

$|V(G)| = p = 3n$
$|E(G)| = q = 4n$
Hence, $f$ is Geometric Mean Labeling of $G$.

**Example 2.11.** Geometric Mean Labeling of $C_4 \otimes P_2$ is shown in figure 6.
References


