Topologies on Dual Spaces and Spaces of Linear Mappings

Dr.P.A.S.Naidu
D.B.Science College, Gondia(Maharashtra)

B.G.Mapari
PhD Scholar, R.T.M. Nagpur University

P. Jha
Govt.J.Y.Chhattisgarh College, Raipur (C.G.)

Abstract:

Let X and Y be two convex spaces over the same (real or complex) field F. We consider a general method of defining convex topologies on the dual of a convex space, taking as neighborhoods of the origin the polars of certain sets in the convex space. Here we have proved that any finite sum of compact sets is compact. Also the sum of a compact and a closed set in a convex space is closed. Let V be the vector space of all continuous linear mappings of X into Y. Let A be any bounded subsets of X and B a base of absolutely convex neighborhoods in Y. Define \( W_{A, B} \) for each A \( \in \) A and B \( \in \) B.

Then \( W_{A, B} \) is absolutely convex and absorbent. The topology is then called the topology of \( A \) –convergence. If X is a barreled space, then any point wise bounded set of continuous linear mappings of X into Y is equicontinuous.

1. Introduction and Necessary Preliminaries

A topological vector space (tvs) is a set with two compatible structures, one, it has the algebraic structure of the vector space, and the other, it has a topology so that the notions of convergence and continuity are meaningful.

A subset V of a tvs X is convex if \( \lambda x + \mu y \in V \) for all \( x, y \in V \) whenever \( \lambda + \mu = 1 \). V is balanced if \( x \in V \) for all \( x \in V \) whenever \( |\lambda| \leq 1 \). V is called absolutely convex if it is both convex and balanced. The set of all finite linear combinations \( \sum \lambda_i x_i \) with...
\[ \lambda_i \geq 0, \sum_i \lambda_i = 1 \text{ and each } x_i \in V \text{ is called convex envelope of } A. \] The absolutely convex envelope of \( V \) is the set of all finite linear combinations \( \sum \lambda_i x_i \) with \( \sum |\lambda_i| \leq 1 \) and each \( x_i \in V \) and is the smallest convex set containing \( V \). The set \( V \) is absorbent if \( x \in V \) there is some \( \lambda > 0 \) such that \( x \in \mu V \) for all \( \mu \) with \( |\mu| \geq \lambda \).

A topology \( \xi \) on \( X \) is said to be compatible with the algebraic structure of \( X \) if the algebraic operations ‘+’ and ‘.’ are continuous in \( X \).

A topological vector space is locally convex if it has a base of convex neighborhood of the origin.

In a convex space, a subset is called a barrel if it is absolutely convex, absorbent and closed. A convex space is called barreled if every barrel is a neighborhood.

A non-negative real valued function \( p \) on a tvs \( X \) is called a semi norm if

(i) \( p(x) \geq 0 \);
(ii) \( p(\xi x) \geq |\xi| p(x) \);
(iii) \( p(x + y) \leq p(\xi x) + p(\xi y) \).

In addition if \( p(x) = 0 \) implies \( x = 0 \), then \( p \) is called a norm.

The semi norm \( p \) corresponding to the absolutely convex absorbent set \( V \) defined by \( p(x) = \inf_{\lambda} \lambda : x \in \lambda V \) along with the property that

\[ A : p(\xi ) \leq 1 \subseteq V \subseteq A : p(\xi ) \leq 1 \]

is called the gauge of \( V \).

The space \( X \) is called normable if its topology can be determined from a norm \( p \).

The dual of a vector space is the vector space of all continuous linear mappings of the space into the scalar field.

A topological space is compact if its each open cover has a finite sub cover.

A topological space \( X \) is said to be supercompact if there is a subbase \( \wp \) for its open sets such that each open cover of \( X \) by elements of \( \wp \) have a sub cover consisting of at most two elements of \( \wp \).

2. General Method of Defining Convex Topologies and Compactness

Let \( (X, X') \) be a dual pair and \( A \) any set of weakly bounded subsets of \( X \). Then the sets \( A^0 (A \in A) \), where is \( A^0 \) called a polar of \( A \) given by

\[ A^0 = A : \sup_{[x, x']} |A x| : x \in A \subseteq 1 \]
is absolutely convex and absorbent and so there is a coarsest topology $\tau'$ on $X'$ in which they are neighborhoods. A base in neighborhoods in $\tau'$ is formed by the sets
\[ \mathcal{E} \bigcap A_i^0 = \left( \mathcal{E}^{-1} \bigcup A_i \right)^0 \quad \xi > 0, A_i \in A \]
This topology $\tau'$ on $X'$ is called the topology of uniform convergence on the sets of $A$ or the topology of $A$-convergence or the polar topology.

Now suppose $X$ is a vector space over the field of real or complex numbers. Let $U$ be a non-empty set of subsets of $X$ such that
(i) if $U \subseteq U, V \subseteq U$, there is a $W \subseteq U$ with $W \subseteq U \cap V$;
(ii) if $U \subseteq U$ and $\alpha \neq 0, \alpha U \subseteq U$;
(iii) each $U \subseteq U$ is absolutely convex and absorbent.

Then there exists a topology $\tau$ making $X$ a convex space with $U$ as a base of neighborhoods. This topology $\tau$ on $X$ is called the convex topology.

**Theorem 2.1** Any finite sum of compact sets in a convex space is compact.

**Proof:** Let $A$ and $B$ be two compact sets and $C$ an open covering of $A + B$. Then for each $x \in A$ and each $y \in B$, there is an open absolutely convex neighborhood $U(x,y)$ of the origin for which $x + y + U(x,y)$ is contained in some set of $C$. Now keeping $x$ fixed, the sets $y + \frac{1}{2} U(x,y)$ form an open covering of $B$. As $B$ is compact, let
\[ \{ y_j + \frac{1}{2} U \xi, y_j \}^{1 \leq j \leq n} \]
be finite sub cover of $y + \frac{1}{2} U(x,y)$. Let
\[ V \xi = \bigcap_{1 \leq j \leq n} \frac{1}{2} U \xi, y_j \]
Then the sets $x + V(x)$ form an open covering of $A$. Again as $A$ is compact, let
\[ \{ x_i + V \xi \}^{1 \leq i \leq m} \]
be a finite sub covering of $A$. Then
\[ A + B \subseteq \bigcup_{1 \leq i \leq m} x_i + V \xi \]
\[ \subseteq \bigcup_{1 \leq i \leq m} \bigcup_{1 \leq j \leq n} x_i + y_j + \frac{1}{2} U \xi, y_j \]
Hence $A + B$ is compact.

**Theorem 2.2** The sum of a compact set and a closed set in a convex space is closed.

**Proof:** Let $A$ be compact and $B$ closed. Let $a \notin A + B$. Then for each $x \in A$, $x + B$ is closed, for $+$ is continuous and $B$ is closed. Hence there is an absolutely convex neighborhood $U(x)$ of the origin with $(a + U(x)) \cap (x + B) = \emptyset$. Then
\[ a \notin x + U \xi \bigcup B \]. Now \[ \{ x + \frac{1}{2} U \xi \}^{x \in A} \]
form an open cover of $A$. Since $A$ is compact,
let $\left\{ x_i + \frac{1}{2} U \xi_i \right\}_{i \in A}, 1 \leq i \leq n$, form a finite sub cover of $A$. Let $V = \bigcap_{i \in S} \frac{1}{2} U \xi_i$.

Then

$$A + V \subseteq \bigcup_{x \in \xi, i \in S} \left( x + \frac{1}{2} U \xi_i \right)$$

$$\subseteq \bigcup_{x \in \xi} x + U \xi_i$$

Hence $a \notin A + V + B$. Thus

$$(A + V) \cap (A + B) = \emptyset$$

and so $a \notin (A + B)$. Hence $A + B$ is closed.

If $X$ is separated convex space, then the set of compact subsets of $X$ can be used to define a polar topology on $X'$. Now if $A$ is bounded, then the absolutely convex envelope of $A$ is also bounded. However, it is not true in general that the closed absolutely convex envelope of a compact set is compact.

Remark: One may check whether the closed absolutely convex envelope of a super compact set is super compact or not?

References