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On Three-Point Finite Difference Techniques for SPBVPs

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Abstract

In this paper, three finite difference three-point techniques for singularly perturbed boundary value problems (SPBVPs) are discussed. These techniques are developed over unevenly spaced grid points aided mathematical symbolic language Maple. Local truncation error, uniqueness and stability conditions are discussed.

Keywords: Finite difference three-point techniques; unevenly spaced grid; convergence; stability.

I. INTRODUCTION

Singularly perturbed boundary value problems (SPBVPs) arise frequently in applied sciences and engineering and have been extensively studied in recent years [1-16]. It is well known that away from the boundary layers, upwind difference methods can be used and accurate results are obtained. Otherwise, other schemes are to be preferred such as difference schemes on a non-uniform mesh [1-6, 8-10]. But in this case one must face the drawbacks related to the use of difference schemes on highly non-uniform meshes, since a fine mesh with maximum step-size $h<\varepsilon$ is required over a domain containing the layer region at which the solution varies rapidly, while for reasons of efficiency a coarse grid with $h \gg \varepsilon$ should be used in the outer region at which the solution behaves regularly and changes slowly. The main difficulty in global discretization of these problems is the restriction on the step- size that to have a unique stable and accurate solution. Therefore stability and order of convergence act as the major achieved requirements. Many authors deal with some of these challenges in global discretization for these problems especially the convection diffusion problems [2-6]. Segal [2] analyzed and compared various methods for solving the convection diffusion equation with small ε. While Il'in's [3] method is a very accurate example of an upwind scheme for a homogeneous, one-dimensional convection-diffusion equation with constant coefficients. It loses accuracy when variable coefficients are used. Dekema and Schultz [4] developed high-order methods to solve elliptic singular perturbation problems and obtained remarkably good numerical results. Later, Choo and Schultz [5] developed the so-called stable central difference methods. They modified the central difference approximations for the first- and second-order derivatives by rewriting its error terms as a combination of the lower-order derivative terms and approximating them. This process reinforced the diagonal dominance of the coefficient matrix and had a stabilizing effect. However, they could not achieve as high accuracy as the method of Dekema and Schultz. Ilicasu and Schultz [6] developed high-order methods to solve singular perturbation problems. They rewrote higher order derivatives in Taylor expansion in terms of the lower-order derivative terms. However, they also used constant coefficients only. Most the above techniques go a way from using non-uniform grid points. The main reason is the complexity of driving general formulas that will solve these problems. Moreover, this leads to more complicated studying of uniqueness, stability, and convergence. Now, using mathematical symbolic language such as Maple, Drive and Matlab makes the mission easier than earlier. In this paper, following the idea in [6] three finite difference three-point techniques for singularly perturbed boundary value problems (SPBVPs) are suggested. These techniques are developed over unevenly spaced grid points aided mathematical symbolic language Maple. Local truncation error, uniqueness and stability conditions are discussed

II. FINITE DIFFERENCE TECHNIQUES

Consider the following linear SPBVP

$$-\varepsilon y'' + p(x)y' + q(x)y = f(x), \qquad a \le x \le b \ , \qquad (1)$$

with boundary conditions

$$y(a)=\alpha \text{ and } y(b)=\beta ,$$

where ε is a small positive parameter $(0<\varepsilon \ll 1)$, α and β are given constants, $p(x)$, $q(x)$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions on $[a,b]$, Moreover assume $q(x)>0$, $p(x)<P<0$ for all $x\in[a,b]$, where P is some negative constant. Under these assumptions, SPBVP (1) has a unique solution which in general displays a boundary layer of width $O(\varepsilon)$ at $x=a$ [2, 4, 6-16]. First, $[a,b]$ is divided into N non-equal subintervals such that $\pi : x_0 = a < x_1 < x_2 < ...... < x_N = b$ with $h_i = x_i - x_{i-1}$,

$i=1,2,.....N$. For the sake of simplicity, we will use $p_i = p(x_i),\ q_i = q(x_i),\ f_i = f(x_i),\ y_{i-1} = y(x_{i-1}),$ $y_{i+1} = y(x_{i+1}),$ and $y_i' = y'(x_i)$, etc.

The solution of SPBVP (1) is approximated over subintervals with unevenly spaces three grid points as shown in figure 1.

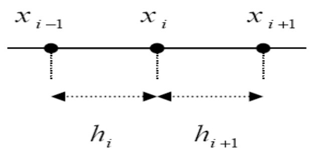

Figure 1. Unevenly spaces grid points over sub-domains

Equation (1) is divided by $-\varepsilon$ and we let $\omega = 1/\varepsilon$. At each x_i , we want to find E_i , F_i , G_i , H_i such that

$$y_i'' - \omega p_i y_i' - \omega q_i y_i = E_i y_{i-1} + F_i y_i + G_i y_{i+1} + H_i = -\omega f_i \tag{2}$$

These terms are obtained using Taylor series expansions of y_{i+1} and y_{i-1} around x_i

$$\begin{aligned} y_i'' - \omega p_i y_i' - \omega q_i y_i &= F_i y_i + H_i + G_i\left[y_i + h_i y_i' + \frac{h_{i+1}^2}{2} y_i'' \right. \\ &\left. + \frac{h_{i+1}^3}{6} y_i''' + .. \right] + E_i\left[y_i - h_i y_i' + \frac{h_i^{\,2}}{2} y_i'' - \frac{h_i^{\,3}}{6} y_i''' + .. \right] \end{aligned} \tag{3}$$

From Eq. (1) we have

$$\begin{aligned} y_i''' &= \omega p_i y_i'' + \omega(p_i' + q_i) y_i' + \omega q_i' y_i - \omega f_i' \\ y_i^{(4)} &= \left[\omega^2 p_i^2 + 2\omega p_i' + \omega q_i \right] y_i'' - \omega f_i'' + \left[\omega^2 p_i (p_i' + q_i) + \right. \\ &\left. + \omega(p_i'' + 2q_i') \right] y_i' + \left[\omega^2 p_i q_i' + \omega q_i'' \right] y_i - \omega^2 p_i f_i' \\ y_i^{(5)} &= \left[\omega^3 p_i^3 + 5\omega^2 p_i' p_i + 3\omega p_i'' + 3\omega q_i' + 2\omega^2 p_i q_i \right] y_i'' \\ &+ \left[\omega^3 p_i^2 p_i' + \omega^3 p_i^2 q_i + 3\omega^2 p_i'^2 + 4\omega^2 p_i' q_i + 3\omega q_i'' + \right. \\ &\left. + 2\omega^2 p_i q_i' + \omega^2 p_i p_i'' + \omega^2 q_i^2 + \omega p_i''' \right] y_i' + \left[\omega^3 p_i^2 q_i' + \right. \\ &\left. + 3\omega^2 p_i' q_i' + \omega^2 q_i q_i' + \omega^2 p_i q_i'' + \omega q_i''' \right] y_i - \\ &- \left[\omega^3 p_i^2 f_i' + 3\omega^2 p_i' f_i' + \omega^2 q_i f_i' + \omega^2 p_i f_i'' + \omega f_i''' \right]. \end{aligned} \tag{4}$$

II.I. Technique-I

Substituting Eq.(4) in Eq.(3) and equating the coefficients of y_i, y_i' and y_i'' , taking the third order derivative terms are the largest contributors to the error, we get

$$\begin{aligned} E_i &= \frac{2 + h_{i+1}\omega p_i}{h_i(h_i + h_{i+1})},\ F_i = -G_i - E_i - \omega q_i , \\ G_i &= \frac{2 - h_i \omega p_i}{h_{i+1}(h_i + h_{i+1})},\ H_i = 0 \end{aligned} \quad , \tag{5}$$

Then, the difference equation of technique I and local truncation error τ_i are introduced as

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = -H_i - w f_i + \tau_i , \tag{6}$$

$$\tau_i = \left(G_i \frac{h_{i+1}^3}{6} y^{(3)}(\xi) - E_i \frac{h_i^3}{6} y^{(3)}(\zeta) \right), \tag{7}$$

where $\xi \in [x_i, x_{i+1}],\ \zeta \in [x_{i-1}, x_i]$.

II.II. Technique-II

Substituting Eq. (4) in Eq. (3) and equating the coefficients of y_i, y_i' and y_i'' , taking the fourth order derivative terms are the largest contributors to the error, we get

$$\left.\begin{aligned} E_i &= \frac{2(h_{i+1}^2 \omega p_i' + h_{i+1}^2 \omega q_i + 3h_{i+1}\omega p_i + h_{i+1}^2 \omega^2 p_i^2 + 6)}{h_i T} \\ G_i &= \frac{2(h_i^2 \omega p_i' + h_i^2 \omega q_i - 3h_i \omega p_i + h_i^2 \omega^2 p_i^2 + 6)}{h_{i+1} T} \\ F_i &= -G_i (1 + h_{i+1}^3 q_i' \omega / 6) - E_i (1 - h_i^3 q_i' \omega / 6) - \omega q_i \\ H_i &= G_i (h_{i+1}^3 f_i' \omega / 6) - E_i (h_i^3 f_i' \omega / 6) \end{aligned}\right\}, \tag{8}$$

where $T = (h_i + h_{i+1})\left[h_{i+1} h_i \omega (p_i' + q_i) + 2 p_i \omega (h_{i+1} - h_i) + 6 \right]$.

Then, the difference equation of technique II and local truncation error τ_i are introduced as

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = -H_i - w f_i + \tau_i , \tag{9}$$

$$\tau_i = \left(G_i \frac{h_{i+1}^4}{24} y^{(4)}(\xi) + E_i \frac{h_i^4}{24} y^{(4)}(\zeta) \right), \tag{10}$$

where $\xi \in [x_i, x_{i+1}],\ \zeta \in [x_{i-1}, x_i]$.

II.III. Technique-III

Substituting Eq.(4) in Eq.(3) and equating the coefficients of y_i, y_i' and y_i'' , taking the fifth order derivative terms are the largest contributors to the error, we get

$$\left.\begin{aligned} E_i &= \frac{6(h_{i+1}^3 \omega\theta_1 + 4h_{i+1}^2 \omega (p_i' + p_i^2 \omega + q_i) + 12 h_{i+1} p_i \omega + 24)}{h_i T} \\ G_i &= \frac{6(-h_i^3 \omega\theta_1 + 4h_i^2 \omega (p_i' + p_i^2 \omega + q_i) - 12 h_i p_i \omega + 24)}{h_{i+1} T} \\ F_i &= -G_i \left[1 + \frac{h_{i+1}^4\, \omega\theta_2}{24} + \frac{h_{i+1}^3 q_i' \omega}{6} \right] - E_i \left[1 + \frac{h_i^{\,4}\, \omega\theta_2}{24} - \right. \\ &\left. - \frac{h_i^{\,3} q_i' \omega}{6} \right] - \omega q_i \\ H_i &= G_i \left[\frac{h_{i+1}^4 \omega\theta_3}{24} + \frac{h_{i+1}^3 f_i' \omega}{6} \right] + E_i \left[\frac{h_i^4 \omega\theta_3}{24} - \frac{h_i^4 f_i' \omega}{6} \right] \end{aligned}\right\},$$

(11)

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where

$$\theta_1 = p_i'' + 2q_i' + p_i\,\omega(3p_i' + p_i^2\omega + 2q_i)$$
$$\theta_2 = q_i' p_i\,\omega + q_i''$$
$$\theta_3 = \omega\left(f_i' p_i\,\omega + f_i''\right)$$
$$\begin{aligned} T &= 6\omega(h_i^3 + h_{i+1}^3)(2p_i' + p_i^2\omega + q_i) + 3\omega(h_{i+1}^2 - h_i^2)(8p_i + \\ &\quad + h_i h_{i+1}(2q_i' + p_i'\ p_i\ \omega + p_i'' + p_i\ q_i\ \omega)) - (h_{i+1} + h_i) \\ &\quad (\omega^2 h_i^2\ h_{i+1}^2(2q_i'\ p_i - 2p_i'^2 - 3p_i'\ q_i + p_i''\ p_i - q_i^2) + \\ &\quad + 12\omega h_i\ h_{i+1}(p_i' + q_i) + 72) \end{aligned}$$

Then, the difference equation of technique III and local truncation error τ_i are introduced as

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = -H_i - \omega f_i + \tau_i\ , \tag{12}$$
$$\tau_i = \left(G_i\,\frac{h_{i+1}^5}{120}\,y^{(5)}(\xi) - E_i\,\frac{h_i^5}{120}\,y^{(5)}(\zeta)\right) \tag{13}$$

where $\xi \in [x_i, x_{i+1}]$, $\zeta \in [x_{i-1}, x_i]$.

III. UNIQUENESS AND STABILITY ANALYSIS

The existence and uniqueness of the solution for the difference techniques defined in section II is shown by establishing that the tridiagonal coefficient matrix of the result algebraic system is diagonally dominant with negative main diagonal elements and positive super-diagonal and sub-diagonal elements.

III. I. Technique I.

It clear that E_i and G_i in (5) are positive under the condition

$$h_i, h_{i+1} < \frac{2}{\omega|p_i|}\ . \tag{14}$$

And since $q > 0$, we have

$$F_i = -\left(G_i + E_i + \omega q_i\right) < 0 \tag{15}$$

and

$$|F_i| = |G_i + E_i + \omega q_i| \geq |\ G_i + E_i\ |\ . \tag{16}$$

Thus the numerical technique I is stable and has a unique solution under condition (14).

III. II. Technique II.

It can be easily shown from (8) with constant coefficient q, that the nominators of E_i and G_i are positive with no restrictions on the step size while the denominator T is positive when

$$h_{i+1}h_i\,\omega(p_i' + q\ ) + 2p_i\,\omega(h_{i+1} - h_i) + 6 > 0\ ,$$

thus

$$h_{i+1} > \left(h_i - \frac{3}{\omega p_i} - \frac{(q_i + p_i')h_i h_{i+1}}{2p_i}\right)\ , \tag{17}$$

Thus the numerical technique II is stable and has a unique solution under condition (17).

III. III. Technique III.

The nominators of E_i and G_i, in (11) with constant coefficients p and q, are positive when $12h_{i+1}p\omega \leq 24$ and $h_{i+1}^3\omega(p\omega(p^2\omega + 2q)) - 4h_{i+1}^2\omega(p^2\omega + q) \leq 0$. Now, let $h_{i+1} = \dfrac{k}{\omega p}$, then the first condition yields $k \leq 2$, and the second condition yields

$$\begin{aligned} &h_{i+1}^3\omega(p\omega(p^2\omega + 2q)) - 4h_{i+1}^2\omega(p^2\omega + q) \leq 0 \\ &h_i\ (p\omega(p^2\omega + 2q))\ \leq\ 4(p^2\omega + q)\ , \\ &k\,(p^2\omega + 2q))\ \leq\ 4(p^2\omega + q) \end{aligned} \tag{18}$$

Thus

$$k \leq \frac{4(p^2\omega + q)}{(p^2\omega + 2q))},\ \text{ or }\ \ k \leq 2\ , \tag{19}$$

The denominator will be

$$\begin{aligned} T &= 6\omega(h_i^3 + h_{i+1}^3)(\ p^2\omega + q) + 3\omega(h_{i+1}^2 - h_i^2)(8p + h_i h_{i+1} pq\omega\ )) \\ &\quad + (h_{i+1} + h_i)(\omega^2 h_i\ h_{i+1}^2 q^2 + 12\omega h_i\ h_{i+1}\ q + 72) \end{aligned}$$

If we substitute by $h_i = \dfrac{L}{\omega p}$ and $h_{i+1} = \dfrac{M}{\omega p}$ in the denominator:

$$\begin{aligned} T &= 6\omega(h_i^3 + h_{i+1}^3)(\ p^2\omega + q) + 3\omega(h_{i+1}^2 - h_i^2)(8p + h_i h_{i+1} pq\omega\ )) \\ &\quad + (h_{i+1} + h_i)(\omega^2 h_i\ h_{i+1}^2 q^2 + 12\omega h_i\ h_{i+1}\ q + 72) \end{aligned}$$

we get $M \leq 2$,or $M \geq 3$, and $L \leq 3$, or $L \geq 4$, which means that there is no restrictions on the step size obtained from the denominator. Thus the numerical scheme is stable and has a unique solution under the condition (20).

$$h_i\,; h_{i+1} \leq \frac{2}{\omega|p|}\ . \tag{20}$$

IV. CONCLUSION AND DISCUSSION

In this paper, we have presented three finite difference three-point techniques for singularly perturbed boundary value problems (SPBVPs). These techniques are developed over unevenly spaced grid points aided mathematical symbolic language Maple. Local truncation error, uniqueness and stability conditions are discussed. The paper draws the attention of researchers to drive general formulas over arbitrary grid points and perform deeply more complicated studying of uniqueness, stability, and convergence aided mathematical symbolic language.

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