Metric Dimension in Product Graphs

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Abstract

In Ars. Combinatoria, [2(1976), 191-195], Harary and Melter mentioned that the metric dimension of wheel $W_{1,n}$ is 2. This result was improved by B. Shanmukha, B. Sooryanarayana and K.S. Harinath in Far East Journal of Applied Mathematics, [8(3)(2002), 217-229] where it was proved that the metric dimension of $W_{1,n}$ is 3 for $n=3, 6$; 2 for $n=4,5$; 3+2$k$ for $n=7+5k$ or $8+5k$ for all $k=0,1,2,....$ and 4+2$k$ for $n=9+5k$ or $10+5k$ or $11+5k$ for all $k=0,1,2,....$ In this paper we compute the metric dimension of some product graphs involving the wheel $W_{1,n}$.

Keywords: Metric dimension, Landmarks, Metric Basis.

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Introduction:

Graphs can be assigned to study various concepts of navigation in space. Each node of the graph denotes a place where the work is to be done, and edges denote the connections between the places. The problem of minimum number of machines (or robots) to be placed at certain nodes to trace each and every node uniquely is a classical one. This problem can be solved by a graph structural framework in which the navigating agent moves from one node to another. The places or nodes of a graph where we place the machines are called landmarks. The minimum number of machines required to locate uniquely each and every node is termed the metric dimension, with the set of all minimum possible number of landmarks constituting a metric basis.

All the graphs considered here are finite, undirected and without multiple edges. We use standard terminology. The terms not defined here may be found in [2], [3] and [4].
Definition 1: The metric dimension of a graph $G$, denoted by $\beta(G)$, is defined as the cardinality of a minimal subset $S$ of $V(G)$ having the property that for each pair of vertices $u,v$ there exists a $w$ in $S$ such that $d(w,u) \neq d(w,v)$. The co-ordinate of each vertex $v$ of $V(G)$ with respect to each landmark $b \in S$ is defined as usual with $i^{th}$ component as $d(v,b_i)$ for each $i$.

The problem of finding the metric dimension of a graph was first studied by Harary and Melter [2]. They gave a characterization for the metric dimension of trees which was later improved by Samir Khuller et.al., in [3]. In [2], along with the computation of the metric dimension of complete graphs, cycles, etc., it was mentioned that the metric dimension of the wheel $W_{1,n}$ is 2. This result was improved by Shanmukha, et.al., in [4] where the actual metric dimension of the wheel was computed.

Theorem 2 [Harary and Melter]: The metric dimension of a complete graph of order $n$ is $n-1$, for $n > 1$.

Theorem 3 [Harary and Melter]: The metric dimension of a graph $G$ is 1 if and only if $G$ is a path.

Theorem 4 [Shanmukha, Sooryanarayana and Harinath]:

If $W_{1,n}$ is the wheel for $n \geq 3$, then

$\beta(W_{1,3}) = \beta(W_{1,6}) = 3$
$\beta(W_{1,4}) = \beta(W_{1,9}) = 2$
$\beta(W_{1,x+5k}) = \begin{cases} 
3 + 2k & \text{when } x = 7 \text{ or } 8 \\
4 + 2k & \text{when } x = 9 \text{ or } 10 \text{ or } 11
\end{cases}$

For all $k = 0, 1, 2, 3, \ldots$

Theorem 5 [Khuller et.al.]: Let $G = (V,E)$ be a graph with metric dimension two and let $a, b \in V$ be a metric basis in $G$. The following are true:

There is a unique shortest path $P$ between $a$ and $b$.

The degree of $a$ and $b$ are at most three.

Every other node on $P$ has degree at most five.

Theorem 6[5]: Let $G = C_n \times P_m$. Then $\beta(G) = 3$, when $n$ is even.

Results:

The graph of $G(W_{1,n}, C_n \times P_m)$:

Let $G=G(W_{1,n}, C_n \times P_m)$ be the graph obtained by taking the wheel at the initial copy of cycle $C_n$ in the product graph $C_n \times P_m$ with $v_{i,j}$ as the $i^{th}$ vertex of $j^{th}$ copy of a path $P_{mj}$ for $1 \leq j \leq n$ (or $j^{th}$ vertex of $i^{th}$ copy of a cycle $C_{ni}$ for $1 \leq i \leq m$) and vertices $v_{i,j}$ for $1 \leq j \leq n$ which lie on the rim of the wheel $W_{1,n}$ with $v_1$ as the central
vertex of the wheel $W_{1,n}$.

**Example:** The following figure shows the graph of $G = (W_{1,6}, C_6 \times P_5)$:

![Graph](image)

**Theorem 7:** Let $G = G(W_{1,n}, C_n \times P_m)$. Then $\beta(G) = 3$, when $3 \leq n \leq 6$.

**Proof:** We first show that $\beta(G) > 2$. In fact, if only two vertices constitute a metric basis, then as the degree of these vertices cannot exceed three (by theorem 5), the vertices which constitute a metric basis are to be in the outermost circuit $C_{n,m}$ of $G$.

Now the innermost circuit together with central vertex of $G$ induce a wheel $W_{1,n}$ and metric dimension of $W_{1,n}$ is 3 for $n = 3$ or 6 (by theorem 4), it easily follows that no two vertices in the outer circuit will form a metric basis. However for $n = 4$ or 5, as $\beta(W_{1,n}) = 2$, to obtain distinct coordinates (unique address) for the vertices of $W_{1,n}$ in $G$ with respect to two vertices (landmarks) that are in the outermost circuit $C_{n,m}$ (which are adjacent vertices in $C_{n,m}$), without loss of generality we take $v_{m,1}$ and $v_{m,2}$ be two such vertices in $C_{n,m}$. Further, for $m > 1$, the vertices $v_{m-1,2}$ and $v_{m,3}$ are at distance two from both of the vertices $v_{m,1}$ and $v_{m,2}$ in $G$, and hence their coordinates are equal which is inadmissible. Thus $\beta(G) > 2$.

Now similar to the proof above we can explicitly show that the set $M = \{v_{m,1}, v_{m,2}, v_{m,4}\}$ constitutes a metric basis and hence $\beta(G) = 3$.

**Theorem 8:** Let $G = G(W_{1,n}, C_n \times P_m)$. Then $\beta(G) = \beta(W_{1,n})$ when $n = x + 5k$, $7 \leq x \leq 11$, for $k = 0, 1, 2, 3, \ldots$.
Proof: Choose a landmark at the vertex that lies on rim of the wheel $W_{1,n}$ in $G$ for optimality (since diameter of the wheel $W_{1,n}$ is 2, and choice of a landmark at $C_{ni}$ for some $2 \leq i \leq m$ in $G$ yields large number of vertices lies on rim of the wheel $W_{1,n}$ receive same coordinate hence it is easy to estimate metric dimension of $G$ by choosing landmark at the vertices lies on rim of the wheel $W_{1,n}$ in $G$).

For $x=7$, $n = 7+5k$, $k = 0, 1, 2, 3, \ldots$

Let $v_{i,j}$ be $i^{th}$ vertex of $j^{th}$ copy of path $P_{m_j}$ for $1 \leq j \leq n$ [or $j^{th}$ vertex of $i^{th}$ copy of cycle $C_n$ (for $1 \leq i \leq m$)] with vertices $v_{i,j}$ for $1 \leq j \leq n$ lie on rim of the wheel $W_{1,n}$ in $G$.

Let $M = \{v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_{1,3+2k}\}$ be a metric basis of $G$ (by theorem 4). Then, there are $7+5k$ vertices that lie on rim of the wheel $W_{1,n}$, out of these $4+3k$ vertices are not in the metric basis $M$ and all vertices of $W_{1,n}$ will receive distinct coordinates w.r.t. $M$, (by theorem 4). Also vertices that lie on rim of the wheel are initial vertices of the path $P_{m_j}$ and they have distinct coordinates and hence each vertex of the path $P_{m_j}$ in $G$ is distinct w.r.t. $M$ in $G$ for each $j$, $1 \leq j \leq m$. Thus the metric basis of $G$ is given by $\beta(G) = |M| = 3+2k = \beta(W_{1,n})$ when, $n = 7+5k$, $k = 0, 1, 2, 3, \ldots$

In a similar way it can be proved for $n = 8+5k$, $9+5k$, $10+5k$, $11+5k$, $k=0,1,2,3\ldots$

The graph of $G(W_{1,n}, C_nXP_m, W_{1,n})$

Let $G=G(W_{1,n}, C_nXP_m, W_{1,n})$ be the graph obtained by taking wheel at the terminal copy of cycle $C_{n_i}$ (or $m^{th}$ copy of cycle) in the graph $G(W_{1,n}, C_nXP_m)$ with $v_{i,j}$ as the $i^{th}$ vertex of $j^{th}$ copy of a path $P_{m_j}$ for $1 \leq j \leq n$ (or $j^{th}$ vertex of $i^{th}$ copy of a cycle $C_{n_i}$ for $1 \leq i \leq m$) and vertices $v_{i,j}$ for $1 \leq j \leq n$ which lie on the rim of the wheel $W_{1,n}$ with $v_i$ as the central vertex of the wheel $W_{1,n}$, the vertices $v_{m,j}$ for $1 \leq j \leq n$ which lie on the rim of the initial wheel $W_{1,n}$ with $v_m$ as the central vertex of the terminal wheel $W_{1,n}$.

Example: The graph of $G(W_{1,3}, C_3XP_5, W_{1,3})$ is shown in figure 5.3:
**Theorem 9:** Let $G = G(W_{1,n} C_n X P_m, W_{1,n})$. Then $\beta(G) = \beta(W_{1,n})$, when $n = 3$, 6, $x + 5k$, for $x = 7, 9, 10$ and $k = 0, 1, 2, 3, \ldots$

**Proof:** Let $n = x + 5k$, and $x = 7$, when $k = 0, 1, 2, \ldots$ Let $v_{ij}$ be $i^{th}$ vertex of $j^{th}$ the path $P_{m_j}$ for $1 \leq j \leq n$.

Let $M = \{v_{1,1}, v_{1,2}, v_{1,3}, \ldots v_{1,3+2k}\}$ be a metric basis of $G$. Then, there are $7 + 5k$ vertices lies on rim of the wheel $W_{1,n}$, out of these $4 + 3k$ vertices are not in the metric basis $M$ and all will receive distinct coordinate w.r.t $M$ (by theorem 4). Also the vertices which lie on the rim of the wheel are initial vertices of the path $P_{m_j}$, for $1 \leq j \leq n$, and they receive distinct coordinates and hence every vertex of the path $P_{m_j}$, $1 \leq j \leq n$, in $G$ is distinct with respect to $M$ in $G$. Thus the metric dimension of $G$ is:

$$\beta(G) = |M| = 3 + 2k = \beta(W_{1,n})$$

when, $n = 7 + 5k$, $k = 0, 1, 2, 3, \ldots$

In a similar way it can be proved for $n = 3, 6, 9 + 5k, 10 + 5k$.

**Theorem 10:** Let $G = G(W_{1,n} C_n X P_m, W_{1,n})$. Then $\beta(G) = \beta(W_{1,n}) + 1$, when $n = 4, 5, x + 5k$, for $x = 8, 11$, and $k = 0, 1, 2, 3, \ldots$

**Proof:** Let $n = 8 + 5k$, for $x = 8$, and $k = 0, 1, 2, 3, \ldots$

Let $v_{ij}$ be $i^{th}$ vertex of $j^{th}$ path $P_{m_j}$ for $1 \leq j \leq n$. Then, if $i = 1$ or $m$, the vertex $v_{ij}$ is on rim of the initial/terminal wheel $W_{1,n}$.

Let $M = \{v_{1,1}, v_{1,2}, v_{1,3}, \ldots v_{1,3+2k}\}$ be a metric basis of the initial wheel $W_{1,n}$ in $G$. Then, there exists a vertex $v_{l,r}$, for $1 \leq r \leq 8 + 5k$, which lies on the rim of the wheel $W_{1,n}$ and also is an initial vertex of a copy of the path $P_{m_r}$. This vertex has coordinates $(a_1 + 2, a_2 + 2, \ldots a_{3+2k} + 2)$ if $a_s = 0$ for each $1 \leq s \leq 3 + 2k$. All the other vertices that lie on rim of the wheel $W_{1,n}$ are of distinct coordinates w.r.t $M$. Now the vertex $v_{m,r}$ is on the rim of the terminal wheel $W_{1,n}$ with coordinates $(a_1 + m + 2, a_2 + m + 2, \ldots a_{3+2k} + m + 2)$ if $a_s = 0$ for each $1 \leq s \leq 3 + 2k$. Hence it follows that the coordinates of the vertex $v_{m-1,r}$ are $(a_1 + m + 1, a_2 + m + 1, \ldots, a_{3+2k} + m + 1)$ if $a_s = 0$ for each $1 \leq s \leq 3 + 2k$ which is identical with the coordinates of central vertex $v_m$ that lies on the terminal wheel $W_{1,n}$ in $G$ w.r.t $M$. Therefore, choose the vertex $v_{l,r}$ as the landmark which lies on the rim of the initial wheel $W_{1,n}$ in $G$. This gives vertices $v_{m-1,r}$ and $v_{m}$ with distinct coordinates w.r.t $M \cup \{v_{l,r}\}$, so that all the vertices in $G$ are of distinct coordinates w.r.t $M \cup \{v_{l,r}\}$. Thus the metric dimension of $G$ is $|M \cup \{v_{l,r}\}|$, that is,

$$\beta(G) = |M| + |\{v_{l,r}\}| = (3 + 2k) + 1 = \beta(W_{1,n}) + 1$$

when, $n = 8 + 5k$, for $k = 0, 1, 2, 3, \ldots$

In a similar manner it can be proved for $n = 4, 5, 11 + 5k$.

**References**