i-Hamiltonian Laceability in Product Graphs

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Abstract

For a connected graph G, let h(G) be the length of a Hamiltonian walk in G and call it the Hamiltonian number of G. Let i be a non-negative integer. A connected graph G of order n is called i-Hamiltonian if h(G)=n+i. In this paper, we define i-Hamiltonian-t-laceable graphs and i-Hamiltonian-t*-laceable graphs. We explore i-Hamiltonian-t*-laceability properties in the cartesian product of graphs involving paths and cycles.

Keywords: Connected graph, Hamiltonian-t-laceable, Hamiltonian-t*-laceable, i-Hamiltonian-t-laceable, i-Hamiltonian-t*-laceability

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Introduction

Let G be a finite, simple, connected and undirected graph. Let u and v be two vertices in G. The distance between u and v denoted by d(u,v) is the length of a shortest u-v path in G. In [1] Goodman and Hedetniemi introduced the concept of a Hamiltonian walk in a connected graph G, defined as a closed spanning walk of minimum length in G. They denoted the length of a Hamiltonian walk in G by h(G) and called h(G) as the Hamiltonian number of G. Therefore, for a connected graph of order n≥3, it follows that h(G)=n if and only if G is Hamiltonian. Figure 1 below shows a connected graph G with h(G)=6.

Let i be a non-negative integer. A connected graph G of order n is called i-Hamiltonian [2] if h(G)=n+i. Thus a 0-Hamiltonian graph is Hamiltonian. An almost Hamiltonian graph is a graph G of order n and h(G)=n+1.

A graph G is Hamiltonian-t-laceable [3] if there exists in G a Hamiltonian path between every pair of vertices u and v with d(u,v)=t, 1≤ t ≤ diamG, where t is a positive integer.
A graph $G$ is Hamiltonian-$t^*$-laceable [4] if there exist in $G$ a Hamiltonian path between at least one pair of distinct vertices $u$ and $v$ such that $d(u,v)=t, 1 \leq t \leq \text{diam}G$.

With the concepts of $i$-Hamiticnicity and Hamiltonian Laceability, we define the following:

**Definition 1:** Let $G$ be a connected graph of order $n$, let $h_p(G)$ be the length of a Hamiltonian path between any two distinct vertices in $G$. A Hamiltonian path in $G$ is called a $0$-Hamiltonian path if $h_p(G)=n-1$ and a path in $G$ is called $1$-Hamiltonian path if $h_p(G)=n$.

**Definition 2:** Let $i$ be a non-negative integer. A connected graph $G$ of order $n$ is called $i$-Hamiltonian-$t$-laceable if there exists in $G$, a $i$-Hamiltonian path between every pair of distinct vertices $u$ and $v$ with the property $d(u,v)=t, 1 \leq t \leq \text{diam}G$.

**Definition 3:** A connected graph $G$ of order $n$ is called $i$-Hamiltonian-$t^*$-laceable if there exists in $G$, a $i$-Hamiltonian path between at least one pair of distinct vertices $u$ and $v$ with the property $d(u,v)=t, 1 \leq t \leq \text{diam}G$.

Figure 1 below illustrates a $1$-Hamiltonian graph $G$ with $h(G)=6$. With respect to the vertices $v_1$ and $v_2$ this graph is $1$-Hamiltonian-$2^*$-laceable.

![Figure 1: A graph with $h(G)=6$](image)

**Results**

**Theorem 1:** Let $G=P_m$ and $H=P_n$. If $m$ and $n$ are odd integers such that $m, n \geq 3$, the Cartesian-product $G \times H$ is $1$-Hamiltonian-$t^*$-laceable, for $t=1, 3$ and $5$.

**Proof:** Let $G_1=G \times H$. In $G_1$ there are $mn$ vertices and diameter of $G \times H$ is $(m+n)-1$. Let the vertices of $G_1$ be denoted by $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Let $B_i$ denote the $m$ paths in $G_1$ given by; $B_i: a_{i1}-a_{i2}-a_{i3}-\ldots-a_{im}$ and let $P_j$ denote the $n$ paths in $G_1$ given by; $P_j: a_{j1}-a_{j2}-a_{j3}-\ldots-a_{mj}$.

Then, in $G_1$, $d(a_{i1},a_{i2})=1$ and the path $P$: $\{P_1 \cup B_m \cup (a_{mn},a_{m-1n}) \cup (a_{m-2},a_{m-2}) \cup$
\[(B_{m-1} - (a_{m-11}, a_{m-12})) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_{4} - (a_{41}, a_{42})) \cup (a_{42}, a_{32}) \cup (B_{3} - (a_{31}, a_{32})) \cup (a_{3n}, a_{2n}) \cup (B_{2} - (a_{2n}, a_{2n-1})) \cup \ldots \cup (a_{23}, a_{22}) \cup (a_{22}, a_{21})) \cup (B_{1} - (a_{1n-1}, a_{1n-2}) \ldots \ldots (a_{14}, a_{13}) \cup (a_{13}, a_{12})) \cup (a_{12}, a_{11})) \cup (a_{2n-2}, a_{1n-2}) \cup \ldots \cup (a_{24}, a_{23}) \cup (a_{23}, a_{13}) \cup (a_{13}, a_{12}) \cup (a_{12}, a_{11}) \cup (a_{1n-1}, a_{2n-1}) \cup (B_{1} - (a_{1n}, a_{1n-1}) \ldots \ldots (a_{14}, a_{13}))\]

is a 1-Hamiltonian path. Hence \(G_{1}\) is 1-Hamiltonian-1*-laceable.

**Figure 2:** Cartesian product of \(G = P_{m}\) and \(H = P_{n}\), \(d(a_{11}, a_{12}) = 1\)

Also, in \(G_{1}\), \(d(a_{11}, a_{14}) = 3\) and the path \(P:\{P_{1} \cup B_{m} \cup (a_{m-12}, a_{m-22}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_{4} - (a_{41}, a_{42})) \cup (a_{42}, a_{32}) \cup (B_{3} - (a_{31}, a_{32})) \cup (a_{3n}, a_{2n}) \cup (a_{2n}, a_{2n-1}) \cup (B_{2} - (a_{2n}, a_{2n-1})) \cup (a_{2n-2}, a_{2n-2}) \ldots \ldots (a_{24}, a_{23}) \cup (B_{1} - (a_{1n}, a_{1n-1}) \ldots \ldots (a_{13}, a_{12}))\}\) is a 1-Hamiltonian path. Hence \(G_{1}\) is 1-Hamiltonian-3*-laceable.

**Figure 3:** Cartesian product of \(G = P_{m}\) and \(H = P_{m}\), \(d(a_{11}, a_{14}) = 3\)
Further, in $G_1$ $d(a_{11}, a_{1n-1}) = 5$ and the path $P: \{P_1 \cup B_m \cup (a_{m,12}, a_{m-22}) \cup (B_m-1 - (a_{m-11}, a_{m-12})) \cup (B_m-2 - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_4 - (a_{41}, a_{42})) \cup (a_{42}, a_{32}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{3n}, a_{2n}) \cup (B_{2n}, a_{1n}) \cup (a_{1n}, a_{2n}) \cup (a_{22}, a_{12}) \cup (B_2 - (a_{21}, a_{2n-1}) \cup \ldots \cup (B_{31}, a_{22})) \cup (B_1 - (a_{11}, a_{12}) \cup \ldots \cup (a_{1n}, a_{1n-1})) \}$ is a 1-Hamiltonian path. Hence $G_1$ is $1$-Hamiltonian-5*-laceable.

**Figure 4:** Cartesian product of $G=P_m$ and $H=P_n$, $d(a_{11}, a_{1n-1}) = 5$

Hence the proof.

**Theorem 2:** Let $G=P_m$ and $H=P_n$. If $m$ and $n$ are odd integers such that $m, n \geq 3$, the Cartesian-product $G \times H$ is 1-Hamiltonian-2*-laceable, for $t=2, 4$ and 6.

**Proof:** Let $G_1 = G \times H$. In $G_1$ there are $mn$ vertices and diameter of $G \times H$ is $(m+n)-1$. Let the vertices of $G_1$ be denoted by $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Let $B_i$ denote the $m$ paths in $G_1$ given by: $B_i: a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}$ and let $P_j$ denote the $n$ paths in $G_1$ given by: $P_j: a_{j1}, a_{j2}, a_{j3}, \ldots, a_{jm}$.

Then, in $G_1$, $d(a_{11}, a_{13}) = 2$ and the path $P: \{P_1 \cup B_m \cup (a_{m,12}, a_{m-22}) \cup (B_m-1 - (a_{m-11}, a_{m-12})) \cup (B_m-2 - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_4 - (a_{41}, a_{42})) \cup (a_{42}, a_{32}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{3n}, a_{2n}) \cup (B_2 - (a_{21}, a_{2n-1}) \cup \ldots \cup (a_{22}, a_{21})) \cup (a_{2n}, a_{1n}) \cup (B_1 - (a_{1n}, a_{1n-1}) \ldots a_{14}, a_{13}) \cup (a_{11}, a_{12}) \cup (a_{1n-1}, a_{2n-1}) \cup (a_{2n}, a_{1n-2}) \cup \ldots \cup (a_{14}, a_{24}) \cup (a_{22}, a_{12})) \}$ is a 0-Hamiltonian path. Hence $G_1$ is 0-Hamiltonian-2*-laceable.
Figure 5: Cartesian product of $G=P_m$ and $H=P_n$, $d(a_{11}, a_{13}) = 2$

Also, in $G_1$, $d(a_{11}, a_{1n-2}) = 4$ and the path $P$: $\{P_1 \cup B_m \cup (a_{mn}, a_{m-1m}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup (a_{m-12}, a_{m-22}) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_4 - (a_{41}, a_{42})) \cup (a_{42}, a_{32}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{3n}, a_{2n}) \cup (B_2 - (a_{21}, a_{22})) \cup \ldots \cup (a_{2m}, a_{2m-1}) \cup (B_{l-1} - (a_{l1}, a_{l2})) \cup (a_{l1}, a_{l2}))\}$ is a 0-Hamiltonian path. Hence $G_1$ is 0-Hamiltonian-$4^*$-laceable.

Figure 6: Cartesian product of $G=P_m$ and $H=P_n$, $d(a_{11}, a_{1n-2}) = 4$

Further, in $G_1$, $d(a_{11}, a_{1n}) = 6$ and the path $P$: $\{P_1 \cup B_m \cup (a_{m12}, a_{m22}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_4 - (a_{41}, a_{42})) \cup (a_{42}, a_{32}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{3n}, a_{2n}) \cup (B_2 - (a_{21}, a_{22})) \cup (a_{2n}, a_{2n-1}) \cup (B_{l-1} - (a_{l1}, a_{l2})) \cup (a_{l1}, a_{l2}))\}$ is a 0-Hamiltonian path. Hence $G_1$ is 0-Hamiltonian-$6^*$-laceable.
Figure 7: Cartesian product of $G=P_m$ and $H=P_n$, $d(a_{11}, a_{1n}) = 6$

Hence the proof

**Theorem 3:** Let $G=P_m$ and $H=P_n$. If $m$ and $n$ are even integers such that $m, n \geq 3$, the Cartesian-product $G \times H$ is 1-Hamiltonian-$t^*$-laceable, for $t=2, 4$ and $6$.

**Proof:** Let $G_1 = G \times H$. In $G_1$ there are $mn$ vertices and diameter of $G \times H$ is $(m + n) - 1$. Let the vertices of $G_1$ be denoted by $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Let $B_i$ denote the $m$ paths in $G_1$ given by $B_i: a_{i1} - a_{i2} - a_{i3} - \ldots - a_{im}$ and let $P_j$ denote the $n$ paths in $G_1$ given by $P_j: a_{1j} - a_{2j} - a_{3j} - \ldots - a_{nj}$.

Then in $G_1$, $d(a_{11}, a_{13}) = 2$ and the path $P: \{P_1 \cup B_m \cup (a_{mn}, a_{m-1, n}) \cup (B_{m-1} - (a_{m-1, 1}, a_{m-1, 2})) \cup (a_{m-1, 2}, a_{m-2, 2}) \cup (B_{m-2} - (a_{m-2, 1}, a_{m-2, 2})) \cup \ldots \cup (B_4 - (a_{41}, a_{42})) \cup (a_{42}, a_{32}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{32}, a_{22}) \cup (B_2 - (a_{2n}, a_{2n-1})) \cup \ldots \cup (a_{21}, a_{22})) \cup (B_1 - (a_{1n-1}, a_{1n-2} - \ldots - a_{11}, a_{12})) \cup (a_{2n, a_{2n}}) \cup (a_{1m-1, a_{1m-2}}) \cup (a_{2m-2, a_{1m-2}}) \cup \ldots \cup (a_{14, a_{24}}) \cup (a_{22, a_{12}})\}$ is a $1$-Hamiltonian path. Hence $G_1$ is 1-Hamiltonian-$2^*$-laceable.

Figure 8: Cartesian product of $G=P_m$ and $H=P_n$, $d(a_{11}, a_{13}) = 2$
Also, in \( G_1 \), \( d(a_{11}, a_{1n-2}) = 4 \) and the path \( P: \{P_1 \cup B_m \cup (a_{mn}, a_{m-1n}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup (a_{m-12}, a_{mn}) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (a_{2n}, a_{2n}) \cup (B_2 - (a_{21}, a_{22})) \cup (B_1 - (a_{11}, a_{12})) \} \) is a 1-Hamiltonian path. Hence \( G_1 \) is 1-Hamiltonian-4*-laceable.

Further in \( G_1 \), \( d(a_{11}, a_{1n}) = 6 \) and the path \( P: \{P_1 \cup B_m \cup (a_{mn}, a_{m-1n}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup (a_{m-12}, a_{mn}) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (a_{2n}, a_{2n}) \cup (B_2 - (a_{21}, a_{22})) \cup (a_{12}, a_{22}) \cup (B_1 - (a_{11}, a_{12})) \} \) is a 1-Hamiltonian path. Hence \( G_1 \) is 1-Hamiltonian-6*-laceable.
Hence the proof.

**Theorem 4:** Let $G = P_m$ and $H = P_n$. Then the Cartesian product $G \times H$ is 0-Hamiltonian-$t^*$-laceable, for $t=1, 3, 5$ such that $1 \leq t \leq (m+n)-2$ where $m$ and $n$ be even for $m, n \geq 3$.

**Proof:** Let $G_1 = G \times H$. In $G_1$ there are $mn$ vertices and diameter of $G \times H$ is $(m + n) - 1$. Let the vertices of $G_1$ be denoted by $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Let $B_i$ denote the $m$ paths in $G_1$ given by $B_i$: $a_{i1}a_{i2}a_{i3} \ldots \ldots a_{in}$ and $P_j$ denote the $n$ paths in $G_1$ given by $P_j$: $a_{1j}a_{2j}a_{3j} \ldots \ldots a_{mj}$.

Then in $G_1$, $d(a_{11}, a_{12}) = 1$ and the path $P$: $\{P_1 \cup B_m \cup (a_{mn}, a_{m-1n}) \cup (a_{m-12}, a_{m-22}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_5 - (a_{51}, a_{52})) \cup (a_{53}, a_{54}) \cup (B_4 - (a_{41}, a_{42})) \cup (a_{43}, a_{44}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{33}, a_{34}) \cup (B_2 - (a_{21}, a_{22})) \cup (a_{23}, a_{24}) \cup (B_1 - (a_{11}, a_{12}))\}$ is a 0-Hamiltonian path. Hence $G_1$ is 0-Hamiltonian-$I^*$-laceable.

![Figure 11: Cartesian product of $G = P_m$ and $H = P_n$, $d(a_{11}, a_{12}) = 1$](image)

Also, in $G_1$, $d(a_{11}, a_{14}) = 3$ and the path $P$: $\{P_1 \cup B_m \cup (a_{mn}, a_{m-1n}) \cup (a_{m-12}, a_{m-22}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_5 - (a_{51}, a_{52})) \cup (a_{53}, a_{54}) \cup (B_4 - (a_{41}, a_{42})) \cup (a_{43}, a_{44}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{33}, a_{34}) \cup (B_2 - (a_{21}, a_{22})) \cup (a_{23}, a_{24}) \cup (B_1 - (a_{11}, a_{12}) \cup (a_{13}, a_{14}))\}$ is a 0-Hamiltonian path. Hence $G_1$ is 0-Hamiltonian-$3^*$-laceable.
Further in $G_1$, $d(a_{11}, a_{1n,l}) = 5$ and the path $P$: $\{P_1 \cup B_m \cup (a_{mn}, a_{m-1n}) \cup (a_{m-12}, a_{m-22}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup \ldots \cup (B_5 - (a_{51}, a_{52})) \cup (a_{52}, a_{42}) \cup (B_4 - (a_{41}, a_{42})) \cup (a_{43}, a_{3n}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{32}, a_{22}) \cup (B_2 - (a_{21}, a_{22})) \cup (a_{22}, a_{23}) \cup (a_{24}, a_{25}) \ldots \ldots \cup (a_{2n-3}, a_{2n-2}) \cup (a_{2n-2}, a_{2n-1}) \cup (B_1 - (a_{11}, a_{12})) \cup (a_{12}, a_{13}) \cup (a_{13}, a_{14}) \cup \ldots \ldots \cup (a_{1n-3}, a_{1n-2}) \cup (a_{12}, a_{13}) \cup (a_{13}, a_{14}) \cup (a_{24}, a_{25}) \ldots \ldots (a_{2n-3}, a_{2n-2}) \}$ is a $0$-Hamiltonian path. Hence $G_1$ is $0$-Hamiltonian-$t^*$-laceable.

Hence the proof

**Theorem 5**: Let $G=C_m$ and $H=P_n$. If $n \geq 2$ is an integer and $m \geq 3$ is an odd integer, the Cartesian-product $G \times H$ is $0$-Hamiltonian-$t^*$-laceable for $t=1,2$ and $3$. 
**Proof:** Let $G_1 = G \times H$. Let the vertices of $G_1$ be denoted by $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Let $B_i$ denote the $m$ paths in $G_1$ given by $B_i: a_{i1} - a_{i2} - a_{i3} - \ldots - a_{in}$ and $P_j$ denote the $n$ paths in $G_1$ given by $P_j: a_{1j} - a_{2j} - a_{3j} - \ldots - a_{nj}$. Where $n$ is an integer and $m$ is odd.

Then in $G_1$, $d(a_{11}, a_{1n}) = 1$ and the path $P: P_1 \cup B_m \cup (a_{mn}, a_{m-1n}) \cup (B_{m-1} - (a_{m-11}, a_{m-12})) \cup (a_{m-12}, a_{m-22}) \cup (B_{m-2} - (a_{m-21}, a_{m-22})) \cup \ldots \cup (B_2 - (a_{21}, a_{22})) \cup (a_{22}, a_{32}) \cup (B_3 - (a_{31}, a_{32})) \cup (a_{32}, a_{42}) \cup (B_4 - (a_{41}, a_{42})) \cup (a_{42}, a_{52}) \cup \ldots \cup (B_{n-1} - (a_{n-11}, a_{n-12})) \cup (a_{n-12}, a_{n-22})$ is a 0-Hamiltonian path. Hence $G_1$ is a 0-Hamiltonian-1*-laceable.

**Figure 14:** Cartesian product of $G = C_m$ and $H = P_n$, $d(a_{11}, a_{1n}) = 1$

Also, in $G_1$, $d(a_{11}, a_{1n-1}) = 2$ and the path $P: (a_{11}, a_{1n}) \cup (a_{1n}, a_{2n}) \cup (a_{2n}, a_{21}) \cup (a_{21}, a_{31}) \cup (a_{31}, a_{3n}) \cup (a_{3n}, a_{4n}) \cup (a_{4n}, a_{41}) \cup \ldots \cup (a_{m-1n}, a_{m-1n-1}) \cup (a_{m-1n-1}, a_{m-1n-2}) \cup (B_{m-1} - (a_{m-1n-2}, a_{m-1n-1})) \cup (a_{m-1n-1}, a_{m-2n}) \cup (B_{m-2} - (a_{m-2n}, a_{m-2n-1})) \cup (a_{m-2n}, a_{m-3n}) \cup (B_{m-3} - (a_{m-3n}, a_{m-3n-1})) \cup (a_{m-3n}, a_{m-4n}) \cup (B_{m-4} - (a_{m-4n}, a_{m-4n-1})) \cup \ldots \cup (P_{m-2} - (a_{mn-2}, a_{mn-1})) \cup (a_{mn-1}, a_{mn}) \cup (B_{m-1} - (a_{mn-1}, a_{mn-2})) \cup (a_{mn-2}, a_{mn}) \cup \ldots \cup (P_{m-1} - (a_{mn-1}, a_{mn-2}))$ is a 0-Hamiltonian path. Hence $G_1$ is a 0-Hamiltonian-2*-laceable.

**Figure 15:** Cartesian product of $G = C_m$ and $H = P_n$, $d(a_{11}, a_{1n-1}) = 2$
Further, in $G_1$, $d(a_{11},a_{1n-2})=3\) and the path 
\[ P: (a_{11}, a_{1n}) \cup (a_{1n}, a_{2n}) \cup (a_{2n}, a_{3n}) \cup (a_{3n}, a_{4n}) \cup \ldots \ldots \cup (a_{m-2n}, a_{m-1n}) \cup (a_{m-1n}, a_{mn}) \cup (B_{m-1}^{-1}(a_{mn}, a_{mn-1}) \cup (a_{mn-1}, a_{mn-2}) \cup \ldots \ldots \cup (a_{2n-1}, a_{2n}) \cup (a_{2n-1}, a_{1n-1}) \cup (B_1^{-1}(a_{1n}, a_{1n-1}) \cup \ldots \ldots \cup (a_{13}, a_{14}) \cup (a_{11}, a_{12})) \] is a 0-Hamiltonian path. Hence $G_1$ is 0-Hamiltonian-3*-laceable.

**Figure 16:** Cartesian product of $G=C_m$ and $H=P_n$, $d(a_{11}, a_{1n-2})=3$

Hence the proof

**Theorem 6:** Let $G=C_m$ and $H=P_n$. If $n \geq 2$ is an integer and $m \geq 3$ is an even integer, the Cartesian-product $G \times H$ is (i) 0-Hamiltonian-$t^*$-laceable for $t=1$ and 3 (ii) 1-Hamiltonian-$t^*$-laceable for $t=2$ and 4.

**Proof:** Let $G_1=G \times H$. Let the vertices of $G_1$ be denoted by $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Let $B_i$ denote the $m$ paths in $G_1$ given by $B_i$: $a_{i1}-a_{i2}-a_{i3}-\ldots-a_{im}$ and $P_j$ denote the $n$ paths in $G_1$ given by $P_j$: $a_{j1}-a_{j2}-a_{j3}-\ldots-a_{mj}$. Where $n$ is any integer and $m$ is even.

Then in $G_1$, $d(a_{11}, a_{1n})=1$ and the path 
\[ P: P_1 \cup (a_{m1}, a_{m2}) \cup P_2 \cup (a_{n1}, a_{n2}) \cup P_3 \cup (a_{m3}, a_{m4}) \cup P_4 \cup \ldots \ldots \cup P_{n-1} \cup (a_{mn-1}, a_{mn}) \cup P_n \] is a 0-Hamiltonian path. Hence $G_1$ is 0-Hamiltonian-$I^*$-laceable.
Figure 17: Cartesian product of $G=C_m$ and $H=P_n$, $d(a_{11},a_{1n})=1$

Also, in $G_1$, $d(a_{11},a_{2n})=2$ and the path

$P: P_1 \cup (a_{m1},a_{m2}) \cup P_2 \cup (a_{12},a_{13}) \cup P_3 \cup (a_{m3},a_{m4}) \cup P_4 \cup \ldots \cup (P_{n-1} \cup (a_{mn-1},a_{mn})) \cup (P_n \cup (a_{3n},a_{2n})) \cup (a_{3n},a_{1n})$ is a 1-Hamiltonian path. Hence $G_1$ is 1-Hamiltonian-2*-laceable.

Figure 18: Cartesian product of $G=C_m$ and $H=P_n$, $d(a_{11},a_{2n})=2$

Further in $G_1$, $d(a_{11},a_{3n})=3$ and the path $P: P_1 \cup (a_{m1},a_{m2}) \cup P_2 \cup (a_{12},a_{13}) \cup P_3 \cup (a_{m3},a_{m4}) \cup P_4 \cup \ldots \cup (P_{n-1} \cup (a_{1n-1},a_{2n-1})) \cup (a_{1n-1},a_{1n}) \cup (a_{2n-1},a_{2n}) \cup (a_{mn-1},a_{mn}) \cup (P_n \cup (a_{3n},a_{2n}))$ is a 0-Hamiltonian path. Hence $G_1$ is 0-Hamiltonian-3*-laceable.
Next in $G_1$, $d(a_{11},a_{4n})=4$ and the path 
\[ P: P_1 \cup (a_{m1},a_{m2}) \cup P_2 \cup (a_{n1},a_{n2}) \cup P_3 \cup (a_{m3},a_{m4}) \cup P_4 \cup \ldots \cup (P_{n-1},a_{2n-1},a_{2n}) \cup (a_{2n-1},a_{3n-1}) \cup (a_{mn-1},a_{mn}) \cup (P_{n},a_{1n},a_{2n}) \cup (a_{3n},a_{4n}) \cup (a_{1n},a_{2n-1}) \] 
is a 1-Hamiltonian path. Hence $G_1$ is 1-Hamiltonian-4-laceable.

\[ \text{Figure 19: Cartesian product of } G=C_m \text{ and } H=P_n, \quad d(a_{11},a_{3n})=3 \]

\[ \text{Figure 20: Cartesian product of } G=C_m \text{ and } H=P_n, \quad d(a_{11},a_{4n})=4 \]

Hence the proof.

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