Edge-Odd Gracefulness of the Graph $P_m + P_n$ for $m = 2, 3, 4, 5,$ and $6$

A. Solairaju, A. Sasikala and C. Vimala

Abstract

A $(p, q)$ connected graph is edge-odd graceful graph if there exists an injective map $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ so that induced map $f^+: V(G) \rightarrow \{0, 1, 2, 3, \ldots, (2k-1)\}$ defined by $f^+(x) \equiv \sum_{y \sim x} f(y) \pmod{2k}$, where the vertex $x$ is incident with other vertex $y$ and $k = \max \{p, q\}$ makes all the edges distinct and odd. In this article, the Edge-odd gracefulness of $P_m + P_n$ for $m = 2, 3, 4, 5,$ and $6$ is obtained.

Keywords: Graceful Graphs, Edge-odd graceful labeling, Edge-odd Graceful Graph

Introduction

A. Solairaju and K. Chitra [2009] obtained edge-odd graceful labeling of some graphs related to paths. A. Solairaju et. al. [2009, 2010] proved that the graphs $C_5 \Theta P_n$ and $C_5 \Theta 2P_n$ are edge-odd graceful.

Here the edge-odd graceful labeling of $P_m + P_n$ for $m = 2, 3, 4, 5,$ and $6$ is obtained.

Edge-odd graceful labeling of $P_m + P_n$ for $m = 2, 3, 4, 5,$ and $6$

Definition 2.1: Graceful Graph: A function $f$ of a graph $G$ is called a graceful labeling with $m$ edges, if $f$ is an injection from the vertex set of $G$ to the set $\{0, 1, 2, \ldots, m\}$ such that when each edge $uv$ is assigned the label $|f(u) - f(v)|$ and the resulting edge labels are distinct. Then the graph $G$ is graceful.
Definition 2.2: Edge-odd graceful graph: A \((p, q)\) connected graph is edge-odd graceful graph if there exists an injective map \(f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}\) so that induced map \(f_+: V(G) \rightarrow \{0, 1, 2, \ldots, (2k-1)\}\) defined by \(f_+(x) \equiv \Sigma f(x, y) \pmod{2k}\), where the vertex \(x\) is incident with other vertex \(y\) and \(k = \max\ \{p, q\}\) makes all the edges distinct and odd. Hence the graph \(G\) is edge-odd graceful.

Edge-odd Gracefulness of the graph \(P_2 + P_n\)

Definition 3.1: \(P_2 + N_n\) is a connected graph such that every vertex of \(P_2\) is adjacent to every vertex of null graph \(N_n\) together with adjacency in both \(P_2\) and \(P_n\). It has \(n + 2\) vertices and \(3n\) edges.

Theorem 3.1: The connected graph \(P_2 + P_n\) is edge-odd graceful.

Proof: The figure 1 is connected graph \(P_2 + P_n\) with \(n + 2\) vertices and \(3n\) edges, with some arbitrary labeling to its vertices and edges as follows.

\[
\begin{align*}
\text{Figure 1: Edge-odd graceful Graph } P_2 + P_n
\end{align*}
\]

Hence define \(f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}\) by

For \(n\) is odd
\[
\begin{align*}
f(e_i) = (2i-1), & \text{ for } i = 1, 2, \ldots, (3n) \quad \text{(Rule 1)}
\end{align*}
\]

For \(n\) is even and \(i \neq 6\)
\[
\begin{align*}
f(e_i) = (2i-1), & \text{ for } i = 1, 2, \ldots, (2n+1), \\
f(e_{3n-i}) = f(e_{2n+1}) + 2i + 2, & \text{ for } i = 0, 1, 2, \ldots, (n-2).
\end{align*}
\]

Define \(f_+: V(G) \rightarrow \{0, 1, 2, \ldots, (2k-1)\}\) by \(f_+(v) \equiv \Sigma f(uv) \pmod{2k}\), where this sum run over all edges through \(v\) (Rule 2)

Hence the map \(f\) and the induced map \(f_+\) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \(\{0, 1, 2, \ldots, (2k-1)\}\). Hence the graph \(P_2 + P_n\) is edge-odd graceful.
Lemma 3.1: The connected graph $P_2 + P_6$ is edge – odd graceful.

Proof: The following graph in figure 2 is a connected graph with 8 vertices and 18 edges with some arbitrary distinct labeling to its vertices and edges.

![Figure 2: Edge – odd graceful Graph $P_2 + P_6$](image)

Edge-odd Gracefulness of the graph $P_3 + P_n$

Definition 4.1: $P_3 + N_n$ is a connected graph such that every vertex of $K_3$ is adjacent to every vertex of null graph $N_n$ together with adjacency in both $P_3$ and $P_n$. It has $n + 3$ vertices and $4n+1$ edges.

Theorem 4.1: The connected graph $P_3 + P_n$ for $n = 1, 2, \ldots, (4n + 1)$ is edge – odd graceful.

Proof: The figure 3 is connected graph $P_3 + P_n$ with $n + 3$ vertices and $4n+1$ edges, with some arbitrary labeling to its vertices and edges.

Case i: $n = 1, 2, \ldots, (4n + 1)$ and $n \neq 8, 14, 20, 26, \ldots (6m + 2)$

![Figure 3: Edge – odd graceful Graph $P_3 + P_n$](image)
Hence define \( f: E(G) \to \{1, 3, \ldots, 2q-1\} \) by

For \( n \equiv 0 \pmod{6} \)
\[ f(e_i) = (2i-1), \text{ for } i = 3, 4, 5, \ldots, (4n+1) \]
\[ f(e_1) = 3; f(e_2) = 1 \]

For \( n \equiv 1 \pmod{6} \)
\[ f(e_i) = (2i-1), \text{ for } i = 1, 4, 5, 6, \ldots, (4n+1) \]
\[ f(e_2) = 5; f(e_3) = 3 \]

(Rule 3)

For \( n \equiv 3, 5 \pmod{6} \)
\[ f(e_i) = (2i-1), \text{ for } i = 2, 4, 5, 6, \ldots, (4n+1) \]
\[ f(e_1) = 5; f(e_3) = 1 \]

For \( n \equiv 4 \pmod{6} \)
\[ f(e_i) = (2i-1), \text{ for } i = 2, 3, \ldots, (4n) \]
\[ f(e_1) = 2q-1; f(e_{4n+1}) = 1 \]

Case ii: \( n \neq 8, 14, 20, 26, \ldots (6m + 2), m = 1, 2, \ldots, \)

Define \( f: E(G) \to \{1, 3, \ldots, 2q-1\} \) by

For \( n \equiv 2 \pmod{6} \)
\[ f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (n-1), (n+1), \ldots, (3n-1), (3n+1), \ldots, (4n+1) \]
\[ f(e_n) = 6n-1; f(e_{3n}) = 2n-1. \]

(Rule 5)
Define \( f_+ : V(G) \to \{0, 1, 2, \ldots, (2k-1)\} \) by
\[ f_+(v) \equiv \sum f(uv) \mod (2k), \]
where this sum runs over all edges through \( v \) (Rule 6).

Hence the map \( f \) and the induced map \( f_+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \( \{0, 1, 2, \ldots, (2k-1)\} \).
Hence the graph \( P_3 + P_n \) is edge-odd graceful.

**Lemma 4.1:** The connected graph \( P_3 + P_3 \) is edge–odd graceful.

**Proof:** The following graph in figure 5 is a connected graph with 6 vertices and 13 edges with some arbitrary distinct labeling to its vertices and edges.

![Figure 5: Edge–odd graceful Graph P₃ + P₃](image)

**Lemma 4.2:** The connected graph \( P_3 + P_4 \) is edge–odd graceful.

The following graph in figure 6 is a connected graph with 7 vertices and 17 edges with some arbitrary distinct labeling to its vertices and edges.

![Figure 6: Edge–odd graceful Graph P₃ + P₄](image)

**Lemma 4.3:** The connected graph \( P_3 + P_5 \) is edge–odd graceful.
The following graph in figure 7 is a connected graph with 8 vertices and 21 edges with some arbitrary distinct labeling to its vertices and edges.

![Graph](image)

**Figure 7**: Edge – odd graceful Graph $P_3 + P_5$

**Edge-odd Gracefulness of the graph $P_4 + P_n$**

**Definition 5.1**: $P_4 + N_n$ is a connected graph such that every vertex of $P_4$ is adjacent to every vertex of null graph $N_n$ together with adjacency in both $P_4$ and $P_n$. It has $n + 4$ vertices and $5n+2$ edges.

**Theorem 5.1**: The connected graph $P_4 + P_n$ for $n = 1, 2, 3, 4, \ldots, (5n + 2)$ is edge – odd graceful.

**Proof**: The figure 8 is connected graph $P_4 + P_n$ with $n + 4$ vertices and $5n+2$ edges, with some arbitrary labeling to its vertices and edges.

**Case i**: $n = 1, 2, \ldots, (5n + 2)$ and $n \neq 8, 14, 20, 26, \ldots (6m + 2)$

![Graph](image)

**Figure 8**: Edge – odd graceful Graph $P_4 + P_n$
Hence define \( f : E(G) \rightarrow \{1, 3, \ldots, 2q-1\} \) by

For \( n \equiv 0 \pmod{6} \)
\[
f(e_i) = (2i-1), \text{ for } i = 4, 5, \ldots, (5n+2)
\]
f(e_1) = 3; f(e_2) = 5; f(e_3) = 1

For \( n \equiv 1 \pmod{6} \)
\[
f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (5n+2)
\]

For \( n \equiv 3 \pmod{6} \)
\[
f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (5n+2); i \neq 4 \text{ & } 4n + 3
\]
f(e_4) = 8n + 5; f(e_{4n + 3}) = 7

For \( n \equiv 4 \pmod{6} \)
\[
f(e_i) = (2i-1), \text{ for } i = 4, 5, 6, \ldots, (5n+2)
\]
f(e_1) = 5; f(e_2) = 1; f(e_3) = 3

For \( n \equiv 5 \pmod{6} \)
\[
f(e_i) = (2i-1), \text{ for } i = 1, 2, 5, 6, \ldots, (5n+2)
\]
f(e_3) = 7; f(e_4) = 5

Case ii: \( n \equiv 8, 14, 20, 26, \ldots (6m + 2), m = 1, 2, \ldots, \)

Figure 9: Edge – odd graceful Graph \( P_4 + P_n \)

Define \( f : E(G) \rightarrow \{1, 3, \ldots, 2q-1\} \) by

For \( n \equiv 2 \pmod{6} \)
\[
\] (Rule 8)
f(e_i) = (2i-1), for i = 1, 2, ..., (5n+2); i ≠ 2n & 4n-1
f(e_{2n}) = 8n - 3; f(e_{4n-1}) = 4n - 3

Define \( f_+: V(G) \rightarrow \{0, 1, 2, ..., (2k-1)\} \) by \( f_+(v) \equiv \sum f(uv) \mod (2k) \), where this sum run over all edges through \( v \) (Rule 9)

Hence the map \( f \) and the induced map \( f_+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \{0, 1, 2,..., (2k-1)\}. Hence the graph \( P_4 + P_n \) is edge-odd graceful.

**Lemma 5.1:** The connected graph \( P_4 + P_4 \) is edge – odd graceful.

**Proof:** The following graph in figure 10 is a connected graph with 8 vertices and 22 edges with some arbitrary distinct labeling to its vertices and edges.

![Figure 10: Edge – odd graceful Graph P4 + P4](image)

**Lemma 5.2:** The connected graph \( P_4 + P_5 \) is edge – odd graceful.
The following graph in figure 11 is a connected graph with 9 vertices and 27 edges with some arbitrary distinct labeling to its vertices and edges.

![Figure 11: Edge – odd graceful Graph P4 + P5](image)
**Edge-odd Gracefulness of the graph P₅ + Pₙ**

**Definition 6.1:** P₅ + Nₙ is a connected graph such that every vertex of P₅ is adjacent to every vertex of null graph Nₙ together with adjacency in both P₅ and Pₙ. It has n + 5 vertices and 6n+3 edges.

**Theorem 6.1:** The connected graph P₅ + Pₙ for all n ≠ 7is edge – odd graceful.

**Proof:** The figure 12 is connected graph P₅ + Pₙ with n + 5 vertices and 6n+3 edges, with some arbitrary labeling to its vertices and edges.

![Figure 12: Edge – odd graceful Graph P₅ + Pₙ](image)

Hence define f: E(G) → {1, 3, …, 2q-1} by
\[ f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (6n+3) \]  

Define \( f_+ \): V(G) → {0, 1, 2, …, (2k-1)} by \( f_+(v) = \sum f(uv) \mod (2k) \), where this sum run over all edges through v  

Hence the map f and the induced map \( f_+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \{0, 1, 2,\ldots, (2k-1)\}. Hence the graph \( P₅ + Pₙ \) is edge-odd graceful.

**Lemma 6.1:** The connected graph P₅ + P₇ is edge – odd graceful.

The graph in figure 13 is a connected graph with 12 vertices and 45 edges with some arbitrary distinct labeling to its vertices and edges.
Definition 7.1: \( P_6 + N_n \) is a connected graph such that every vertex of \( P_6 \) is adjacent to every vertex of null graph \( N_n \) together with adjacency in both \( P_6 \) and \( P_n \). It has \( n + 6 \) vertices and \( 7n + 4 \) edges.

Theorem 7.1: The connected graph \( P_6 + P_n \) for all \( n \neq 7 \) and 8 is edge – odd graceful.

Proof: The figure 14 is connected graph \( P_6 + P_n \) with \( n + 6 \) vertices and \( 7n + 4 \) edges, with some arbitrary labeling to its vertices and edges.

Hence define \( f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\} \) by
\[
f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (7n+4) \]  
[Rule 12]

Define \( f+: V(G) \rightarrow \{0, 1, 2, \ldots, (2k-1)\} \) by \( f_+(v) = \Sigma f(uv) \mod (2k) \), where this sum run over all edges through \( v \)  
[Rule 13]
Hence the map \( f \) and the induced map \( f_+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \( \{0, 1, 2, \ldots, (2k-1)\} \).

Hence the graph \( P_6 + P_n \) is edge-odd graceful.

**Lemma 7.1:** The connected graph \( P_6 + P_7 \) is edge–odd graceful.

The graph \( P_6 + P_7 \) is a connected graph with 13 vertices and 53 edges. All the edges of the graph are labeled with distinct odd numbers in such a way that there will be distinct labeling for all its vertices.

That is, define \( f: E(G) \to \{1, 3, \ldots, 2q-1\} \) by
\[
f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, 53
\]

Define \( f_+: V(G) \to \{0, 1, 2, \ldots, (2k-1)\} \) by
\[
f_+(v) \equiv \sum f(uv) \mod (2k), \text{ where this sum runs over all edges through } v
\]

Hence the graph \( P_6 + P_7 \) is edge-odd graceful.

The graph with edge-odd graceful labeling is given in the figure 15.

![Figure 15: Graph of \( P_6 + P_7 \)](image)

**Lemma 7.2:** The connected graph \( P_6 + P_8 \) is edge–odd graceful.

The graph \( P_6 + P_8 \) is a connected graph with 14 vertices and 60 edges. All the edges of the graph are labeled with distinct odd numbers in such a way that there will be distinct labeling for all its vertices.

That is, define \( f: E(G) \to \{1, 3, \ldots, 2q-1\} \) by
\[
f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, 60
\]

Define \( f_+: V(G) \to \{0, 1, 2, \ldots, (2k-1)\} \) by
\[
f_+(v) \equiv \sum f(uv) \mod (2k), \text{ where this sum runs over all edges through } v
\]

Hence the graph \( P_6 + P_8 \) is edge-odd graceful.

The graph with edge-odd graceful labeling is given in the figure 16.
Figure 16: Graph of $P_6 + P_8$

Reference

[1] A.Solairaju and K.Chitra  
Edge-odd graceful labeling of some graphs 
‘Electronics Notes in Discrete Mathematics’ Volume 33, April 2009, Pages 15 - 20


(communicated to Serial Publications)