Pebbling Number of the Sequential Join of Complete Graphs

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Abstract

Given a configuration of pebbles on the vertices of a graph G, a pebbling move consists of taking two pebbles off some vertex v and putting one of them back on a vertex adjacent to v. A graph is called pebbleable if for each vertex v there is a sequence of pebbling moves that would place at least one pebble on v. The pebbling number of graph G, is the smallest integer m such that G is pebbleable for every configuration of m pebbles on G. Let G1 and G2 be graphs such that G1 and G2 have disjoint vertex sets V1 and V2 and edges sets E1 and E2 respectively. Their join G1+G2 consists of G1∪G2 and all edges joining V1 with V2. For three or more disjoint graphs G1, G2……, Gn, the sequential join G1 + G2 +…..+ Gn is the graph (G1 + G2) ∪ (G2 + G3) ∪…..∪ (Gn-1 + Gn). In this paper we find the pebbling number of the sequential join of complete graphs.

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Introduction

A pebbling configuration of a graph G is a distribution of pebbles on G. A pebbling move consists of removing two pebbles lying on the same vertex v and placing one of these pebbles on some vertex that is adjacent to v. A distribution (configuration) p is “v-solvable” (v is reachable under p), if v has a pebble after some (possibly empty) pebbling sequence (sequence of pebbling moves) starting from p. The pebbling number of a graph G is the minimum number m of pebbles that ensure that every
vertex of $G$ is pebbleable, no matter what initial distribution of $m$ pebbles we start with. Let $f(G, v)$ denotes the pebbling number of a vertex $v$ of $G$, and $f(G)$ the pebbling number of $G$.

T. Clarke et al [3] defined class 0 and class 1 graphs. They defined a graph $G$ to be class 0 if $f(G) = n(G)$, the number of vertices in $G$ and of class 1 if $f(G) = n(G) + 1$.

Let $G_1$ and $G_2$ be graphs such that $G_1$ and $G_2$ have disjoint vertex sets $V_1$ and $V_2$ and edges sets $E_1$ and $E_2$ respectively. Their join $G_1 + G_2$ consists of $G_1 \cup G_2$ and all edges joining $V_1$ with $V_2$. For three or more disjoint graphs $G_1, G_2, ..., G_n$, the sequential join $G_1 + G_2 + ... + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup ... \cup (G_{n-1} + G_n)$.

**Theorem 1[1]**: If $\text{diam}(G) = 2$ and $K(G) \geq 3$, then $G$ is of class 0.

The graph $H$ in figure 1 is $K_1 + K_2 + K_2 + K_1$. It is clear that $H$ is not class 0. More generally, the graph $K_1 + K_r + K_r + K_1$ is class 0 if $r \geq 3$ and not otherwise.

![Figure 1: A graph H which is not class 0.](image)

**Theorem 2**: The graph $K_1 + K_{n-2} + K_1$ is class 0, if $n \geq 4$.

**Proof**: The proof is obvious. We say that a vertex $v$ in graph $G$ is substituted by $H$ if $v$ is replaced by $H$ and if $uv$ is an edge in $G$, $\{uw/w \in V(H)\}$ will be edges in the new graph. If $G^1$ is the graph obtained, then

$$G^1 = \{G - v\} \cup H \cup \{ux : x \in V(H), u \in N(v)\}$$

**Lemma 3**: Let $G$ be a connected graph. Let $v \in G$ be substituted by $K_n$, a complete graph on $n$ vertices. Let $G^1$ be the new graph obtained. Let $V^1 = V(K_n)$ and $w \in V^1$, then $f(G^1, w) = f(G, v) + n - 1$.

**Proof**: Let $f(G, v) = r$. Let $p$ be a non $v$-solvable distribution of pebbles for $G$ with $|p| = r - 1$. Clearly $p(v) = 0$. 


Let $p^1$ be the pebbling distribution of $G^1$ with $p/G - \{v\} = p^1/G - \{v\}$ and one pebble on each vertex of $K_n$ except $w$. Then $|p^1| = r + (n-2)$ and is, clearly, not $w$ solvable.

Let $Q$ be any distribution of $r + n - 1$ pebbles on the vertices of $G^1$. Then, either $Q/\{v\} \geq n$ or $Q/\{V(G) - V\} \geq f(G, v)$

So, $Q$ is $w$ solvable.

Hence the lemma.

**Lemma 4:** Let $G, V, G^1$ be as in lemma 3. Let $uv \in E(G)$. Then $f(G^1, u) \leq f(G, u) + |V(H)|$.

If there is at least one non $u$-solvable distribution of $f(G, u) - 1$ pebbles on $G$, with $p(v) = 1$, then $f(G^1, u) \geq f(G, u) + |V(H)| - 1$.

(In particular, if $\text{deg} v = 1$, the equality holds)

**Theorem 5:** The $K_{n_1} + K_{n_2} + K_{n_3}$ is class of 0, if $\sum_{i=1}^{3} n_i \geq 4$ and $n_2 \geq 2$.

**Proof:** The graph is 3-connected and is of diameter 2 and hence a class 0 graph.

**Theorem 6:** Let $G$ denotes the sequential join of the four complete graphs $K_{n_1}, K_{n_2}, K_{n_3}, K_{n_4}$. Then $f(G) = \sum_{i=1}^{4} n_i$, where $n_2 \geq 2$, $n_3 \geq 2, n_2 + n_3 \geq 5$. That is, $G$ is a diameter 3, class 0, graph.

**Proof:** Consider the path $P_4$ with vertex set $\{u, v, w, x\}$. We have $G \cong K_{n_1} + K_{n_2} + K_{n_3} + K_{n_4}$.

$G$ can be obtained by substituting $u, v, w$ and $x$ by $K_{n_1}, K_{n_2}, K_{n_3}$ and $K_{n_4}$ respectively. We calculate the pebbling number of $G$ in four steps.

**Step 1:** Replace one pendant vertex of $P_4$ say $u$ by $K_{n_1}$. Let $G_1$ be the new graph obtained. It is easy to see that $f(G_1) = n_1 + 7$.

**Step 2:** We now replace $x$ in $G_1$ by $K_{n_4}$. Let $G_2$ be the new graph obtained. Then $f(G_2) = n_1 + n_4 + 6$.

**Step 3:** In $G_2$, we substitute $K_{n_2}$ in the place of $v$. Let $G_3$ be the new graph obtained.

We have $f(G_3) = \begin{cases} n + 3, & \text{if } n_2 = 2 \\ n + 2, & \text{if } n_2 = 3 \\ n + 1, & \text{if } n_2 \geq 4. \end{cases}$
Step 4: The vertex $w$ in $G_3$ is replaced by $K_{n_3}$. Let $G_4$ be the new graph obtained. We see that $G_4$ is required graph $G$. Again it is easy to prove that

$$f(G) = \begin{cases} n & \text{if } n_2+n_3 \geq 6 \\ n+1 & \text{if } n_2+n_3 = 5 \\ n+2 & \text{if } n_2-n_3 = 2 \end{cases}$$

Hence the theorem

We now have the following conjecture.
Conjecture 7: If $G = K_{n_1} + K_{n_2} + \ldots + K_{n_r}$, $r \geq 4$, then $G$ is class 0 if $n_2$, $n_3$, \ldots, $n_{r-1} \geq \frac{2r}{r-1}$

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References