Characterization of Two Domination Number and Chromatic Number of a Graph

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Abstract

A Subset S of V is called a dominating set in G if every vertex in V-S is adjacent to at least one vertex in S. A Dominating set is said to be two dominating set if every vertex in V-S is adjacent to at least two vertices in S. The minimum cardinality taken over all, the minimal two dominating set is called two domination number and is denoted by $\gamma_2(G)$. The minimum number of colors required to colour all the vertices such that adjacent vertices do not receive the same colour is the chromatic number $\chi(G)$. In this paper, we characterize the classes of graphs whose sum of two domination number and chromatic number is equals to 2n-5 and 2n – 6.

Keywords: Two Domination Number, Chromatic Number.

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Introduction

Let $G= (V, E)$ be a simple undirected graph. The degree of any vertex $u$ in $G$ is the number of edges incident with $u$ and is denoted by $d(u)$. The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively; $P_n$ denotes the path on $n$ vertices. The vertex connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph. A colouring of a graph is an assignment of colours to its vertices so that two adjacent vertices have the same color. An $n$-colouring of a graph $G$ uses $n$ colours. The Chromatic Number $\chi$ is defined to be
the minimum n for which G has an n-colouring. If $\chi(G) = k$ but $\chi(G) < k$, for every proper subgraph H of G, then G is k-critical.

A subset S of V is called a dominating set in G if every vertex in V-S is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets in G is called the domination number of G and is denoted by $\gamma$. A dominating set is said to be two dominating set if every vertex in V-S is adjacent to at least two vertices in S. The minimum cardinality taken over all the minimal two dominating set is called two domination number and is denoted by $\gamma_2(G)$.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [10], Paulraj Joseph J and Arumugam S proved that $\gamma + k \leq p$. In [11], Paulraj Joseph J and Arumugam S proved that $\gamma + \chi = p + 1$. They also characterized the class of graphs for which the upper bound is attained. They also proved similar results for $\gamma$ and $\gamma_t$. In [7], Paulraj Joseph J and Mahadevan G, proved that $\gamma_{cc} + \chi \leq 2n - 1$ and characterized the corresponding extremal graphs. In [12], Paulraj Joseph J and Mahadevan G proved that $\gamma_{pr} + \chi \leq 2n - 1$ and characterized the corresponding extremal graphs. In [8], Paulraj Joseph J and Mahadevan G introduced the concept of complementary perfect domination number $\gamma_{cp}$ and proved that $\gamma_{cp} + \chi \leq 2n - 2$, and characterized the corresponding extremal graphs. They also obtained the similar results for the induced complementary perfect domination number and chromatic number of a graph. In this paper, we obtain sharp upper bound for the sum of the two domination number and chromatic number and characterize the corresponding extremal graphs. Terms not defined here, are used in the sense of Hanary[1].

Notations: $K_n (P_m)$ denotes the graph obtained from $K_n$ by attaching the end vertex of $P_m$ to any one of the vertices of $K_n$. $K_n (m_1, m_2, m_3, \ldots, m_k)$ denotes the graph obtained from $K_n$ by attaching $m_1$ edges to the vertex $u_i$ of $K_n$, $m_2$ edges to the vertex $u_j$ for $i \neq j$ of $K_n$, ..., $m_k$ edges to all the distinct vertices of $K_n$.

Previous Results

Theorem 1.1: For any connected graph G, $\gamma_2(G) \leq n$

Theorem 1.2: For any connected graph G, $\chi(G) \leq \Delta(G) + 1$

Theorem 1.3: For any connected graph G, $\gamma_2(G) + \chi(G) \leq 2n$ and the equality holds if and only if $G \cong K_2$.

Main Results

Theorem 2.1: For any connected graph G, $\gamma_2(G) + \chi(G) = 2n - 5$ if and only if $G \cong K_2(4,0,0)$, $K_2(3,1,0)$, $K_2(2,2,0)$, $S(K_1,3)$, $K_4(1,1,1,0)$, $K_4(3,0,0,0)$, $K_4(2,1,0,0)$, $K_5(P_3)$, $K_3(P_3,P_2,0)$, $K_4(P_3)$, $K_5(2,0,0,0,0)$, $K_5(2,2,0,0,0)$, $K_6(P_2)$, $K_7$ or one of the following graphs in the figure 2.1.
Proof: If \( G \) is anyone of the graph given in the figure, then clearly \( \gamma_2(G) + \chi(G) = 2n - 5 \). Conversely assume that \( \gamma_2(G) + \chi(G) = 2n - 5 \). This is possible only if \( \gamma_2 = n, \chi = n - 5 \) (or) \( \gamma_2 = n - 1, \chi = n - 4 \) (or) \( \gamma_2 = n - 2, \chi = n - 3 \) (or) \( \gamma_2 = n - 3, \chi = n - 2 \) (or) \( \gamma_2 = n - 4, \chi = n - 1 \) (or) \( \gamma_2 = n - 5, \chi = n \).

Case (i) Let \( \gamma_2 = n \) and \( \chi = n - 5 \), since \( \chi = n - 5 \), \( G \) contains a clique \( K \) on \( n - 5 \) vertices. Let \( S = \{x_1, x_2, x_3, x_4, x_5\} \in V - S \). Then \( <S> = K_5, \bar{K}_5, P_5, K_4 \cup K_1, P_3 \cup K_2, K_4, P_2 \cup K_3, K_2, K_3 \cup K_2 \). In all the above cases, it can be verified that no new graph exists.

Case (ii) Let \( \gamma_2 = n - 1 \) and \( \chi = n - 4 \), since \( \chi = n - 4 \), \( G \) contains a clique \( K \) on \( n - 4 \) vertices. Let \( S = \{x_1, x_2, x_3, x_4\} \). Then \( <S> = K_4, \bar{K}_4, P_4, K_3 \cup K_1, K_3 \cup K_2, K_2 \cup K_3 \). If \( <S> = K_4 \), then no graph exists.

Subcase (a) Let \( <S> = \bar{K}_4 \), since \( G \) is connected. One of the vertices of \( K_{n-4} \) say \( u_i \) is adjacent to all the vertices of \( S \) (or) three vertices of \( S \) are adjacent to the vertex \( u_i \) and the fourth one is adjacent to \( u_i \) for \( i \neq j \) (or) two vertices of \( S \) are adjacent to the vertex \( u_i \) and the remaining vertices of \( S \) are adjacent to \( u_j \) (or) two vertices of \( S \) are adjacent to the vertex \( u_i \) and in the remaining vertices one is adjacent to \( u_j \) and another one is adjacent to \( u_k \) for \( i \neq j \neq k \) (or) all the vertices of \( S \) are adjacent to the distinct vertices of \( K_{n-4} \). Then in all the cases, \( \gamma_2 = 6 \), since \( \gamma_2 = n - 1 \). So that \( n = 7 \). Since \( \chi = n - 4 = 3 \). Hence \( K = K_3 \). Let \( u_1, u_2, u_3 \) be the vertices of \( K_3 \). If all the vertices of \( S \) are adjacent to \( u_1 \) then \( \gamma_2 = 6 \) and \( d(x_1) = d(x_2) = d(x_3) = d(x_4) = 1 \). Hence \( G \cong G_1 \). If three vertices of \( S \) are adjacent to \( u_1 \) and the fourth one is adjacent to \( u_2 \) then \( \gamma_2 = 6 \) and \( d(x_1) = d(x_2) = d(x_3) = d(x_4) = 1 \). Hence \( G \cong G_3 \). If two vertices of \( S \) are adjacent to \( u_1 \) and the remaining two vertices are adjacent to \( u_2 \), then \( \gamma_2 = 6 \) and \( d(x_1) = d(x_2) = d(x_3) = d(x_4) = 1 \). Hence \( G \cong G_3 \). If two vertices of \( S \) are adjacent to \( u_1 \) and the remaining two vertices are adjacent to \( u_2 \), then \( \gamma_2 = 6 \) and \( d(x_1) = d(x_2) = d(x_3) = d(x_4) = 1 \). Hence \( G \cong G_3 \).

Subcase (b) Let \( <S> = P_4 = (x_1, x_2, x_3, x_4) \), since \( G \) is connected. There exists a vertex say \( u_i \) in \( K_{n-4} \) is adjacent to \( x_1 \) (or equivalently \( x_4 \)) (or) \( x_2 \) (or equivalently \( x_3 \)). Let \( u_i \) be

![Figure 2.1](image-url)
adjacent to \(x_1\). Then \(\{x_2, x_4, u_i, u_j\}\) for \(i \neq j\) is a \(\gamma_2\) set. So that \(\gamma_2 = 4\), since \(\gamma_2 = n-1\) implies that \(n=5\). Since \(\chi = n - 4 = 1\) which is a contradiction. Since \(G\) is totally disconnected. Hence no graph exists. Let \(u_i\) be adjacent to \(x_2\). Then \(\{x_1, x_2, x_4, u_i, u_j\}\) for \(i \neq j\) is a \(\gamma_2\) set. So that \(\gamma_2 = 5\), since \(\gamma_2 = n-1\) implies that \(n=6\). Since \(\chi = n - 4 = 2\). Hence \(K = K_2 = uv\).

If \(u\) is adjacent to \(x_1\) then \(\gamma_2 = 4\) which is a contradiction. Hence no graph exists.

Subcase(c) Let \(<S> = K_{4,3}\). Let the vertex \(x_1\) be adjacent to \(x_2, x_3, x_4\). Since \(G\) is connected, there exists a vertex \(u_i\) in \(K_n-4\) which is adjacent to \(x_1\) or any one of \(\{x_2, x_3, x_4\}\). Let \(u_i\) be adjacent to \(x_1\), then \(\{x_2, x_3, x_4, u_i, u_j\}\) for \(i \neq j\) is a \(\gamma_2\) set so that \(\gamma_2 = 5\), since \(\gamma_2 = n-1\) implies that \(n=6\). Since \(\chi = n - 4 = 2\). Hence \(K = K_4\). Let \(u_1, u_2, u_3\) be the vertices of \(K_4\). Without loss of generality, \(u_1\) is adjacent to \(x\) and \(u_2\) is adjacent to \(y\) and \(u_3\) is adjacent to \(z\) since \(\gamma_2 = 4\). If \(d(x) = d(y) = d(z) = 1\). Then \(G \cong G_4\).

Subcase(d) Let \(<S> = K_2 \cup K_1\). Let \(xy\) be the edge in \(K_2 \cup K_1\), since \(G\) is connected. There exists a \(u_i\) in \(K_{n-3}\) is adjacent to \(x\). If \(z\) is adjacent to same \(u_i\) or \(z\) is adjacent to \(u_j\) for \(i \neq j\) in \(K_{n-3}\). Then \(\{y, z, u_i, u_j\}\) for \(i \neq j\) is a \(\gamma_2\) set. So that \(n=6\). Hence \(K = K_3\). Let \(u_1, u_2, u_3\) be the vertices of \(K_3\). If \(u_1\) is adjacent to \(x\), then \(\gamma_2 = 4\). Hence \(G \cong G_3\). If \(u_1\) is adjacent to \(y\), then \(\gamma_2 = 4\). Hence \(G \cong G_2\).

Case (iv) Let \(\gamma_2 = n-3\) and \(\chi = n-2\), since \(\chi = n-2\), \(G\) contains a clique \(K\) on \(n-2\) vertices. Let \(S = \{x, y\} \in V-S\). \(<S> = K_2\) or \(K_2\).
Characterization of Two Domination Number

Subcase (a) Let $\langle S \rangle = K_2$, since $G$ is connected. There exists a vertex $u_i$ in $K_{n-2}$ is adjacent to $x$. Then $\{y, u_i, u_j\}$ is a $\gamma_2$ set. So that $n=6$. Hence $K=K_4$. Let $u_1, u_2, u_3, u_4$ be the vertices of $K_4$. Let $u_i$ be adjacent to $x$, if $d(x) = 2$ and $d(y) = 1$, then $G \cong K_4(P_3)$.

Subcase (b) Let $\langle S \rangle = \overline{K}_2$, since $G$ is connected. There exists a vertex $u_i$ in $K_{n-2}$ is adjacent to $x$ and $y$ (or) $x$ is adjacent to $u_i$ and $y$ is adjacent to $u_j$. In both the cases, $\{x, y, u_i, u_j\}$ is a $\gamma_2$ set. So that $n=7$, hence $K=K_5$. Let $u_1, u_2, u_3, u_4, u_5$ be the vertices of $K_5$. Let $x$ and $y$ be adjacent to $u_1$ then $\gamma_2 = 4$. If $d(x) = d(y) = 1$, then $G \cong K_5(2,0,0,0,0)$. Let $u_1$ be adjacent to $x$ and $u_2$ be adjacent to $y$, then $\gamma_2 = 4$. If $d(x) = d(y) = 1$, then $G \cong K_5(1,1,0,0,0)$. 

Case (v) Let $\gamma_2 = n-4$ and $\chi = n-1$, since $\chi = n-1$, $G$ contains a clique $K$ on $n-1$ vertices or does not contain a clique on $n-1$ vertices. There exists a vertex $u_i$ in $K_{n-1}$ is adjacent to $x$. Then $\{x, u_i, u_j\}$ for $i \neq j$ is a $\gamma_2$ set. So that $n = 7$. Hence $K=K_6$. Let $u_1, u_2, u_3, u_4, u_5, u_6$ be the vertices of $K_6$. Let $u_1$ is adjacent to $x$, so that $\gamma_2 = 3$. If $d(x) = 1$, then $G \cong K_6(P_2)$.

Case (vi) Let $\gamma_2 = n-5$ and $\chi = n$, since $\chi = n$. Then $G = K_n$, but for $K_n$, $\gamma_2 = 2$ implies that $n=7$. Hence $G \cong K_7$.

Theorem 2.2: For any connected $G$, $\gamma_2(G) + \chi(G) = 2n-6$ if and if only $G \cong K_3(5,0,0)$, $S(K_1,6)$, $K_4(4,0,0,0)$, $K_4(3,1,0,0)$, $K_4(2,2,0,0)$, $P_6$, $K_5(1,3,0)$, $K_5(3,0,0,0,0)$, $K_5(2,1,0,0,0)$, $K_5(1,1,1,0,0)$, $K_4(P_4)$, $K_5(P_3)$, $K_6(2,0,0,0,0,0)$, $K_6(1,1,0,0,0,0)$, $K_7(1,0,0,0,0,0)$, $K_8$ or any one of the following graphs in the figure 2.2.
Proof: If \( G \) is anyone of the graph given in the figure, then clearly \( \gamma_2(G) + \chi(G) = 2n-6 \), conversely assume that \( \gamma_2(G) + \chi(G) = 2n-6 \). This is possible only if \( \gamma_2(G) = n \), \( \chi(G) = n-6 \)(or)\( \gamma_2(G) = n-1 \), \( \chi(G) = n-5 \)(or)\( \gamma_2(G) = n-2 \), \( \chi(G) = n-4 \)(or)\( \gamma_2(G) = n-3 \), \( \chi(G) = n-3 \)(or)\( \gamma_2(G) = n-4 \), \( \chi(G) = n-2 \)(or)\( \gamma_2(G) = n-5 \), \( \chi(G) = n-1 \)(or)\( \gamma_2(G) = n-6 \), \( \chi(G) = n \).

Case (i) Let \( \gamma_2(G) = n \), \( \chi(G) = n-6 \), since \( \chi = n-6 \), \( G \) contains a clique \( K \) on \( n-6 \) vertices. Let \( S = \{x_1, x_2, x_3, x_4, x_5, x_6\} \in V-S \). Then \( \langle S \rangle = K_6, K_6, P_6, K_3 \cup K_3, K_2 \cup K_2, P_3 \cup K_3, K_2 \cup K_4, K_1, K_3, K_3, K_2, 4 \).

If \( \langle S \rangle = K_6 \), then no graph exists.

Subcase(a) Let \( \langle S \rangle = K_6 \). Since \( G \) is connected, one of the vertices of \( K_{n-6} \) say \( u_i \) is adjacent to all the vertices of \( S \). Then \( \gamma_2 = 8 \). Hence \( n = 8 \). So that \( K = K_2 = uv \). In all the cases, no new graph exists.

For all the remaining cases, no new graph exists.

Case (ii) Let \( \gamma_2(G) = n-1 \), \( \chi(G) = n-5 \). Since \( \chi = n-5 \), \( G \) contains a clique \( K \) on \( n-5 \) vertices or does not contain a clique \( K \) on \( n-5 \) vertices. Let \( S = \{x_1, x_2, x_3, x_4, x_5\} \in V-S \). Then \( \langle S \rangle = K_5, K_5, P_5, P_3 \cup K_2, K_3 \cup K_4, K_4 \cup K_1, P_4 \cup K_1, K_1, K_2, 3 \).

If \( \langle S \rangle = K_5 \), then no graph exists.
Subcase(a) Let \( \langle S \rangle = \overline{K}_5 \). Since \( G \) is connected. There exists a vertex \( u_i \) in \( K_{n-5} \) is adjacent to all the vertices of \( S \) (or) four vertices of \( S \) (or) three vertices of \( S \) (or) two vertices on \( S \) (or) one vertex of \( S \). Then in all the cases, \( \{ x_1, x_2, x_3, x_4, x_5, u_i, u_j \} \) for \( i \neq j \) is a \( \gamma \) set. So that \( \gamma_2 = 7 \). Hence \( n = 8 \). So that \( K = K_4 \). Let \( u_1, u_2, u_3, u_4 \) be the vertices of \( K_3 \). If \( u_1 \) is adjacent to all the vertices of \( S \) and if \( d(x_1) = d(x_2) = d(x_3) = d(x_4) = d(x_5) = 1 \), then \( G \cong K_4(5,0,0) \). In all other cases, no new graph exists.

Subcase(b) Let \( \langle S \rangle = P_4 \cup K_1 \). Since \( G \) is connected. Let \( P_4 = (x_1, x_2, x_3, x_4) \) and \( x_5 \) be the vertex of \( K_1 \). There exists a \( u_i \) in \( K_{n-5} \) is adjacent to \( x_1 \) and \( x_5 \) (or) \( u_i \) is adjacent to \( x_1 \) and \( u_j \) for \( i \neq j \) is adjacent to \( x_5 \). Then in both the cases \( \{ x_2, x_4, x_5, u_i, u_j \} \) is a \( \gamma \) set. So that \( n = 6 \), since \( \chi = n-5 = 1 \), for which \( G \) is totally disconnected. Hence no graph exists. If \( u_i \) is a adjacent to \( x_2 \) (or equivalently \( x_3 \)) and \( x_5 \). Then \( \{ x_1, x_3, x_4, x_5, u_i, u_j \} \) is a \( \gamma \) set. So that \( n = 7 \). Hence \( K = K_2 = uv \). If \( u \) is adjacent to \( x_2 \) and \( x_5 \), then \( G \cong G_1 \). In all other cases, no new graph exists.

Subcase(c) Let \( \langle S \rangle = K_1, 4 \). Since \( G \) is connected. Let the vertex \( x_1 \) be adjacent to \( x_2, x_3, x_4, x_5 \). There exists a vertex \( u_i \) in \( K_{n-5} \) which is adjacent to \( x_1 \) or any one of \( \{ x_2, x_3, x_4, x_5 \} \). Then \( \{ x_2, x_3, x_4, x_5, u_i, u_j \} \) for \( i \neq j \) is a \( \gamma \) set. So that \( n = 7 \). Hence \( K = K_2 = uv \).

For all the remaining cases, no new graph exists.

Case (iii) Let \( \gamma_2 (G) = n-2 \), \( \chi (G) = n-4 \), since \( \chi = n-4 \), \( G \) contains a clique \( K \) on \( (n-4) \) vertices or does not contain a clique \( K \) on \( n-4 \) vertices. Let \( S = \{ x_1, x_2, x_3, x_4 \} \in V-S \).

Then \( \langle S \rangle = K_4, K_4, P_4, K_3 \cup K_1, K_1, K_2 \cup K_2, P_3 \cup K_1 \).

If \( \langle S \rangle = K_4 \), then no graph exists.

Subcase(a) Let \( \langle S \rangle = \overline{K}_4 \). Since \( G \) is connected, one of the vertices of \( K_{n-4} \) says \( u_i \) is adjacent to all the vertices of \( S \) (or) three vertices of \( S \) (or) two vertices of \( S \) (or) one vertex of \( S \). Then in all the cases, \( \{ x_1, x_2, x_3, x_4, u_i, u_j \} \) for \( i \neq j \) is a \( \gamma \) set. So that \( n = 8 \). Hence \( K = K_4 \). Let \( u_1, u_2, u_3, u_4 \) be the vertices of \( K_4 \). If all the vertices of \( S \) are adjacent to \( u_1 \), then \( \gamma_2 = 6 \) and \( d(x_1) = d(x_2) = d(x_3) = d(x_4) = 1 \). Hence \( G \cong K_4(4, 0, 0, 0) \). If three vertices of \( S \) are adjacent to \( u_1 \) and the fourth one is adjacent to \( u_2 \) and \( d(x_1) = d(x_2) = d(x_3) = d(x_4) = 1 \), then \( \gamma_2 = 6 \). Hence \( G \cong K_4(3, 1, 0, 0) \). If two vertices of \( S \) are adjacent to \( u_1 \) and the remaining two vertices are adjacent to \( u_2 \) and \( d(x_1) = d(x_2) = d(x_3) = d(x_4) = 1 \), then \( \gamma_2 = 6 \). Hence \( G \cong K_4(2, 2, 0, 0) \).

Subcase(b) Let \( \langle S \rangle = P_4 = (x_1, x_2, x_3, x_4) \). Since \( G \) is connected, there exists a vertex say \( u_i \) in \( K_{n-4} \) is adjacent to \( x_1 \) (or equivalently \( x_4 \)) (or) \( x_2 \) (or equivalently \( x_3 \)). Let \( u_i \) be adjacent to \( x_1 \), then \( \{ x_2, x_3, u_i, u_j \} \) for \( i \neq j \) is a \( \gamma \) set. So that \( n = 6 \). Hence \( K = K_2 = uv \).

If \( u_1 \) is adjacent to \( u \), then \( \gamma_2 = 4 \). Hence \( G \cong P_6 \). Let \( u_i \) be adjacent to \( x_2 \), then \( \{ x_1, x_3, x_4, u_i, u_j \} \) for \( i \neq j \) is a \( \gamma \) set. So that \( n = 7 \). Hence \( K = K_3 \). Let \( u_1, u_2, u_3 \) be the vertices of \( K_3 \). If \( x_1 \) of \( S \) is adjacent to \( u_1 \), then \( \gamma_2 = 4 \), which is a contradiction. Hence no graph exists.

Subcase(c) Let \( \langle S \rangle = K_1, 3 \). Let the vertex \( x_1 \) be adjacent to \( x_2, x_3, x_4 \). Since \( G \) is connected, there exists a vertex \( u_i \) in \( K_{n-4} \) which is adjacent to \( x_1 \) or any one of \( (x_2, x_3, x_4) \). Then in both the cases, \( \{ x_2, x_3, x_4, u_i, u_j \} \) for \( i \neq j \) is a \( \gamma \) set. So that \( n = 7 \).
Hence $K=K_3$. Let $u_1,u_2,u_3$ be the vertices of $K_3$. If $u_1$ is adjacent to $x_1$ then $\gamma_2 = 5$. Hence $G \cong G_3$. If $u_4$ is adjacent to $x_4$ then $\gamma_2 = 5$. Hence $G \cong G_4$.

Subcase (d) Let $<S> = P_3 \cup K_1$. Let $P_3 = (x_2,x_3,x_4)$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $x_1$. Again since $G$ is connected, we consider the following two situations: (i) The vertex $u_i$ is adjacent to $x_2$ (or equivalently $x_4$) or $x_3$. (ii) There exists a vertex $u_i$ for $i \neq j$ in $K_{n-4}$ such that $u_i$ is adjacent to $x_2$ (or equivalently $x_4$) or $x_3$. Then in all the cases, $\{x_1,x_2,x_4,u_i,u_j\}$ for $i \neq j$ is a $\gamma_2$ set. So that $n=7$. Hence $K=K_3$. Let $u_1,u_2,u_3$ be the vertices of $K_3$. Let $u_1$ be adjacent to $x_1$ and $x_2$ (or equivalently $x_3$) and let $u_2$ be adjacent to $x_2$ (or equivalently $x_1$) and let $u_1$ be adjacent to $x_1$ and $x_3$. Then in all the cases, $\gamma_2 = 5$. Hence $G \cong G_5,G_6,G_7,K_3(1,3,0)$.

Subcase (e) Let $<S> = K_2 \cup K_2$. Let $x_1x_2$ and $x_3x_4$ be the edges in $<S>$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ which is adjacent to $x_1$ and $x_3$ in $S$ (or) $u_i$ is adjacent to $x_1$ and $u_j$ is adjacent to $x_3$ for $i \neq j$ in $K_{n-4}$. Then in both the cases, $\{x_2,x_4,u_i,u_j\}$ for $i \neq j$ is a $\gamma_2$ set, hence $\gamma_2 = 4$, so that $n=6$. Hence $K=K_2 = uv$. If $u$ is adjacent to $x_1$ and $x_3$ then $\gamma_2 = 4$. Hence $G \cong G_8$. If $u$ is adjacent to $x_1$ and $v$ is adjacent to $x_3$, then $\gamma_2 = 4$. Hence $G \cong G_9$.

Subcase (f) Let $<S> = K_3 \cup K_1$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-4}$ such that $x_1$ and $x_4$ (or) $u_i$ is adjacent to $x_1$ and $u_j$ is adjacent to $x_4$ for $i \neq j$ in $K_{n-4}$. Then in both the cases, $\{x_2,x_3,x_4,u_i,u_j\}$ for $i \neq j$ is a $\gamma_2$ set of $G$. So that $\gamma_2 = 5$. Hence $n=7$. Since $\chi = n-4 = 3$. Hence $K=K_3$. Let $u_1,u_2,u_3$ be the vertices of $K_3$. If $u_1$ is adjacent to $x_1$ and $x_3$ then $\gamma_2 = 5$. Hence $G \cong G_{10}$. If $u_1$ is adjacent to $x_1$ and $u_2$ is adjacent to $x_3$, then $\gamma_2 = 4$, which is a contradiction. Hence no graph exists.

Case (iv) Let $\gamma_2 = n-3$ & $\chi = n-3$. Since $G$ is connected. Since $\chi = n-3$, $G$ contains a clique $K$ on $(n-3)$ vertices or does not contain a clique $K$ on $(n-3)$ vertices. Let $S = \{x_1,x_2,x_3\} \in V-S$. Then $<S> = K_3,K_3, P_3 , K_2 \cup K_1 , P_2 \cup K_1$.

Subcase (a) Let $<S> = K_3$. Since $G$ is connected, let $x_1$ be adjacent to $u_i$ for some $i$ in $K_{n-3}$. Then $\{x_1,x_2,x_3,u_i,u_j\}$ for $i \neq j$ is a $\gamma_2$ set . So that $\gamma_2 = 4$ implies that $n=7$. Hence $K=K_4$. Let $u_1,u_2,u_3,u_4$ be the vertices of $K_4$. If $u_1$ is adjacent to $x_1$ then $\gamma_2 = 4$. Hence $G \cong G_{11}$. If $u_1$ is adjacent to $x_2$ and $u_2$ is adjacent to $x_3$ and $u_3$ is adjacent to $x_4$, then $\gamma_2 = 4$, which is a contradiction. Hence no graph exists.
There exists an $u_i$ in $K_{n-3}$ is adjacent to $x_1$ and $x_3$ (or) $u_i$ is adjacent to $x_1$ and $u_j$ for $i \neq j$ is adjacent to $x_3$. Then in both the cases, $\{x_2,x_3,u_i,u_j\}$ for $i \neq j$ is a $\gamma_2$ set. So that $\gamma_2=4$, implies that $n=7$. Hence $K=K_7$. Let $u_1,u_2,u_3,u_4$ be the vertices of $K_4$. If $u_1$ is adjacent to $x_1$ and $x_3$, then $\gamma_2=4$ and $d(x_1)=2$ and $d(x_2)=d(x_3)=1$. Hence $G \cong G_{12}$. If $u_1$ is adjacent to $x_1$ and $u_2$ is adjacent to $x_3$, then $\gamma_2=4$. Hence $G \cong G_{13}$.

Case (v): Let $\gamma_2=n-4$ and $\chi=n-2$, since $\chi=n-2$, $G$ contains a clique $K$ on $(n-2)$ vertices or does not contain a clique $K$ on $n-2$ vertices. Let $S=\{x_1,x_2\} \in V-S$. Then $<S>=K_2$ or $K_2$.

Subcase(a) Let $<S>=K_2$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-2}$ is adjacent to $x_1$. Then $\{x_2,u_i,u_j\}$ for $i \neq j$ is a $\gamma_2$ set. So that $n=7$. Hence $K=K_7$. Let $u_1,u_2,u_3,u_4,u_5$ be the vertices of $K_5$. If $u_i$ is adjacent to $x_1$, then $\gamma_2=3$ and $d(x_1)=2$, $d(x_2)=1$. Hence $G \cong K_5(P_3)$.

Subcase(b) Let $<S>=K_2$. Since $G$ is connected, there exists a vertex $u_i$ in $K_{n-2}$ is adjacent to $x_1$ and $x_2$ (or) If $u_i$ is adjacent to $x_1$ and $u_j$ for $i \neq j$ is adjacent to $x_2$. Then in both the cases, $\{x_1,x_2,u_i,u_j\}$ for $i \neq j$ is a $\gamma_2$ set. So that $n=8$. Hence $K=K_8$. Let $u_1,u_2,u_3,u_4,u_5,u_6$ be the vertices of $K_6$. If $x_1$ and $x_2$ be adjacent to $u_1$, then $\gamma_2=4$ and $d(x_1)=d(x_2)=1$. Hence $G \cong K_8(2,0,0,0,0,0)$. If $x_1$ is adjacent to $u_1$ and $x_2$ is adjacent to $u_2$, then $\gamma_2=4$ and $d(x_1)=d(x_2)=1$. Hence $G \cong K_8(1,1,0,0,0,0)$.

Case (vi) Let $\gamma_2=n-5$ and $\chi=n-1$, since $\chi=n-1$, $G$ contains a clique $K$ on $(n-1)$ vertices or does not contain a clique $K$ on $n-1$ vertices. There exists a vertex $u_i$ in $K_{n-1}$ is adjacent to $x$. Then $\{x,u_i,u_j\}$ for $i \neq j$ is a $\gamma_2$ set. So that $n=8$. Hence $K=K_8$. Let $u_1,u_2,u_3,u_4,u_5,u_6,u_7$ be the vertices of $K_7$. If $u_i$ is adjacent to $x$ then $\gamma_2=3$. Hence $G \cong K_8(1,0,0,0,0,0,0)$.

Case (vii) Let $\gamma_2=n-6$ and $\chi=n$, since $\chi=n$ then $G=K_n$, $G$ must be complete. But for $K_n$, $\gamma_2=2$. So that $n=8$. Hence $G \cong K_8$.

References


