A Note on Characterization of Intuitionistic Fuzzy Ideals in $\Gamma$-Near-Rings

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Abstract

In this paper, we study some properties of intuitionistic fuzzy ideals of a $\Gamma$–near-ring and prove some results on these.

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Introduction

The notion of a fuzzy set was introduced by L.A.Zadeh[10], and since then this concept have been applied to various algebraic structures. The idea of “Intuitionistic Fuzzy Set” was first published by K.T.Atanassov[1] as a generalization of the notion of fuzzy set. $\Gamma$- near-rings were defined by Bh.Satyanarayana [9] and G.L.Booth [2, 3] studied the ideal theory in $\Gamma$-near-rings. W. Liu[7] introduced fuzzy ideals and it has been studied by several authors. The notion of fuzzy ideals and its properties were applied to semi groups, BCK- algebras and semi rings. Y.B. Jun [5, 6] introduced the notion of fuzzy left (resp.right) ideals. In this paper, we introduce the notion of intuitionistic fuzzy ideals in $\Gamma$–near- rings and study some of its properties.

Preliminaries

In this section we include some elementary aspects that are necessary for this paper.
Definition 2.1 A non–empty set $R$ with two binary operations “+” (addition) and “.” (multiplication) is called a near-ring if it satisfies the following axioms:

i. $(R, +)$ is a group,

ii. $(R, \cdot)$ is a semigroup,

iii. $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$. It is a right near-ring because it satisfies the right distributive law.

Definition 2.2 A $\Gamma$-near-ring is a triple $(M, +, \Gamma)$ where

i. $(M, +)$ is a group,

ii. is a nonempty set of binary operators on $M$ such that for each $\alpha \in \Gamma$, $(M, +, \alpha)$ is a near–ring,

iii. $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.3 A subset $A$ of a $\Gamma$-near-ring $M$ is called a left (resp. right) ideal of $M$ if

i. $(A, +)$ is a normal divisor of $(M, +)$,

ii. $u\alpha(x + v) - u\alpha v \in A$ (resp. $x\alpha u \in A$) for all $x \in A$, $\alpha \in \Gamma$ and $u, v \in M$.

Definition 2.4 A fuzzy set $\mu$ in a $\Gamma$-near-ring $M$ is called a fuzzy left (resp. right) ideal of $M$ if

i. $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

ii. $\mu(y + x-y) \geq \mu(x)$, for all $x, y \in M$.

iii. $\mu(u\alpha(x + v) - u\alpha v) \geq \mu(x)$ (resp. $\mu(x\alpha u) \geq \mu(x)$) for all $x, u, v \in M$ and $\alpha \in \Gamma$.

Definition 2.5 [1] Let $X$ be a nonempty fixed set. An intuitionistic fuzzy set (IFS) $A$ in $X$ is an object having the form $A = \{ < x, \mu_A(x), \nu_A(x) / x \in X \}$, where the functions $\mu_A : X \rightarrow [0, 1]$ and $\nu_A: X \rightarrow [0, 1]$ denote the degree of membership and degree of non membership of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Notation. For the sake of simplicity, we shall use the symbol $A = < \mu_A, \nu_A >$ for the IFS $A = \{ < x, \mu_A(x), \nu_A(x) / x \in X \}$.

Definition 2.6 [1]. Let $X$ be a non-empty set and let $A = < \mu_A, \nu_A >$ and $B = < \mu_B, \nu_B >$ be IFSs in $X$. Then

1. $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

3. $A^c = < \nu_A, \mu_A >$.

4. $A \cap B = ( \mu_A \land \mu_B , \nu_A \lor \nu_B )$.

5. $A \cup B = ( \mu_A \lor \mu_B , \nu_A \land \nu_B )$.

6. $\Box A = ( \mu_A , 1-\mu_A )$, $\Diamond A = (1-\nu_A , \nu_A )$.

Definition 2.7. Let $\mu$ and $\nu$ be two fuzzy sets in a $\Gamma$-near-ring For $s, t \in [0, 1]$ the set $U(\mu, s) = \{ x \in \mu(x) \geq s \}$ is called upper level of $\mu$. The set $L(\nu, t) = \{ x \in \nu(x) \leq t \}$ is
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called lower level of \( v \).

**Definition 2.8.** Let \( A \) be an IFS in a \( \Gamma \)– ring \( M \). For each pair \( < t, s > \in [0, 1] \) with \( t + s \leq 1 \), the set \( A_{<t, s>} = \{ x \in X / \mu_A(x) \geq t \text{ and } \nu_A(x) \leq s \} \) is called a \( < t, s > \)– level subset of \( A \).

**Definition 2.9.** Let \( A = < \mu_A, \nu_A > \) be an intuitionistic fuzzy set in \( M \) and let \( t \in [0, 1] \). Then the sets \( U(\mu_A; t) = \{ x \in M : \mu_A(x) \geq t \} \) and \( L(\nu_A; t) = \{ x \in M : \nu_A(x) \leq t \} \) are called upper level set and lower level set of \( A \) respectively.

**Intuitionistic fuzzy ideals**

In what follows, let \( M \) denote a \( \Gamma \)-near-ring unless otherwise specified.

**Definition 3.1.** An IFS \( A = < \mu_A, \nu_A > \) in \( M \) is called an intuitionistic fuzzy left (resp. right) ideal of a \( \Gamma \)-near-ring \( M \) if

1. \( \mu_A(x - y) \geq \{ \mu_A(x) \wedge \mu_A(y) \} \),
2. \( \mu_A(y + x - y) \geq \mu_A(x) \),
3. \( \mu_A(ua(x + v) - ua\nu) \geq \mu_A(x) \) (resp. \( \mu_A(xau) \geq \mu_A(x) \)),
4. \( \nu_A(x - y) \leq \{ \nu_A(x) \vee \nu_A(y) \} \),
5. \( \nu_A(y + x - y) \leq \nu_A(x) \),
6. \( \nu_A(ua(x + v) - ua\nu) \leq \nu_A(x) \) (resp. \( \nu_A(xau) \leq \nu_A(x) \)),

for all \( x, y, u, v \in M \) and \( \alpha \in \Gamma \).

**Example 3.2.** Let \( R \) be the set of all integers then \( R \) is a ring. Take \( M = \Gamma = R \). Let \( a, b \in M \), \( \alpha \in \Gamma \), suppose \( a \alpha b \) is the product of \( a, \alpha, b \in R \).

Then \( M \) is a \( \Gamma \)-near-ring.

Define an IFS \( A = < \mu_A, \nu_A > \) in \( R \) as follows.

\[ \mu_A(0) = 1 \text{ and } \mu_A(\pm 1) = \mu_A(\pm 2) = \mu_A(\pm 3) = \ldots = t \quad \text{ and } \quad \nu_A(0) = 0 \text{ and } \nu_A(\pm 1) = \nu_A(\pm 2) = \nu_A(\pm 3) = \ldots = s, \text{ where } t \in [0, 1], s \in [0, 1] \text{ and } t + s \leq 1. \]

By routine calculations, clearly \( A \) is an intuitionistic fuzzy ideal of a \( \Gamma \)-near-ring \( R \).

**Theorem 3.3.** \( A \) is an ideal of a \( \Gamma \)-near-ring \( M \) if and only if \( \tilde{A} = < \mu_{\tilde{A}}, \nu_{\tilde{A}} > \) where

\[
\mu_{\tilde{A}}(x) = \begin{cases} 1 & x \in A \\ 0 & Otherwise \end{cases} \quad \nu_{\tilde{A}}(x) = \begin{cases} 0 & x \in A \\ 1 & Otherwise \end{cases}
\]

is an intuitionistic fuzzy left (resp.right) ideal of \( M \).

**Proof (\( \Rightarrow \))**: Let \( A \) be a left (resp.right) ideal of \( M \).

Let \( x, y, u, v \in M \) and \( \alpha \in \Gamma \).

If \( x, y \in A \), then \( x - y \in A \), \( y + x - y \in A \) and \( (ua(x + v) - ua\nu) \in A \). Therefore
\[
\mu_\lambda(x - y) = 1 \geq \{ \mu_\lambda(x) \land \mu_\lambda(y) \}, \mu_\lambda(y + x - y) = 1 \geq \mu_\lambda(x) \land \mu_\lambda(y)
\]
and
\[
\mu_\lambda(u\alpha(x + v) - u\alpha v) = 1 = \mu_\lambda(x) \land \mu_\lambda(y)
\]

If \(x \not\in \Lambda\) or \(y \not\in \Lambda\) then \(\mu_\lambda(x) = 0\) or \(\mu_\lambda(y) = 0\).

Thus we have
\[
\nu_\lambda(x - y) \leq \{ \nu_\lambda(x) \lor \nu_\lambda(y) \}, \nu_\lambda(y + x - y) \leq \nu_\lambda(x)
\]
and
\[
\nu_\lambda(u\alpha(x + v) - u\alpha v) \leq \nu_\lambda(x)
\]

Hence \(\Lambda\) is an intuitionistic fuzzy left (resp. right) ideal of \(\mathcal{M}\).

\((\Leftarrow)\): Let \(\Lambda\) be an intuitionistic fuzzy left (resp. right) ideal of \(\mathcal{M}\).
Let \(x, y \in \mathcal{M}\) and \(\alpha \in \Gamma\).

If \(x, y, u, v \in \Lambda\), then
\[
\mu_\lambda(x - y) \geq \{ \mu_\lambda(x) \land \mu_\lambda(y) \} = 1
\]
\[
\nu_\lambda(x - y) \leq \{ \nu_\lambda(x) \lor \nu_\lambda(y) \} = 0
\]

So \(x - y \in \Lambda\).
\[
\mu_\lambda(y + x - y) = 1
\]
\[
\nu_\lambda(y + x - y) \leq \nu_\lambda(x) = 0
\]

So \((y + x - y) \in \Lambda\).

Also
\[
\mu_\lambda(u\alpha(x + v) - u\alpha v) \geq \mu_\lambda(x) = 1 \ (\textrm{resp.} \ \mu_\lambda(x\alpha u) = \mu_\lambda(x) = 1)
\]
\[
\nu_\lambda(u\alpha(x + v) - u\alpha v) \leq \nu_\lambda(x) = 0 \ (\textrm{resp.} \ \nu_\lambda(x\alpha u) = \nu_\lambda(x) = 0)
\]

So \((u\alpha(x + v) - u\alpha v) \in \Lambda\).
Hence \(\Lambda\) is a left (resp. right) ideal of \(\mathcal{M}\).

**Theorem 3.4.** Let \(\Lambda\) be an intuitionistic fuzzy left (resp. right) ideal of \(\mathcal{M}\) and \(t \in [0,1]\), then

I. \(U(\mu_\Lambda; t)\) is either empty or an ideal of \(\mathcal{M}\).

II. \(L(\nu_\Lambda; t)\) is either empty or an ideal of \(\mathcal{M}\).

**Proof.** (i) Let \(x, y \in U(\mu_\Lambda; t)\).

Then \(\mu_\Lambda(x - y) \geq \{ \mu_\Lambda(x) \land \mu_\Lambda(y) \} \geq t\),

Hence \(x - y \in L(\nu_\Lambda; t)\).
\[
\mu_\Lambda(y + x - y) \geq \mu_\Lambda(x) \geq t
\]

Hence \((y + x - y) \in U(\mu_\Lambda; t)\).
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Let $x \in \mathcal{M}$, $\alpha \in \Gamma$ and $u, v \in U(\mu A; t)$. Then
$$\mu A(u \alpha (x + v) - u \alpha v) \geq \mu A(x) \geq t$$
and so $(u \alpha (x + v) - u \alpha v) \in U(\mu A; t)$. Hence $U(\mu A; t)$ is an ideal of $\mathcal{M}$.

III. Let $x, y \in L(\nu A; t)$.
Then
$$\nu A(x - y) \leq \{\nu A(x) \lor \nu A(y)\} \leq t.$$ 
Hence $x - y \in L(\nu A; t)$. Let $x \in \mathcal{M}$, $\alpha \in \Gamma$ and $u, v \in L(\nu A; t)$.
Then $\nu A(u \alpha (x + v) - u \alpha v) \leq \nu A(x) \leq t$ and so $(u \alpha (x + v) - u \alpha v) \in L(\nu A; t)$. Hence $L(\nu A; t)$ is an ideal of $\mathcal{M}$.

Theorem 3.5. Let $I$ be the left (resp. right) ideal of $\mathcal{M}$. If the intuitionistic fuzzy set $A= <\mu A, \nu A>$ in $\mathcal{M}$ is defined by
$$\mu A(x) = \begin{cases} p & \text{if } x \in I \\ s & \text{Otherwise} \end{cases}$$ 
and $\nu A(x) = \begin{cases} u & \text{if } x \in I \\ v & \text{Otherwise} \end{cases}$
for all $x \in \mathcal{M}$ and $\alpha \in \Gamma$, where $0 \leq s < p$, $0 \leq v < u$ and $p + u \leq 1$, $s + v \leq 1$, then $A$ is an intuitionistic fuzzy left (resp. right) ideal of $\mathcal{M}$ and $U(\mu A; p) = I = L(\nu A; u)$.

Proof. Let $x, y \in \mathcal{M}$ and $\alpha \in \Gamma$.
If at least one of $x$ and $y$ does not belong to $I$, then
$$\mu A(x - y) \geq s = \{\mu A(x) \land \mu A(y)\},$$
$$\nu A(x - y) \leq v = \{\nu A(x) \lor \nu A(y)\}.$$ 
If $x, y \in I$, then
$$x - y \in I$$ 
and so $\mu A(x - y) = p = \{\mu A(x) \land \mu A(y)\}$ and
$$\nu A(x - y) = v = \{\nu A(x) \lor \nu A(y)\}.$$ 
If $x, y \in I$, then
$$\nu A(y + x - y) \leq s = \mu A(x),$$
$$\nu A(y + x - y) \leq v = \nu A(x).$$ 
If $u, v \in I, \ x \in \mathcal{M}$ and $\alpha \in \Gamma$, then $(u \alpha (x + v) - u \alpha v) \in I$,
$$\mu A(u \alpha (x + v) - u \alpha v) = p = \mu A(x) \land \nu A(u \alpha (x + v) - u \alpha v) = u = \nu A(x).$$ 
(resp. $\mu A(x \alpha u) = p = \mu A(x)$ and $\nu A(x \alpha u) = u = \nu A(x)$.
If $y \not\in I$, then $\mu A(u \alpha (x + v) - u \alpha v) = s = \mu A(x), \nu A(u \alpha (x + v) - u \alpha v) = v = \nu A(x)$.
(resp. $\mu A(x \alpha u) = s = \mu A(x)$ and $\nu A(x \alpha u) = v = \nu A(x)$.
Therefore $A$ is an intuitionistic fuzzy left (resp. right) ideal.

Definition 3.6. Let $f$ be a mapping from a $\Gamma$-near-ring $\mathcal{M}$ onto a $\Gamma$-near-ring $\mathcal{N}$. Let $A$
be an intuitionistic fuzzy ideal of M. Now A is said to be f–invariant if f(x) = f(y) implies μ_A(x) = μ_A(y) and ν_A(x) = ν_A(y).

**Definition 3.7** [2]. A function f : M → N, where M and N are Γ–near-rings, is said to be a Γ–homomorphism if f(a + b) = f(a) + f(b), f(abα) = f(a)f(b), for all a, b ∈ M and α ∈ Γ.

**Definition 3.8** Let f : X → Y be a mapping of a Γ–near-ring and A be an intuitionistic fuzzy set of Y. Then the map f^{-1}(A) is the pre-image of A under f, if μ_{f^{-1}(A)}(x) = μ_A(f(x)) and ν_{f^{-1}(A)}(x) = ν_A(f(x)), for all x ∈ X.

**Definition 3.9.** Let f be a mapping from a set X to the set Y. If A = <μ_A, ν_A> and B = <μ_B, ν_B> are intuitionistic fuzzy subsets in X and Y respectively, then the image of A under f is the intuitionistic fuzzy set f(A) = <μ_f(A), ν_f(A)> defined by

\[ μ_{f^{-1}}(B)(x) = \begin{cases} \bigvee_{y∈f^{-1}(A)} μ_B(y) & \text{if } f^{-1}(y) ≠ φ, \\ 0 & \text{Otherwise,} \end{cases} \]

\[ ν_{f^{-1}}(B)(x) = \begin{cases} \bigwedge_{y∈f^{-1}(A)} ν_B(y) & \text{if } f^{-1}(y) ≠ φ, \\ 1 & \text{Otherwise,} \end{cases} \]

for all y ∈ Y.

(b) the pre image of A under f is the intuitionistic fuzzy set f^{-1}(B) = <μ_{f^{-1}}(B), ν_{f^{-1}}(B)> defined by

\[ μ_{f^{-1}}(B)(x) = \begin{cases} \bigvee_{y∈f(x)} μ_B(y) & \text{if } f^{-1}(x) ≠ φ, \\ 0 & \text{Otherwise,} \end{cases} \]

\[ ν_{f^{-1}}(B)(x) = \begin{cases} \bigwedge_{y∈f(x)} ν_B(y) & \text{if } f^{-1}(x) ≠ φ, \\ 1 & \text{Otherwise,} \end{cases} \]

for all x ∈ X, where μ_{f^{-1}}(B)(x) = μ_B(f(x)) and ν_{f^{-1}}(B)(x) = ν_B(f(x)).

**Theorem 3.10.** Let M and N be two Γ–near-rings and θ : M → N be a Γ–epimorphism and let B = <μ_B, ν_B> be an intuitionistic fuzzy set of N. If θ^{-1}(B) = <μ_θ^{-1}, ν_θ^{-1} > is an intuitionistic fuzzy left (resp.right) ideal of M, then B is an intuitionistic fuzzy left (resp.right) ideal of N.

**Proof.** Let x, y, u, v ∈ N and α ∈ Γ, then there exists a, b, c, d ∈ M such that θ(a) = x, θ(b) = y, θ(c) = u, θ(d) = v.

It follows that μ_B(x−y) = μ_B(θ(a)−θ(b)) = μ_B(θ(a−b))

= μ_θ^{-1}(B)(a−b) ≥ \{ μ_0^{-1}(B)(a) ∧ μ_0^{-1}(B)(b) \}

= \{ μ_B(θ(a)) ∧ μ_B(θ(b)) \}
An intuitionistic fuzzy set $A = \langle \mu_A, \nu_A \rangle$ in a $\Gamma$-near-ring $M$ is an intuitionistic fuzzy left (resp. right) ideal of $M$ for $\mu_A(0) \geq t, \nu_A(0) \leq s$.

**Theorem 3.11.** An intuitionistic fuzzy set $A = \langle \mu_A, \nu_A \rangle$ in a $\Gamma$-near-ring $M$ is an intuitionistic fuzzy left (resp. right) ideal if and only if $A_{<1,\alpha>} = \{ x \in M \mid \mu_A(x) \geq t, \nu_A(x) \leq s \}$ is a left (resp. right) ideal of $M$ for $\mu_A(0) \geq t, \nu_A(0) \leq s$.

**Proof.** $(\Rightarrow)$ Suppose that $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy left (resp. right) ideal of $M$ and let $\mu_A(0) \geq t, \nu_A(0) \leq s$. Let $x, y, u, v \in A_{<1,\alpha>}$ and $\alpha \in \Gamma$.

Then $\mu_A(x) \geq t, \nu_A(x) \leq s$ and $\mu_A(y) \geq t, \nu_A(y) \leq s$.

Hence $\mu_A(x-y) \geq \{ \mu_A(x) \land \mu_A(y) \} \geq t, \nu_A(x-y) \leq \{ \nu_A(x) \lor \nu_A(y) \} \leq s$.

$\mu_A(y+x-y) \geq \mu_A(x) \geq t, \nu_A(y+x-y) \leq \nu_A(x) \leq s$.

$\mu_A(ua(x+v)-uav) \geq \mu_A(x) \geq t$ and $\nu_A(ua(x+v)-uav) \leq \nu_A(x) \leq s$.

Therefore $x-y \in A_{<1,\alpha>}, \ (y+x-y) \in A_{<1,\alpha>}$ and $(ua(x+v)-uav) \in A_{<1,\alpha>}$ for all $x, y \in A_{<1,\alpha>}$ and $\alpha \in \Gamma$.

So $A_{<1,\alpha>}$ is a left (resp. right) ideal of $M$.

$(\Leftarrow)$ Suppose that $A_{<1,\alpha>}$ is a left (resp. right) ideal of $M$ for $\mu_A(0) \geq t$ and $\nu_A(0) \leq s$.

Let $x, y \in M$ be such that $\mu_A(x) = t_1, \nu_A(x) = s_1, \mu_A(y) = t_2$ and $\nu_A(y) = s_2$. 


Then \( x \in A_{t_2, s_2} \) and \( y \in A_{t_1, s_1} \).
We may assume that \( t_2 \leq t_1 \) and \( s_2 \geq s_1 \) without loss of generality.
It follows that 
\[
A_{t_1, s_1} \subseteq A_{t_2, s_2}
\]
so that \( x, y \in A_{t_1, s_1} \).
Since \( A_{t_1, s_1} \) is an ideal of \( M \), we have 
\[
x - y \in A_{t_1, s_1}, \quad (y + x - y) \in A_{t_1, s_1}.
\]
and 
\[
(\alpha x + v - u \alpha v) \in A_{t_1, s_1}
\]
for all \( \alpha \in \Gamma \).
\[
\mu(x - y) \geq t_1 \geq t_2 = \{\mu(x) \land \mu(y)\},
\]
\[
\nu(x - y) \leq s_1 \leq s_2 = \{\nu(x) \lor \nu(y)\}.
\]
\[
\mu(y + x - y) \geq t_1 \geq t_2 = \mu(x),
\]
\[
\nu(y + x - y) \leq s_1 \leq s_2 = \nu(x).
\]
\[
\mu(\alpha x + v - u \alpha v) \geq t_1 \geq t_2 = \mu(x) \quad \text{and}
\]
\[
\nu(\alpha x + v - u \alpha v) \leq s_1 \leq s_2 = \nu(x).
\]

Therefore \( A \) is an intuitionistic fuzzy left (resp.right) ideal of \( M \).

**Theorem 3.12.** If the IFS \( A = < \mu_A, \nu_A > \) is an intuitionistic fuzzy left (resp.right) ideal of a \( \Gamma \)-near-ring \( M \), then the sets \( M\mu_A = \{x \in M / \mu_A(x) = \mu_A(0)\} \) and \( M\nu_A = \{x \in M / \nu_A(x) = \nu_A(0)\} \) are left (resp.right) ideals.

**Proof.** Let \( x, y, u, v \in M\mu_A \) and \( \alpha \in \Gamma \).

Then \( \mu_A(x) = \mu_A(0), \mu_A(y) = \mu_A(0). \)
Since \( A \) is an intuitionistic fuzzy left (resp.right) ideal of a \( \Gamma \)-near-ring \( M \), we get
\[
\mu_A(x - y) \geq \{\mu_A(x) \land \mu_A(y)\} = \mu_A(0).
\]
\[
\mu_A(0) \geq \mu_A(x - y). \quad \text{So} \ x - y \in M\mu_A.
\]
\[
\mu_A(y + x - y) \geq \mu_A(x) = \mu_A(0).
\]
But \( \mu_A(0) \geq \mu_A(y + x - y). \) So \( y + x - y \in M\mu_A. \)

\[
\mu_A(\alpha x + v - u \alpha v) \geq \mu_A(x) = \mu_A(0) \quad (\text{resp.} \ \mu_A(x \alpha u) \geq \mu_A(x) = \mu_A(0)).
\]

Hence \( (\alpha x + v - u \alpha v) \in M\mu_A. \)
Therefore \( M\mu_A \) is a left (resp.right) ideal of \( M. \)

Similarly, let \( x, y, u, v \in M\nu_A \) and \( \alpha \in \Gamma \). Then \( \nu_A(x) = \nu_A(0), \nu_A(y) = \nu_A(0). \)
Since \( A \) is an intuitionistic fuzzy left (resp.right) ideal of a \( \Gamma \)-near-ring \( M, \)
\[
\nu_A(x - y) \leq \{\nu_A(x) \lor \nu_A(y)\} = \nu_A(0).
\]
But \( \nu_A(0) \leq \nu_A(x - y). \) So \( x - y \in M\nu_A. \)
\[
\nu_A(y + x - y) \leq \nu_A(x) = \nu_A(0).
\]
But \( \nu_A(0) \leq \nu_A(y + x - y). \) So \( y + x - y \in M\nu_A. \)

\[
\nu_A(\alpha x + v - u \alpha v) \leq \nu_A(x) = \nu_A(0) \quad (\text{resp.} \ \nu_A(x \alpha u) \leq \nu_A(x) = \nu_A(0) ).
\]
Hence \( (\alpha x + v - u \alpha v) \in M\nu_A. \)
Therefore \( M\nu_A \) is a left (resp.right) ideal of \( M. \)

**Definition 3.13.** A \( \Gamma \)-near-ring \( M \) is said to be regular if for each \( a \in M \) there exists
an \( x \in M \) and \( \alpha, \beta \in \Gamma \) such that \( a = a\alpha x \beta a \).

**Definition 3.14.** Let \( A = < \mu_A, \nu_A > \) and \( B = < \mu_B, \nu_B > \) be two intuitionistic fuzzy subsets of a \( \Gamma \)-near-ring \( M \). The product \( AB \) is defined by

\[
\begin{align*}
\mu_{AB}(x) &= \bigvee_{x = (u\gamma(v + w) - u\gamma w), u, v, w \in M, \gamma \in \Gamma} (\mu_A(u) \wedge \mu_B(v)) \quad \text{if} \quad x = (u\gamma(v + w) - u\gamma w), u, v, w \in M, \gamma \in \Gamma \\
\nu_{AB}(x) &= \bigwedge_{x = (u\gamma(v + w) - u\gamma w), u, v, w \in M, \gamma \in \Gamma} (\nu_A(u) \vee \nu_B(v)) \quad \text{if} \quad x = (u\gamma(v + w) - u\gamma w), u, v, w \in M, \gamma \in \Gamma \\
&= 0 \quad \text{otherwise,}
\end{align*}
\]

**Theorem 3.15.** If \( A = < \mu_A, \nu_A > \) and \( B = < \mu_B, \nu_B > \) are two intuitionistic fuzzy left (resp. right) ideals of \( M \), then \( A \cap B \) is an intuitionistic fuzzy left (resp. right) ideal of \( M \). If \( A \) is an intuitionistic fuzzy right ideal and \( B \) is an intuitionistic fuzzy left ideal, then \( AB \subseteq A \cap B \).

**Proof.** Suppose \( A \) and \( B \) are intuitionistic fuzzy ideals of \( M \) and let \( x, y, z, z' \in M \) and \( \alpha \in \Gamma \).

Then

\[
\begin{align*}
\mu_{A \cap B}(x-y) &= \mu_A(x-y) \wedge \mu_B(x-y) \\
&\geq [\mu_A(x) \wedge \mu_B(y)] \wedge [\mu_B(x) \wedge \mu_B(y)] \\
&= [\mu_A(x) \wedge \mu_B(x)] \wedge [\mu_A(y) \wedge \mu_B(y)] \\
&= \mu_{A \cap B}(x) \wedge \mu_{A \cap B}(y), \\
\nu_{A \cap B}(x-y) &= \nu_A(x-y) \vee \nu_B(x-y) \\
&\leq [\nu_A(x) \vee \nu_A(y)] \vee [\nu_B(x) \vee \nu_B(y)] \\
&= [\nu_A(x) \vee \nu_B(x)] \vee [\nu_A(y) \vee \nu_B(y)] \\
&= \nu_{A \cap B}(x) \vee \nu_{A \cap B}(y), \\
\mu_{A \cap B}(y + x-y) &= \mu_A(y + x-y) \wedge \mu_B(y + x-y) \\
&\geq [\mu_A(x) \wedge [\mu_B(x)] \\
&= \mu_{A \cap B}(x) \wedge \mu_{A \cap B}(y), \\
\nu_{A \cap B}(y + x-y) &= \nu_A(y + x-y) \vee \nu_B(y + x-y) \\
&\leq [\nu_A(x) \vee [\nu_B(x)] \\
&= \nu_{A \cap B}(x) \vee \nu_{A \cap B}(y).
\end{align*}
\]

Since \( A \) and \( B \) are intuitionistic fuzzy ideals of \( M \), we have

\[
\begin{align*}
\mu_A(x \alpha(y+z) - x \alpha z) &\geq \mu_A(x), \quad \nu_A(x \alpha(y+z) - x \alpha z) \leq \nu_A(x) \quad \text{and} \quad \mu_B(y \alpha x) \geq \mu_B(x), \\
\nu_B(y \alpha x) &\leq \nu_B(x).
\end{align*}
\]

Now

\[
\begin{align*}
\mu_{A \cap B}(x \alpha(y+z) - x \alpha z) &= \mu_A(x \alpha(y+z) - x \alpha z) \wedge \mu_B(x \alpha(y+z) - x \alpha z) \\
&\geq \mu_A(x) \wedge \mu_B(x) = \mu_{A \cap B}(x) \quad (\text{resp.} \quad \mu_{A \cap B}(y \alpha x) \geq \mu_{A \cap B}(x)), \\
\nu_{A \cap B}(x \alpha(y+z) - x \alpha z) &= \nu_A(x \alpha(y+z) - x \alpha z) \vee \nu_B(x \alpha(y+z) - x \alpha z) \\
&\leq \nu_A(x) \vee \nu_B(x) = \nu_{A \cap B}(x) \quad (\text{resp.} \quad \nu_{A \cap B}(y \alpha x) \leq \nu_{A \cap B}(x)).
\end{align*}
\]

Hence \( A \cap B \) is an intuitionistic fuzzy left ideal of \( M \).
To prove the second part if $\mu_{\Gamma B}(x) = 0$ and $\nu_{\Gamma B}(x) = 1$, there is nothing to show.

From the definition of $\Gamma B$, $\mu_A(x) = \mu_A(ya(z+z')- y\alpha z') \geq \mu_A(z)$, $\nu_A(x) = \nu_A(ya(z+z')- y\alpha z') \leq \nu_A(z)$.

Since $A$ is an intuitionistic fuzzy right ideal and $B$ is an intuitionistic fuzzy left ideal, we have

$$\mu_A(x) = \mu_A(za_\alpha y) \geq \mu_A(z), \quad \nu_A(x) = \nu_A(za_\alpha y) \leq \nu_A(z),$$

$$\mu_B(x) = \mu_B(za_\alpha y) \geq \mu_B(z), \quad \nu_B(x) = \nu_B(za_\alpha y) \leq \nu_B(z).$$

Hence by Definition 3.5,

$$\mu_{\Gamma B}(x) = \bigvee_{x=za_\alpha y} \{\mu_A(y) \land \mu_B(z)\} \leq \mu_A(x) \land \mu_B(x) = \mu_{A \land B}(x)$$

and

$$\nu_{\Gamma B}(x) = \bigwedge_{x=za_\alpha y} \{\nu_A(y) \lor \nu_B(z)\} \geq \nu_A(x) \lor \nu_B(x) = \nu_{A \lor B}(x)$$

which means that $\Gamma B \subseteq A \cap B$.

**Theorem 3.16.** A $\Gamma$-near-ring $M$ is regular if and only if for each intuitionistic fuzzy right ideal $A$ and each intuitionistic fuzzy left ideal $B$ of $M$, $\Gamma B = A \cap B$.

**Proof.** ($\Rightarrow$) Suppose $R$ is regular. $\Gamma B \subseteq A \cap B$. Thus it is sufficient to show that $A \cap B \subseteq \Gamma B$. Let $a \in M$ and $\alpha, \beta \in \Gamma$. Then, by hypothesis, there exists an $x \in M$ such that $a = a\alpha x \beta a$.

Thus

$$\mu_A(a) = \mu_A(a\alpha x \beta a) \geq \mu_A(a) \alpha \geq \mu_A(a),$$

$$\nu_A(a) = \nu_A(a\alpha x \beta a) \leq \nu_A(a \alpha) \leq \nu_A(a).$$

So $\mu_A(a \alpha x) = \mu_A(a)$ and $\nu_A(a \alpha x) = \nu_A(a)$.

On the other hand,

$$\mu_{\Gamma B}(a) = \bigvee_{a = a\alpha x \beta a} [\mu_A(a \alpha x) \land \mu_B(a)] \geq [\mu_A(a) \land \mu_B(a)] = \mu_{A \cap B}(a)$$

and

$$\nu_{\Gamma B}(a) = \bigwedge_{a = a\alpha x \beta a} [\nu_A(a \alpha x) \lor \nu_B(a)] \leq [\nu_A(a) \lor \nu_B(a)] = \nu_{A \cap B}(a).$$

Thus $A \cap B \subseteq \Gamma B$. Hence $\Gamma B = A \cap B$.

**References**


A Note on Characterization of Intuitionistic Fuzzy Ideals in Γ-Near-Rings


