Edge-Odd Graceful Labeling for Sum of a Path and a Finite Path

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Abstract

Choudum and Kishore [1999] found graceful labelling of the union of paths and cycles. Kaneria et. al. [2014b] found that complete bipartite graphs are graceful. Kaneria et. al. [2014c] got graceful labeling for open star of graphs. Liu et.al. [2012] investigated gracefulness of cartesian product of graphs, like paths, cycles, and stars. A (p, q) connected graph is edge-odd graceful graph if there exists an injective map \( f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\} \) so that induced map \( f+: V(G) \rightarrow \{0, 1, 2, 3, \ldots, (2k-1)\} \) defined by \( f+(x) \equiv \Sigma f(x, y) \) (mod 2k), where the vertex x is incident with other vertex y and \( k = \max \{p, q\} \) makes all the edges distinct and odd. This article, the Edge-odd gracefulfulness of \( P_m + P_n \) for \( m = 2, 3, 4, 5, \) and \( 6 \) is obtained.

Key words: Graceful Graphs, Edge-odd graceful labeling, Edge-odd Graceful Graph

Section 1: Introduction

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gracefulness of some new class of graphs. Vaidya and Lekha [2010] investigated odd graceful labeling of some new graphs. They also [2010] found new families of odd graceful graphs. Vaidya and Shah [2013] got graceful and odd graceful labeling of some graphs. In this paper, the edge-odd graceful labelings of graphs as the sum of a path with \( n \) vertices and each path with 2, 3, 4, 5, and 6 vertices.

Section 2: Edge-odd Gracefulness of the graph \( P_2 + P_n \)

The following definitions are first given.

**Definition 2.1: Graceful graph:** A function \( f \) of a graph \( G \) is called a graceful labeling with \( m \) edges, if \( f \) is an injection from the vertex set of \( G \) to the set \{0, 1, 2, ... , \( m \)\} such that when each edge \( uv \) is assigned the label \( |f(u) - f(v)| \) and the resulting edge labels are distinct. Then the graph \( G \) is graceful.

**Definition 2.2: Edge-odd graceful graph:** A \((p, q)\) connected graph is edge-odd graceful graph if there exists an injective map \( f: E(G) \rightarrow \{1, 3, ... , 2k-1\} \) so that induced map \( f+: V(G) \rightarrow \{0, 1, 2, ... , (2k-1)\} \) defined by \( f+(x) \equiv \Sigma f(x, y) \mod (2k) \), where the vertex \( x \) is incident with other vertex \( y \) and \( k = \max \{p, q\} \) makes all the edges distinct and odd. Hence the graph \( G \) is edge-odd graceful.

**Definition 2.3:** \( P_2 + N_n \) is a connected graph such that every vertex of \( P_2 \) is adjacent to every vertex of null graph \( N_n \) together with adjacency in both \( P_2 \) and \( P_n \). It has \( n + 2 \) vertices and \( 3n \) edges.

**Theorem 3.4:** The connected graph \( P_2 + P_n \) is edge–odd graceful.

**Proof:** The figure 1 is connected graph \( P_2 + P_n \) with \( n + 2 \) vertices and \( 3n \) edges, with some arbitrary labeling to its vertices and edges as follows.

![Figure 1: Edge–odd graceful graph \( P_2 + P_n \)](image)

Define \( f: E(G) \rightarrow \{1, 3, ... , 2q-1\} \) by
Case (i). n is odd
\[ f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (3n) \]  
(Rule 1).

Case (ii). n is even and \( i \neq 6 \)
\[ f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (2n+1). \]
\[ f(e_{3n-i}) = f(e_{2n+1}) + 2i + 2, \text{ for } i = 0, 1, 2, \ldots, (n-2). \]

Define \( f^+: V(G) \rightarrow \{0, 1, 2, \ldots, (2k-1)\} \) by \( f^+(v) \equiv \Sigma f(uv) \mod (2k) \), where this sum run over all edges through \( v \)  
(Rule 2).

Hence the map \( f \) and the induced map \( f^+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \( \{0, 1, 2, \ldots, (2q-1)\} \).

Hence the graph \( P_2 + P_n \) is edge-odd graceful.

Lemma 3.1: The connected graph \( P_2 + P_6 \) is edge – odd graceful.

Proof: The following graph in figure 2 is a connected graph with 8 vertices and 18 edges with some arbitrary distinct labeling to its vertices and edges.

![Figure 2: Edge – odd graceful graph P_2 + P_6](image)

Section 4: Edge-odd gracefulfulness of the graph \( P_3 + P_n \)

Definition 4.1: \( P_3 + N_n \) is a connected graph such that every vertex of \( K_3 \) is adjacent to every vertex of null graph \( N_n \) together with adjacency in both \( P_3 \) and \( P_n \). It has \( n + 3 \) vertices and \( 4n+1 \) edges.

Theorem 4.2: The connected graph \( P_3 + P_n \) for \( n = 1, 2, \ldots, (4n + 1) \) is edge – odd graceful.

Proof: The figure 3 is connected graph \( P_3 + P_n \) with \( n + 3 \) vertices and \( 4n+1 \) edges, with some arbitrary labeling to its vertices and edges for case (i).

Case (i): \( n = 1, 2, \ldots, (4n + 1) \) and \( n \neq 8, 14, 20, 26, \ldots (6m + 2) \)
Define $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ by

**Case (i).** $n \equiv 0 \pmod{6}$
\[ f(e_i) = (2i-1), \text{ for } i = 3, 4, 5, \ldots, (4n+1) \]
\[ f(e_1) = 3; f(e_2) = 1 \]

**Case (ii).** $n \equiv 1 \pmod{6}$
\[ f(e_i) = (2i-1), \text{ for } i = 1, 4, 5, 6, \ldots, (4n+1) \]
\[ f(e_2) = 5; f(e_3) = 3 \quad (\text{Rule 3}) \]

**Case (iii).** $n \equiv 3, 5 \pmod{6}$
\[ f(e_i) = (2i-1), \text{ for } i = 2, 4, 5, 6, \ldots, (4n+1) \]
\[ f(e_1) = 5; f(e_3) = 1 \]

**Case (iii).** $n \equiv 4 \pmod{6}$
\[ f(e_i) = (2i-1), \text{ for } i = 2, 3, \ldots, (4n) \]
\[ f(e_1) = 2q - 1; f(e_{4n+1}) = 1 \]

**Case (iv):** $n \not\equiv 8, 14, 20, 26, \ldots (6m + 2), m = 1, 2, \ldots$,
The figure 4 is connected graph $P_3 + P_n$ with $(n + 3)$ vertices and $(4n + 1)$ edges, with some arbitrary labeling to its vertices and edges for this case (iv).
Define $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ by
\[ n \equiv 2 \pmod{6} \]
\[ f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (n-1), (n+1), \ldots, (3n-1), \ldots, (4n+1) \]
\[ f(e_n) = 6n-1; f(e_{3n}) = 2n-1. \]

Define $f^+: V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\}$ by
\[ f^+(v) \equiv \sum f(uv) \mod (2q), \text{ where this sum run over all edges through } v \]
(Rule 6).

Hence the map $f$ and the induced map $f^+$ provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in $\{0, 1, 2, \ldots, (2q-1)\}$. Hence the graph $P_3 + P_n$ is edge-odd graceful.

**Lemma 4.3:** The connected graph $P_3 + P_3$ is edge-odd graceful.

**Proof:** The following graph in figure 5 is a connected graph with 6 vertices and 13 edges with some arbitrary distinct labeling to its vertices and edges.

**Lemma 4.4:** The connected graph $P_3 + P_4$ is edge-odd graceful.

The following graph in figure 6 is a connected graph with 7 vertices and 17 edges with some arbitrary distinct labeling to its vertices and edges.
Lemma 4.4: The connected graph $P_3 + P_5$ is edge–odd graceful.
The following graph in figure 7 is a connected graph with 8 vertices and 21 edges with some arbitrary distinct labeling to its vertices and edges.

Section 5: Edge-odd Gracefulness of the graph $P_4 + P_n$

Definition 5.1: $P_4 + N_n$ is a connected graph such that every vertex of $P_4$ is adjacent to every vertex of null graph $N_n$ together with adjacency in both $P_4$ and $P_n$. It has $n + 4$ vertices and $5n+2$ edges.

Theorem 5.2: The connected graph $P_4 + P_n$ for $n = 1, 2, 3, 4, \ldots, (5n + 2)$ is edge–odd graceful.

Proof: The figure 8 is connected graph $P_4 + P_n$ with $n + 4$ vertices and $5n+2$ edges, with some arbitrary labeling to its vertices and edges for case (i).

Case (i): $n = 1, 2, \ldots, (5n + 2)$ and $n \neq 8, 14, 20, 26, \ldots (6m + 2)$. 
Define $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ by

**Case (i).** $n \equiv 0 \pmod{6}$

$f(e_i) = (2i-1)$, for $i = 4, 5, \ldots, (5n+2)$

$f(e_1) = 3$; $f(e_2) = 5$; $f(e_3) = 1$

**Case (ii).** $n \equiv 1 \pmod{6}$

$f(e_i) = (2i-1)$, for $i = 1, 2, \ldots, (5n+2)$

**Case (iii).** $n \equiv 3 \pmod{6}$

$f(e_i) = (2i-1)$, for $i = 1, 2, \ldots, (5n+2)$; $i \neq 4 \& 4n + 3$  

(Rule 7)

$f(e_4) = 8n + 5$; $f(e_{4n+3}) = 7$

**Case (iv).** $n \equiv 4 \pmod{6}$

$f(e_i) = (2i-1)$, for $i = 4, 5, 6, \ldots, (5n+2)$

$f(e_1) = 5$; $f(e_2) = 1$; $f(e_3) = 3$

**Case (v).** $n \equiv 5 \pmod{6}$

$f(e_i) = (2i-1)$, for $i = 1, 2, 5, 6, \ldots, (5n+2)$

$f(e_3) = 7$; $f(e_4) = 5$

**Case (v):** $n \equiv 8, 14, 20, 26, \ldots (6m + 2), m = 1, 2, \ldots$.

The figure 9 is connected graph $P_4 + P_n$ with $n + 4$ vertices and $5n+2$ edges, with some arbitrary labeling to its vertices and edges for this case (v).
Define \( f: \text{E}(G) \to \{1, 3, \ldots, 2q-1\} \) by
\[
\begin{align*}
&\text{for } n \equiv 2 \pmod{6} \\
&\quad \text{Rule 8} \\
&f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (5n+2); \ i \neq 2n \ & 4n-1 \\
&f(e_{2n}) = 8n - 3; f(e_{4n-1}) = 4n - 3
\end{align*}
\]

Define \( f^+: \text{V}(G) \to \{0, 1, 2, \ldots, (2q-1)\} \) by
\[
f^+(v) \equiv \sum f(uv) \mod (2q), \text{ where this sum run over all edges through } v \quad \text{(Rule 9).}
\]

Hence the map \( f \) and the induced map \( f^+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \( \{0, 1, 2, \ldots, (2q-1)\} \).

Hence the graph \( P_4 + P_n \) is edge-odd graceful.

**Lemma 5.3:** The connected graph \( P_4 + P_4 \) is edge–odd graceful.

**Proof:** The following graph in figure 10 is a connected graph with 8 vertices and 22 edges with some arbitrary distinct labeling to its vertices and edges.

\[
\text{Figure 10: Edge – odd graceful graph } P_4 + P_4
\]
Lemma 5.4: The connected graph $P_4 + P_5$ is edge–odd graceful.
The following graph in figure 11 is a connected graph with 9 vertices and 27 edges
with some arbitrary distinct labeling to its vertices and edges.

![Figure 11: Edge–odd graceful graph $P_4 + P_5$](image1)

Section 6: Edge-odd gracefulness of the graph $P_5 + P_n$
Definition 6.1: $P_5 + N_n$ is a connected graph such that every vertex of $P_5$ is adjacent to
every vertex of null graph $N_n$ together with adjacency in both $P_5$ and $P_n$. It has $n + 5$
vertices and $6n+3$ edges.
Theorem 6.2: The connected graph $P_5 + P_n$ for all $n \neq 7$ is edge–odd graceful.
Proof: The figure 12 is connected graph $P_5 + P_n$ with $n + 5$ vertices and $6n+3$ edges,
with some arbitrary labeling to its vertices and edges.

![Figure 12: Edge–odd graceful graph $P_5 + P_n$](image2)

Define $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ by
$f(e_i) = (2i-1)$, for $i = 1, 2, \ldots, (6n+3)$

[Rule 10]
Define \( f^+ : V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\} \) by

\[
f^+(v) \equiv \sum f(uv) \mod (2q), \text{ where this sum run over all edges through } v \quad \text{(Rule 11)}
\]

Hence the map \( f \) and the induced map \( f^+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \( \{0, 1, 2, \ldots, (2q-1)\} \).

Hence the graph \( P_5 + P_n \) is edge-odd graceful.

**Lemma 6.3:** The connected graph \( P_5 + P_7 \) is edge-odd graceful.

The graph in figure 13 is a connected graph with 12 vertices and 45 edges with some arbitrary distinct labeling to its vertices and edges.

![Figure 13: Edge-odd graceful graph P5 + P7](image)

**Section 7: Edge-odd Gracefulness of the graph \( P_6 + P_n \)**

**Definition 7.1:** \( P_6 + N_n \) is a connected graph such that every vertex of \( P_6 \) is adjacent to every vertex of null graph \( N_n \) together with adjacency in both \( P_6 \) and \( P_n \). It has \( n + 6 \) vertices and \( 7n + 4 \) edges.

**Theorem 7.2:** The connected graph \( P_6 + P_n \) for all \( n \neq 7 \) and \( 8 \) is edge-odd graceful.

**Proof:** The figure 14 is connected graph \( P_6 + P_n \) with \( n + 6 \) vertices and \( 7n + 4 \) edges, with some arbitrary labeling to its vertices and edges.
Define $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ by
$$f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, (7n+4)$$

[Rule 12]

Define $f^+: V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\}$ by
$$f^+(v) \equiv \sum f(uv) \mod (2q), \text{ where this sum run over all edges through } v$$

[Rule 13]

Hence the map $f$ and the induced map $f^+$ provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in $\{0, 1, 2, \ldots, (2q-1)\}$. Hence the graph $P_6 + P_n$ is edge-odd graceful.

**Lemma 7.3:** The connected graph $P_6 + P_7$ is edge – odd graceful.

The graph $P_6 + P_7$ is a connected graph with 13 vertices and 53 edges. All the edges of the graph are labeled with distinct odd numbers in such a way that there will be distinct labeling for all its vertices.

Define $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ by
$$f(e_i) = (2i-1), \text{ for } i = 1, 2, \ldots, 53$$

Define $f^+: V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\}$ by
$$f^+(v) \equiv \sum f(uv) \mod (2q), \text{ where this sum run over all edges through } v$$

Hence the graph $P_6 + P_7$ is edge-odd graceful.

The graph with edge-odd graceful labeling is given in the figure 15.

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**Figure 14:** graph of $P_6 + P_n$
Lemma 7.4: The connected graph $P_6 + P_8$ is edge–odd graceful.
The graph $P_6 + P_8$ is a connected graph with 14 vertices and 60 edges. All the edges of the graph are labeled with distinct odd numbers in such a way that there will be distinct labeling for all its vertices.

Define $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ by

\[ f(e_i) = (2i-1), \quad \text{for } i = 1, 2, \ldots, 60 \]

Define $f^+: V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\}$ by

\[ f^+(v) = \sum f(uv) \mod (2q), \quad \text{where this sum run over all edges through } v \]

Hence the graph $P_6 + P_8$ is edge-odd graceful.
The graph with edge-odd graceful labeling is given in the figure 16.
REFERENCES


