Contra Harmonic Mean Labeling of Disconnected Graphs

S.S. Sandhya, S.Somasundaram and J. Rajeshni Golda

Assistant Professor in Mathematics, Sree Ayyappa College for Women, Chunkankadai, Kanyakumari District, Tamilnadu, India.
Professor in Mathematics, Manonmaniam Sundaranar University, Tirunelveli, Tamilnadu, India.
Assistant Professor in Mathematics, Women’s Christian College, Nagercoil Kanyakumari District, Tamilnadu, India.

Abstract

A graph G (V,E) is called a Contra Harmonic mean graph with p vertices and q edges, if it is possible to label the vertices x ∈ V with distinct element f(x) from 0,1………………q in such a way that when each edge e = uv is labeled with

\[ f(e=uv) = \left[ \frac{f(u)^2 + f(v)^2}{f(u)+f(v)} \right] \text{ or } \left[ \frac{f(u)^2 + f(v)^2}{f(u)+f(v)} \right] \]

with distinct edge labels. Here f is called a Contra Harmonic mean labeling of G.

Keywords: Graph, Contra Harmonic mean labeling, Contra Harmonic mean graphs, Path, Cycle, Comb, etc

1. INTRODUCTION

All graph in this paper are simple, finite, undirected. Let G be a graph with p vertices and q edges. For a detail survey of graph labeling we refer to Gallian [1]. For all other standard terminology and notation we follow Harray [2]. S. Somasundaram and R. Ponraj introduced mean labeling for some standard graphs in 2013. S.S. Sandhya and S. Somasundaram introduced Harmonic mean labeling of graph. We have introduced Contra Harmonic mean labeling in [5]. In this paper we investigate the Contra
Harmonic mean labeling behaviour of some disconnected graphs. The following
definition are useful for our present study.

**Definition 1.1**
A graph G (V,E) is called a Contra Harmonic mean graph with p vertices and q edges,
if it is possible to label the vertices \( x \in V \) with distinct elements \( f(x) \) from \( 0,1,\ldots\)\( q \)
in such a way that when each edge \( e = uv \) is labeled with \( f(e=uv) = \left\lceil \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right\rceil \) or
\( \left\lfloor \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right\rfloor \) with distinct edge labels. Then \( f \) is called Contra Harmonic mean labeling
of G.

**Definition 1.2:** The union of two graphs \( G_1 = (V_1,E_1) \) and \( G_2 = (V_2, E_2) \) is a graph
\( G = G_1 \cup G_2 \) with vertex set \( V = V_1 \cup V_2 \) and edge set \( E = E_1 \cup E_2 \).

**Definition 1.3:** The corona of two graphs \( G_1 \) and \( G_2 \) is the graph \( G = G_1 \odot G_2 \) formed
by taking one copy of \( G_1 \) and \( |V(G_1)| \) copies of \( G_2 \) where the \( i^{th} \) vertex of \( G_1 \) is
adjacent to every vertex in the \( i^{th} \) copy of \( G_2 \).

**Theorem 1.4:** Any Path is a Contra Harmonic mean graph.

**Theorem 1.5:** Any Cycle is a Contra Harmonic mean graph.

**Theorem 1.6:** Any Comb is a Contra Harmonic mean graph.

**Theorem 1.7:** Any Crown is a Contra Harmonic mean graph.

2. MAIN RESULTS

**Theorem 2.1:** \( C_m \odot P_n \) is a Contra Harmonic mean graph ,for \( m \geq 3 \) and \( n \geq 1 \)
Proof: Let \( C_m \) be the cycle \( u_1,\ldots \)\( u_m \) and \( P_n \) be the path \( v_1,\ldots v_n \)
Let \( G = C_m \odot P_n \).
Define a function \( f: V(G) \to \{0,1,\ldots,q\} \) by
\( f(u_i) = i-1, \ 1 \leq i \leq m-1 \), \( f(u_m) = m \)
\( f(v_i) = m+i, \ 1 \leq i \leq n \)
Then the distinct edge labels are
\( f(u_i, u_{i+1}) = i, \ 1 \leq i \leq m \)
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\[ f(v_i, v_{i+1}) = m+i, \ 1 \leq i \leq n-1 \]

\( C_m \cup P_n \) is Contra Harmonic mean graph

**Example 2.2** The Contra Harmonic mean labeling of \( C_m \cup P_n \) is

![Figure 1](image)

**Theorem 2.3:** \( C_m \cup (P_n \cup K_1) \) is a Contra Harmonic mean graph.

**Proof:**

Let \( C_m \) be a cycle with vertices \( u_1 \ldots u_m \) and Let \( v_1 \ldots v_n \) be the path \( P_n \). and let \( w_i \) be the vertices which is joined to the vertex \( v_i \), \( 1 \leq i \leq n \) of the path \( P_n \). The resultant graph is \( P_n \cup K_1 \). Let \( G = C_m \cup (P_n \cup K_1) \)

Define a function \( f : V(G) \rightarrow \{0,1\ldots q\} \) by

\[ f(u_i) = i-1, \ 1 \leq i \leq m-1 \]
\[ f(u_m) = m \]
\[ f(v_i) = m+2i-1, \ 1 \leq i \leq n \]
\[ f(w_i) = m+2i, \ 1 \leq i \leq n \]

Then the distinct edge labels are

\[ f(u_iu_{i+1}) = i, \ 1 \leq i \leq m \]
\[ f(v_i, v_{i+1}) = m+2i, \ 1 \leq i \leq n-1 \]
\[ f(v_i, w_i) = m+2i-1, \ 1 \leq i \leq n \]
Then f is a Contra Harmonic mean graph of G.

**Example 2.4:** The Contra Harmonic mean labeling of $C_5 \cup (P_3 \cup K_1)$ is

![Figure: 2](image)

**Theorem 2.5:** $(C_m \circ K_1) \cup P_n$ is a Contra Harmonic mean graph.

Proof: Let $u_1, u_2, \ldots, u_m$ be a cycle $C_m$ and $v_i$ be the vertex which is joined to the vertex $u_i$ of the cycle $C_m$, $1 \leq i \leq m$.

The resultant graph is $C_m \circ K_1$. Let $w_1, w_2, \ldots, w_n$ be the path $P_n$.

Let $G = (C_m \circ K_1) \cup P_n$.

Define a function

$$f: V(G) \rightarrow \{0, 1, \ldots, q\}$$

by

- $f(u_i) = 2i-2, 1 \leq i \leq m-1$
- $f(u_m) = 2m$
- $f(v_i) = 2i-1, 1 \leq i \leq m$
- $f(w_i) = 2m+i, 1 \leq i \leq n$

Then the distinct edge labels are

- $f(u_i, u_{i+1}) = 2i, 1 \leq i \leq m$
- $f(u_i, v_i) = 2i-1, 1 \leq i \leq m$
- $f(w_i, w_{i+1}) = 2m+i, 1 \leq i \leq n-1$

$(C_m \circ K_1) \cup P_n$ is a Contra Harmonic mean graph
**Example 2.6**: The Contra Harmonic mean labeling of \((C_6 \odot K_1) \cup P_3\)

![Figure: 3]

**Theorem 2.7**

\((C_m \odot K_1) \cup (P_n \odot K_1)\) is a Contra Harmonic mean graph.

Proof: Let \(u_1 u_2 \ldots u_m\) be the cycle \(C_m\) and let \(v_i\) be the pendant vertex joined to the vertex \(u_i\) of \(C_m\) \(1 \leq i \leq m\). The resultant graph is \(C_m \odot K_1\). Let \(w_1 \ldots w_n\) be the path \(P_n\) and \(t_i\) be the vertex which is joined to the vertex \(w_i\), \(1 \leq i \leq n\) of the path \(P_n\). The resultant graph is \(P_n \odot K_1\).

Let \(G = (C_m \odot K_1) \cup (P_n \odot K_1)\).

Define a function \(f: V(G) \rightarrow \{0, 1, \ldots, q\}\) by

\[
\begin{align*}
f(u_i) &= 2i - 2, \ 1 \leq i \leq m - 1, f(u_m) = 2m, \\
f(v_i) &= 2i - 1, \ 1 \leq i \leq m \\
f(w_i) &= 2m + 2i - 1, \ 1 \leq i \leq n \\
f(t_i) &= 2m + 2i, \ 1 \leq i \leq n
\end{align*}
\]

Then the distinct edge labels are

\[
\begin{align*}
f(u_i u_{i+1}) &= 2i, \ 1 \leq i \leq m \\
f(u_i v_i) &= 2i - 1, \ 1 \leq i \leq m \\
f(w_i w_{i+1}) &= 2m + 2i, \ 1 \leq i \leq n - 1 \\
f(w_i t_i) &= 2m + 2i - 1, \ 1 \leq i \leq n
\end{align*}
\]

\(\therefore (C_m \odot K_1) \cup (P_n \odot K_1)\) is a Contra Harmonic mean graph.
Example: 2.8

The Contra Harmonic mean labeling of \((C_6 \odot K_1) \cup (P_5 \odot K_1)\)

![Diagram](image)

**Figure: 4**

**Theorem 2.9:** \((C_m \odot K_1) \cup (P_n \odot \overline{K}_2)\) is a Contra Harmonic mean graph.

**Proof:** Let \(u_1 \ u_2 \ldots u_m\) be the cycle \(C_m\) and let \(v_i\) be the vertex joined to the vertex \(u_i\) of \(C_m\) \(1 \leq i \leq m\). The resultant graph is \(C_m \odot K_1\). Let \(w_1 \ldots w_n\) be the path \(P_n\) and let \(t_i\) and \(s_i\) be the vertices which are joined to the vertex \(w_i\) of path \(P_n\), \(1 \leq i \leq n\). The resultant graph in \(P_n \odot \overline{K}_2\)

Let \(G = (C_m \odot K_1) \cup (P_n \odot \overline{K}_2)\)

Define \(f:\ V(G) \rightarrow \{0,1,\ldots,q\}\) by

\[
\begin{align*}
  f(u_i) &= 2i-2, \quad 1 \leq i \leq m-1, f(u_m) = 2m \\
  f(v_i) &= 2i-1, \quad 1 \leq i \leq m \\
  f(w_i) &= 2m+3i-2, \quad 1 \leq i \leq n \\
  f(t_i) &= 2m+3i-1, \quad 1 \leq i \leq n \\
  f(s_i) &= 2m+3i, \quad 1 \leq i \leq n
\end{align*}
\]

Then the distinct edge labels are

\[
\begin{align*}
  f(u_i, u_{i+1}) &= 2i, \quad 1 \leq i \leq m \\
  f(u_i, v_i) &= 2i-1, \quad 1 \leq i \leq m \\
  f(w_i, w_{i+1}) &= 2m+3i, \quad 1 \leq i \leq n-1 \\
  f(w_i, t_i) &= 2m+3i-2, \quad 1 \leq i \leq n
\end{align*}
\]
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\[ f(w_i) = 2m+3i-1, \quad 1 \leq i \leq n \]

\( I(C_m \odot K_1) \cup (P_n \odot K_2) \) is a Contra Harmonic mean graph of \( G \).

**Example 2.10** Contra Harmonic mean labeling of \( (C_6 \odot K_1) \cup (P_4 \odot K_2) \)

![Graph](image)

**Figure: 6**
Theorem 2.11 : \((C_m \odot \overline{K}_2) \cup P_n\) is a Contra Harmonic mean graph

Proof: \(u_1u_2...u_mu_1\) be a cycle \(C_m\) and let \(v_i, w_i\) be the vertices that are joined to the vertex \(u_i, 1 \leq i \leq m\) of the cycle \(C_m\).

Let \(G = (C_m \odot \overline{K}_2) \cup P_n\)

Define \(f : V(G) \to \{0, 1, ..., q\}\)

by

\[
\begin{align*}
    f(u_i) &= 3i-2, \quad 1 \leq i \leq m-1, f(u_m) = 3m-1 \\
    f(v_i) &= 3i-3, \quad 1 \leq i \leq m \\
    f(w_i) &= 3i-1, \quad 1 \leq i \leq m-1, f(w_m) = 3m \\
    f(s_i) &= 3m+i, \quad 1 \leq i \leq n
\end{align*}
\]

Then the distinct edge labels are

\[
\begin{align*}
    f(u_i u_{i+1}) &= 3i, \quad 1 \leq i \leq m-1, f(u_m u_1) = 3m-1 \\
    f(u_i v_i) &= 3i-2, \quad 1 \leq i \leq m \\
    f(u_i w_i) &= 3i-1, \quad 1 \leq i \leq m-1, f(u_m w_m) = 3m \\
    f(s_i s_{i+1}) &= 3m+i, \quad 1 \leq i \leq n-1
\end{align*}
\]

\((C_m \odot \overline{K}_2) \cup P_n\) is a Contra Harmonic mean graph.
Example 2.12 The Contra Harmonic mean labeling of \((C_5 \odot K_2) \cup P_6\) is

![Graph Image]

Figure 5

Theorem 2.13 \((C_m \odot K_2) \cup (P_n \odot K_1)\) is a Contra Harmonic mean graph.

Proof: Let \(u_1, u_2, \ldots, u_m\) be the cycle \(C_m\). Let \(v_i, w_i\) be the vertices that are joined to the vertex \(u_i, 1 \leq i \leq m\) of the cycle \(C_m\).

Let \(s_1s_2 \ldots s_n\) be the path \(P_n\) and \(t_i\) be the vertex that are joined to the vertex \(s_i, 1 \leq i \leq n\) of \(P_n\).

Let \(G = (C_m \odot K_2) \cup (P_n \odot K_1)\)

Define \(f: V(G) \to \{0, 1, 2, \ldots, q\}\) by

\[
\begin{align*}
    f(u_i) &= 3i - 2, 1 \leq i \leq m-1, f(u_m) = 3m-1 \\
    f(v_i) &= 3i - 3, 1 \leq i \leq m \\
    f(w_i) &= 3i - 1, 1 \leq i \leq m-1, f(w_m) = 3m \\
    f(s_i) &= 3m + 2i - 1, 1 \leq i \leq n \\
    f(t_i) &= 3m + 2i, 1 \leq i \leq n
\end{align*}
\]

Then the distinct edge labels are

\[
\begin{align*}
    f(u_i u_{i+1}) &= 3i, 1 \leq i \leq m-1, f(u_m, u_1) = 3m-1 \\
    f(u_i v_i) &= 3i - 2, 1 \leq i \leq m \\
    f(u_i w_i) &= 3i - 1, 1 \leq i \leq m-1, f(u_m w_m) = 3m
\end{align*}
\]
\[ f(s_i, s_{i+1}) = 3m+2i, \; 1 \leq i \leq n-1 \]
\[ f(s_i) = 3m+2i-1, \; 1 \leq i \leq n \]

\[ l \left( (C_m \odot \overline{K}_2) \cup (P_n \odot K_1) \right) \] is a Contra Harmonic mean graph.

Example : 2.14

The Contra Harmonic mean labeling of \( (C_5 \odot \overline{K}_2) \cup (P_6 \odot K_1) \) is

Figure: 7
Theorem : 2.15

Proof: \((C_m \odot \overline{K}_2) \cup (P_n \odot \overline{K}_2)\) is a Contra Harmonic mean graph.

Let \(u_1 \ldots u_m\) be the cycle \(C_m\) and let \(v_i, w_i\) be the vertices that are joined to vertex \(u_i\) \(1 \leq i \leq m\) of \(C_m\). Let \(z_1 \ldots z_n\) be the path \(P_n\) and let \(s_i, t_i\) be the vertices that are joined to the vertex \(z_i\) of the path \(P_n\) \(1 \leq i \leq n\).

Let \(G = (C_m \odot \overline{K}_2) \cup (P_n \odot \overline{K}_2)\)

Define a function \(f: V(G) \rightarrow \{0, 1, \ldots, q\}\) by

\[
\begin{align*}
f(u_i) &= 3i-2, \ 1 \leq i \leq m-1, f(u_m) = 3m-1 \\
f(v_i) &= 3i-3, \ 1 \leq i \leq m \\
f(w_i) &= 3i-1, \ 1 \leq i \leq m-1, f(u_m) = 3m \\
f(z_i) &= 3m+3i-2, \ 1 \leq i \leq n \\
f(s_i) &= 3m+3i-1, \ 1 \leq i \leq n \\
f(t_i) &= 3m+3i, \ 1 \leq i \leq n
\end{align*}
\]

Then the distinct edge labels are

\[
\begin{align*}
f(u_i u_{i+1}) &= 3i, \ 1 \leq i \leq m-1, f(u_m u_1) = 3m-1 \\
f(u_i v_i) &= 3i-2, \ 1 \leq i \leq m \\
f(u_i w_i) &= 3i-1, \ 1 \leq i \leq m-1, f(u_m w_i) = 3m \\
f(z_i z_{i+1}) &= 3m+3i-2, \ 1 \leq i \leq n \\
f(z_i s_i) &= 3m+3i-1, \ 1 \leq i \leq n \\
f(z_i t_i) &= 3m+3i, \ 1 \leq i \leq n
\end{align*}
\]

\((C_m \odot \overline{K}_2) \cup (P_n \odot \overline{K}_2)\) is a Contra Harmonic mean graph.
Example 2.16:

The Contra Harmonic mean labeling of $(C_5 \odot K_2) \cup (P_4 \odot K_2)$

**Figure: 8**

**Theorem 2: 17**

$(C_m \odot K_2) \cup (P_n \odot K_3)$ is a Contra Harmonic mean graph.

Proof: Let $u_1, \ldots, u_m$ be a cycle $C_m$ and let $v_i, w_i$ be the vertices joined to the vertex $u_i$, $1 \leq i \leq m$. 
Let $z_1\ldots z_n$ be the path $P_n$ and let $s_i, t_i$ be the vertex of $K_3$ that are joined to the vertex $z_i$ of the path $P_n$ $1 \leq i \leq n$.

Let $G = (C_m \odot K_2) \cup (P_n \odot K_3)$

Define $f: V(G) \rightarrow \{0,1,\ldots,q\}$ by

- $f(u_i) = 3i-2$ for $1 \leq i \leq m-1$, $f(u_m) = 3m-1$
- $f(v_i) = 3i-3$, for $1 \leq i \leq m$
- $f(w_i) = 3i-1$, for $1 \leq i \leq m-1$, $f(w_m) = 3m$
- $f(z_i) = 3m+4i-3$, for $1 \leq i \leq n$
- $f(s_i) = 3m+4i-2$, for $1 \leq i \leq n$
- $f(t_i) = 3m+4i-1$, for $1 \leq i \leq n$

Then the distinct edge labels are

- $f(u_iu_{i+1}) = 3i$, for $1 \leq i \leq m-1$, $f(u_mu_1) = 3m-1$
- $f(u_i, v_i) = 3i-2$, for $1 \leq i \leq m$
- $f(u_1w_i) = 3i-1$, for $1 \leq i \leq m-1$, $f(u_mw_m) = 3m$
- $f(z_i, z_{i+1}) = 3m+4i$, for $1 \leq i \leq n-1$
- $f(z_i, s_i) = 3m+4i-3$, for $1 \leq i \leq n$
- $f(z_i, t_i) = 3m+4i-1$, for $1 \leq i \leq n$
- $f(s_i, t_i) = 3m+4i-2$, for $1 \leq i \leq n$

$(C_m \odot K_2) \cup (P_n \odot K_3)$ is a Contra Harmonic mean graph.
Example 2: 18: The Contra Harmonic mean labeling of \((C_5 \odot \overline{K}_2) \cup (P_4 \odot K_3)\) is

![Diagram of labeling](image)

Figure: 9

REFERENCES


Contra Harmonic Mean Labeling of Disconnected Graphs


