Approximating the STIELTJES Integral by using the Generalized Simpson's Rule

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Abstract

Accurate approximations for the Stieltjes integral by the generalized Simpson's rule. The generalized Simpson's rule is established on the notion of the derivative of function with respect to the strictly increasing function, defined in [9].

Keywords: Stieltjes integral, Generalired Simpson's rule, the derivative of the function with respect to the strictly increasing function, generalized formula of Taylor.

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Introduction and the Derivative of Function with respect to the strictly increasing function

Our aim is to describe the generalized Simpson's rule for the approximation of Stieltjes integral

\[ I = \int_a^b f(x)du(x), a < b, a, b \in R, \]

where \( f(x) \) is a given continuous function on \([a, b]\) and \( u(x) \) is a given function of bounded variation on \([a, b]\).

It is known [7, 205 p.], that the function \( u(x) \) presented in the form

\[ u(x) = \varphi(x) - \psi(x), x \in [a, b] \]

where \( \varphi(x) \) and \( \psi(x) \) are the known increasing functions, on \([a, b]\).
It is suggested different methods of the approximate calculation of the Stieltjes integral in works [1-6]. Particularly, in 1998 Dragomir and Fedotov [2], in order to approximate the Stieltjes integral (1) with the simpler expression

$$\frac{1}{b-a}[u(b) - u(a)] \int_a^b f(x)dx,$$

introduced the following error functional

$$D(f,u;a,b) = \int_a^b f(x)dx - \frac{1}{b-a}[u(b) - u(a)] \int_a^b f(x)dx.$$

In this work for the approximate calculation of the Stieltjes integral (1) it is suggested the generalized Simpson's rule which is based on the notion of the derivative of function with respect to the strictly increasing function [9]. The generalized Simpson's rule summarizes the Simpson's rule [8]. Then we need the concept of the derivative defined in the work [9] and the theorems with proofs connected with it. Apparently the first notion of the derivative, with respect to the strictly increasing function, was introduced in [9].

**Definition:** The derivative of a function $f(x)$ with respect to $\varphi(x)$ is the function $f'_\varphi(x)$ whose value at $x \in (a, b)$ is the number:

$$f'_\varphi(x) = \lim_{\Delta \to 0} \frac{f(x + \Delta) - f(x)}{\varphi(x + \Delta) - \varphi(x)},$$

where $\varphi(t)$ is the given strictly increasing continuous function in $(a, b)$.

If the limit in equation (3) exists, we say that $f(x)$ has a derivative (is differentiable) with respect to $\varphi(x)$. The first derivative $f'_\varphi(x)$ may also be differentiable function with respect to $\varphi(x)$ at every point $x \in (a, b)$. If so, its derivative

$$f''_\varphi(x) = (f'_\varphi(x))'_\varphi,$$

is called the second derivative of $f(x)$ with respect to $\varphi(x)$. The names continue as you imagine they would, with

$$f^{(n)}_\varphi(x) = (f^{(n-1)}_\varphi(x))'_\varphi,$$

denoting the $n-th$ derivative of $f(x)$ with respect to $\varphi(x)$.

**Theorem 1:** (generalized Fermat's Theorem). Let $\varphi(x)$ is strictly increasing continuous function in $[a,b]$, the function $f(x)$ is continuous at every point of the closed interval $[a,b]$ and the function $f(x)$ has a local maximum or a local minimum value in an interior point $x_0 \in (a,b)$ and $\exists f''_\varphi(x_0)$. Then $f''_\varphi(x_0) = 0$. 
Approximating the STIELTJES Integral

Proof: The point $x_0$ cannot be the point of increase (or decrease). Because if the point $x_0$ is the point of increase (decrease) then in some deleted neighborhood of this point

$$\frac{f(x) - f(x_0)}{\varphi(x) - \varphi(x_0)} > 0 \left( \frac{f(x) - f(x_0)}{\varphi(x) - \varphi(x_0)} < 0 \right).$$

That is why inequality $f'_\varphi(x_0) > 0$ or $f'_\varphi(x_0) < 0$ is impossible.

It remains to accept, that $f'_\varphi(x_0) = 0$. The theorem 1 is proved.

Theorem 2: (generalized Rolle's Theorem). Let $\varphi(x)$ is strictly increasing continuous function in $[a,b]$ , the function $f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable with respect to $\varphi(x)$ at every point of its interior $(a,b)$ and $f(a) = f(b)$. Then there is at least one number $\xi$ between $a$ and $b$ at which $f'_\varphi(\xi) = 0$.

Proof: On the ground of the theorem of Weierstrass the function $f(x)$, which is continuous in $[a,b]$, gets the maximum $M$ and the minimum $m$ in it. If both of these values are got on the ends of $[a,b]$, then by the data they are equal ($M = m$). And it means that the function $f(x)$ is identically constant in $[a,b]$. Then the derivative $f'_\varphi(x)$ is equal to 0 in all points of $[a,b]$. Even if one of these values - maximum or minimum - is got in point $\xi \in (a,b)$ (that is $m < M$), then on the strength of the theorem 1 $f'_\varphi(\xi) = 0$. The theorem 2 is proved.

Theorem 3: Let $\varphi(x)$ is the strictly increasing continuous function on $[a,b]$ and $f'_{\varphi}^{(IV)}(x) \in C[a,b]$. Then

$$f(x) = H_3(x) + \frac{f'_{\varphi}^{(IV)}(\xi)}{24}(\varphi(x) - \varphi(a))(\varphi(x) - \varphi(d))^2(\varphi(x) - \varphi(b)), \forall x \in [a, b],$$

(4)

where $\xi = \xi(x) \in (a, b)$, $\varphi(d) = \frac{1}{2}[\varphi(a) + \varphi(b)]$,

$$H_3(x) = \frac{(\varphi(x) - \varphi(d))(\varphi(x) - \varphi(b))}{(\varphi(a) - \varphi(d))(\varphi(a) - \varphi(b))}f(a) + \frac{(\varphi(x) - \varphi(a))(\varphi(x) - \varphi(b))}{(\varphi(d) - \varphi(a))(\varphi(d) - \varphi(b))}f(d) + \frac{(\varphi(x) - \varphi(a))(\varphi(x) - \varphi(d))^2}{(\varphi(b) - \varphi(a))(\varphi(b) - \varphi(d))^2}f(b) + \frac{(\varphi(x) - \varphi(a))(\varphi(x) - \varphi(d))(\varphi(x) - \varphi(b))}{(\varphi(d) - \varphi(b))(\varphi(d) - \varphi(a))}f'_{\varphi}(d).$$

(5)
Proof: We consider the function \( g(s) \) defined on \([a, b]\) by the correlation
\[
g(s) = f(s) - H_3(s) - k\omega(s), s \in [a, b],
\]
where \( k \) is constant,
\[
\omega(s) = \left( \varphi(s) - \varphi(a) \right) \left( \varphi(s) - \varphi(d) \right)\left( \varphi(s) - \varphi(b) \right).
\]

For each fixed \( x \in [a, b], x \neq a, x \neq d, x \neq b \), we suppose
\[
k = \frac{f(x) - H_3(x)}{\omega(x)}.
\]

The points \( s = a, s = d, s = b, \) and \( s = x \) are the roots of the equation \( g(s) = 0 \), besides the point \( s = d \) is the root of the equation \( g'_\varphi(s) = 0 \). Then on the strength of the theorem 2 the equation \( g'_\varphi(s) = 0 \) have at least 4 distinct roots on \((a, b)\).

Therefore, on the strength of the theorem 2 the equation \( g''_\varphi(s) = 0 \) have at least 3 distinct roots on \((a, b)\). Then on the strength of the theorem 2 the equation \( g'''_\varphi(s) = 0 \) have at least 2 distinct roots on \((a, b)\). Therefore, on the strength of the theorem 2 the equation \( g''''_\varphi(s) = 0 \) have at least one root \( s = \xi(x) \in (a, b) \). Then taking into account (5) from (6) we have
\[
g''''_\varphi(s) = f''''_\varphi(s) - k\omega''''_\varphi(s), s \in [a, b].
\]

But \( \omega''''_\varphi(s) = 4! = 24 \). Therefore taking into account \( g''''_\varphi(\xi) = 0 \), from (8) we obtain
\[
k = \frac{f''''_\varphi(\xi)}{24}.
\]

Then on the strength (9) from (7) we have (4). The theorem 3 is proved.

Theorem 4: Let function \( f(x) \) is continuous function in \([a, b]\), \( \varphi(x) \) is strictly increasing continuous function in \([a, b]\) and
\[
F(x) = \int_a^x f(t) \varphi(t), x \in [a, b].
\]

Then
\[
F'_\varphi(x) = \left( \int_a^x f(t) \varphi(t) \right)'_{\varphi} = f(x), x \in [a, b],
\]
where
Approximating the STIELTJES Integral

\[ F'_\varphi(a) = \lim_{\Delta x \to 0^+} \frac{F(a + \Delta x) - F(a)}{\varphi(a + \Delta x) - \varphi(a)}, \quad F'_\varphi(b) = \lim_{\Delta x \to 0^-} \frac{F(b + \Delta x) - F(b)}{\varphi(b + \Delta x) - \varphi(b)} \]

**Proof:** From definition of \( F'_\varphi(x) \), we have

\[
F'_\varphi(x) = \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} d\varphi(t) - \int_x^{x+\Delta x} (f(x) - f(t))d\varphi(t)}{[\varphi(x + \Delta x) - \varphi(x)]} = f(x) - \lim_{\Delta x \to 0} \psi(x, \Delta x),
\]

where

\[
\psi(x, \Delta x) = \frac{\left( \int_x^{x+\Delta x} (f(x) - f(t))d\varphi(t) \right)}{[\varphi(x + \Delta x) - \varphi(x)]}.
\]

Then

\[
|\psi(x, \Delta x)| \leq \omega_f(\Delta x) \left( \int_x^{x+\Delta x} d\varphi(t) \right) \left| \frac{\varphi(x + \Delta x) - \varphi(x)}{\varphi(x + \Delta x) - \varphi(x)} \right| = \omega_f(\Delta x)
\]

where

\[
\omega_f(\delta) = \sup_{|t-x| \leq \delta} |f(x) - f(t)|,
\]

and \( \lim_{\delta \to 0} \omega_f(\delta) = 0 \). Therefore

\[
\lim_{\Delta x \to 0} |\psi(x, \Delta x)| \leq \lim_{\Delta x \to 0} \omega_f(|\Delta x|) = 0.
\]

Hence: \( F'_\varphi(x) = f(x) \). Analogously the other cases are proved. The theorem 4 is proved.

**Corollary:** Let \( F_0(x) = f(x) \in C[a, b], \varphi(x) \) is strictly increasing continuous function on \([a, b]\) and

\[
F_i(x) = \int_a^x F_{i-1}(t) \varphi(t) \, dx, i = 1, ..., n.
\]

Then \( F_n(x) \in C^{(n)}_\varphi[a, b] \), where \( C^{(n)}_\varphi[a, b] \) is the linear space of all continuous functions \( v(x) \) defined in \([a, b]\) such that \( v^{(n)}_\varphi(x) \in C[a, b] \).

**The generalized Simpson's rule**

Let
\[ h = \frac{b - a}{2n}, \quad x_j = a + jh, \quad j = 0,1,\ldots, 2n, n \in \mathbb{N}. \]

Then the approximate value of the integral (1) we define by the formula

\[
A_n = \frac{1}{6} \sum_{j=0}^{n-1} \left\{ [\varphi(x_{2j+2}) - \varphi(x_{2j})][f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})] - [\psi(x_{2j+2}) - \psi(x_{2j})][f(x_{2j}) + 4f(x'_{2j+1}) + f(x_{2j+2})] \right\}.
\] (10)

where

\[
\varphi(x_{2j+1}) = \frac{1}{2} [\varphi(x_{2j}) + \varphi(x_{2j+2})],
\]
(11)

\[
\psi(x'_{2j+1}) = \frac{1}{2} [\psi(x_{2j}) + \psi(x_{2j+2})].
\]
(12)

**Theorem 5:** Let \( \varphi(x) \) and \( \psi(x) \) are the strictly increasing continuous functions on \([a, b] \), \( f^{(IV)}(x), f^{(IV)}(x) \in C[a, b] \). Then

\[
|l - A_n| \leq \frac{L_0}{2880} (\varphi(b) - \varphi(a))(\omega_\varphi(2h))^4 + \frac{L_1}{2880} (\psi(b) - \psi(a))(\omega_\psi(2h))^4,
\]
(13)

where

\[
\left\{ \begin{array}{c}
L_0 = \sup_{x \in [a,b]} |f^{(IV)}(x)|, \\
L_1 = \sup_{x \in [a,b]} |f^{(IV)}(x)|,
\end{array} \right.
\]

\[
\omega_\varphi(2h) = \sup_{|x_1 - x_2| \leq 2h} |\varphi(x_1) - \varphi(x_2)|, \omega_\psi(2h) = \sup_{|x_1 - x_2| \leq 2h} |\psi(x_1) - \psi(x_2)|.
\]
(14)

**Proof:** Let us introduce the notations:

\[
\begin{align*}
P_j &= \int_{x_{2j}}^{x_{2j+2}} f(x) d\varphi(x), \\
Q_j &= \int_{x_{2j}}^{x_{2j+2}} f(x) d\psi(x), \\
M_j &= \frac{1}{6} [\varphi(x_{2j+2}) - \varphi(x_{2j})][f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})], \\
N_j &= \frac{1}{6} [\psi(x_{2j+2}) - \psi(x_{2j})][f(x_{2j}) + 4f(x'_{2j+1}) + f(x_{2j+2})],
\end{align*}
\]
(15)

\[
\begin{aligned}
j &= 0,1,\ldots, n-1,
\end{aligned}
\]

On the strength of the theorem 3 for \( a = x_{2j}, b = x_{2j+2}, d = x_{2j+1}^* \) we have

\[
P_j = \int_{x_{2j}}^{x_{2j+2}} f(x) d\varphi(x) = P_{j1} + P_{j2} + P_{j3} + P_{j4} + P_{j5}, \quad j = 0,1,\ldots, n-1,
\]
(16)
Taking into account (11) and (22) from (17), (18), (19) and (20) we have

\[
P_{j1} = \frac{1}{\alpha_1} \int_{x_{j1}}^{x_{j1+2}} \left[ \varphi(x) - \varphi(x_{j1+1}) \right]^2 \left[ \varphi(x) - \varphi(x_{j1+2}) \right] f(x_{j1}) d\varphi(x),
\]

\[
P_{j2} = \frac{1}{\alpha_2} \int_{x_{j2}}^{x_{j2+2}} \left[ \varphi(x) - \varphi(x_{j2}) \right] \left[ \varphi(x) - \varphi(x_{j2+2}) \right] f(x_{j2+1}) d\varphi(x),
\]

\[
P_{j3} = \frac{1}{\alpha_3} \int_{x_{j3}}^{x_{j3+2}} \left[ \varphi(x) - \varphi(x_{j3}) \right] \left[ \varphi(x) - \varphi(x_{j3+1}) \right] f(x_{j3+2}) d\varphi(x),
\]

\[
P_{j4} = \frac{1}{\alpha_4} \int_{x_{j4}}^{x_{j4+2}} \left[ \varphi(x) - \varphi(x_{j4}) \right] \left[ \varphi(x) - \varphi(x_{j4+1}) \right] \left[ \varphi(x) - \varphi(x_{j4+2}) \right] f(\varphi(x)) d\varphi(x),
\]

\[
P_{j5} = \int_{x_{j5}}^{x_{j5+2}} \frac{f^{(IV)}(\xi)}{24} \left[ \varphi(x) - \varphi(x_{j5}) \right] \left[ \varphi(x) - \varphi(x_{j5+1}) \right] \left[ \varphi(x) - \varphi(x_{j5+2}) \right] d\varphi(x),
\]

\[
\begin{align*}
\alpha_1 &= -\frac{1}{4} \left[ \varphi(x_{j1+2}) - \varphi(x_{j1}) \right]^3, \\
\alpha_2 &= -\frac{1}{4} \left[ \varphi(x_{j2+2}) - \varphi(x_{j2}) \right]^3, \\
\alpha_3 &= -\alpha_1, \\
\alpha_4 &= \alpha_2.
\end{align*}
\]

Taking into account (11) and (22) from (17), (18), (19) and (20) we have

\[
P_{j1} = \frac{1}{\alpha_1} \int_{x_{j1}}^{x_{j1+2}} \left[ \varphi(x) - \varphi(x_{j1+1}) \right]^2 \left[ \varphi(x) - \varphi(x_{j1+2}) \right] f(x_{j1}) d\varphi(x),
\]

\[
\varphi(x_{j1+2}) f(x_{j1}) d\varphi(x) - \varphi(x_{j1+1}) - \varphi(x_{j1+2}) f(x_{j1}) d\varphi(x) = \frac{f(x_{j1})}{\alpha_1} \left\{ \left[ \varphi(x) - \varphi(x_{j1+1}) \right]^4 \right\} \bigg|_{x=x_{j1+2}}^{x=x_{j1+2}} +
\]

\[
-\frac{1}{\alpha_1} \cdot 2 \frac{f(x_{j1})}{3} \left[ \varphi(x_{j1+2}) - \varphi(x_{j1}) \right] \bigg|_{x=x_{j1+2}}^{x=x_{j1+2}} = \frac{1}{6} \left[ \varphi(x_{j1+2}) - \varphi(x_{j1}) \right] f(x_{j1}),
\]

\[
P_{j2} = \frac{1}{\alpha_2} \int_{x_{j2}}^{x_{j2+2}} \left[ \varphi(x) - \varphi(x_{j2}) \right] \left[ \varphi(x) - \varphi(x_{j2+2}) \right] f(x_{j2+1}) d\varphi(x),
\]

\[
\varphi(x_{j2+2}) f(x_{j2}) d\varphi(x) - \varphi(x_{j2}) - \varphi(x_{j2+2}) f(x_{j2}) d\varphi(x) = \frac{f(x_{j2})}{\alpha_2} \left\{ \left[ \varphi(x) - \varphi(x_{j2+1}) \right]^4 \right\} \bigg|_{x=x_{j2+2}}^{x=x_{j2+2}} +
\]

\[
-\frac{1}{\alpha_2} \cdot 2 \frac{f(x_{j2})}{3} \left[ \varphi(x_{j2+2}) - \varphi(x_{j2}) \right] \bigg|_{x=x_{j2+2}}^{x=x_{j2+2}} = \frac{1}{6} \left[ \varphi(x_{j2+2}) - \varphi(x_{j2}) \right] f(x_{j2}).
\]
\[
\frac{f(x_{2j+1})}{\alpha_{j2}} \left\{ \left[ \frac{\varphi(x) - \varphi(x_{2j})}{3} - \frac{\varphi(x_{2j+2}) - \varphi(x_{2j})}{2} \right]^3 - \left[ \varphi(x) - \varphi(x_{2j}) \right]^2 \right\}^{x=x_{2j+2}} = \\
-\frac{1}{\alpha_{j2}} \frac{f(x_{2j+1})}{6} [\varphi(x_{2j+2}) - \varphi(x_{2j})]^3 = \frac{2}{3} [\varphi(x_{2j+2}) - \varphi(x_{2j})] f(x_{2j+1}). \quad (24)
\]

\[
P_{j3} = \frac{1}{\alpha_{j3}} \int_{x_{2j}}^{x_{2j+2}} \left[ \varphi(x) - \varphi(x_{2j+1}) - \varphi(x_{2j}) \right] \times \\
\left[ \varphi(x) - \varphi(x_{2j+1}) \right]^2 d[\varphi(x) - \varphi(x_{2j+1})] f(x_{2j+2}) = \\
\frac{1}{\alpha_{j3}} \left\{ \frac{1}{4} [\varphi(x) - \varphi(x_{2j+1})]^4 + \frac{1}{3} \left[ \varphi(x_{2j+1}) - \varphi(x_{2j}) \right] \left[ \varphi(x) - \varphi(x_{2j+1}) \right]^3 \right\}^{x=x_{2j+2}} \times \\
f(x_{2j+2}) = \frac{1}{\alpha_{j3}} \cdot \frac{2}{3} \frac{f(x_{2j+2})}{2 \cdot 8} \left[ \varphi(x_{2j+2}) - \varphi(x_{2j}) \right]^4 = \\
\frac{1}{6} \left[ \varphi(x_{2j+2}) - \varphi(x_{2j}) \right] f(x_{2j+2}), \quad (25)
\]

\[
P_{j4} = \frac{1}{\alpha_{j4}} \int_{x_{2j}}^{x_{2j+2}} \left[ \varphi(x) - \varphi(x_{2j+1}) \right] \left[ \varphi(x) - \varphi(x_{2j+1}) + \varphi(x_{2j+1}) - \varphi(x_{2j}) \right] \left[ \varphi(x) - \varphi(x_{2j+1}) \right]^2 \\
- \varphi(x_{2j+1}) + \varphi(x_{2j+1} - \varphi(x_{2j+2})) f_{\varphi}'(x_{2j+1}) d[\varphi(x) - \varphi(x_{2j+1})] = \\
\frac{1}{\alpha_{j4}} \left\{ \frac{1}{4} [\varphi(x) - \varphi(x_{2j+1})]^4 - \frac{1}{8} \left[ \varphi(x_{2j+2}) - \varphi(x_{2j+2}) \right] \left[ \varphi(x) - \varphi(x_{2j+1}) \right]^2 \\
- \varphi(x_{2j+1}) \right\}^{x=x_{2j+2}} = 0, j = 0, 1, ..., n - 1. \quad (26)
\]

Taking into account (11) and (14) from (21) we obtain

\[
|P_{j5}| \leq \int_{x_{2j}}^{x_{2j+2}} \frac{L_0}{24} \left[ \varphi(x) - \varphi(x_{2j}) \right] \left[ \varphi(x) - \varphi(x_{2j+2}) \right] \left[ \varphi(x) - \varphi(x_{2j+1}) \right]^2 d[\varphi(x) - \varphi(x_{2j+1})] \\
- \varphi(x_{2j+1}) = - \frac{L_0}{24} \int_{x_{2j}}^{x_{2j+2}} \left[ \varphi(x) - \varphi(x_{2j+1}) \right] \left[ \varphi(x) - \varphi(x_{2j+1}) + \varphi(x_{2j+1}) - \varphi(x_{2j}) \right] d[\varphi(x) - \varphi(x_{2j+1})] = \\
- \frac{L_0}{24} \left\{ \left[ \varphi(x) - \varphi(x_{2j+1}) \right]^2 - \frac{1}{4} \left[ \varphi(x_{2j+2}) - \varphi(x_{2j}) \right]^2 \right\} \left[ \varphi(x) - \varphi(x_{2j+1}) \right]^2 \\
- \varphi(x_{2j+1})^2 d[\varphi(x) - \varphi(x_{2j+1})] = - \frac{L_0}{24} \left\{ \frac{1}{5} [\varphi(x) - \varphi(x_{2j+1})]^5 - \frac{1}{12} \left[ \varphi(x_{2j+2}) - \varphi(x_{2j}) \right]^2 [\varphi(x) - \varphi(x_{2j+1})]^3 \right\}^{x=x_{2j+2}} = \\
- \frac{L_0}{24} \left\{ \frac{2}{5} \cdot \frac{1}{32} \left[ \varphi(x_{2j+2}) - \varphi(x_{2j}) \right]^5 - \frac{1}{48} [\varphi(x_{2j+2}) - \varphi(x_{2j})]^5 \right\} = 
\]
Approximating the STIELTJES Integral

\[
\frac{L_0}{2880}[\varphi(x_{2j+2}) - \varphi(x_{2j})]^5, \quad j = 0, 1, \ldots, n - 1.
\]  

(27)

On the strength of (23), (24), (25) and (26) from (15) we have

\[
M_j = P_{j1} + P_{j2} + P_{j3} + P_{j4}, \quad j = 0, 1, \ldots, n - 1.
\]  

(28)

Taking into account (28) and (27) from (16) we obtain

\[
|P_j - M_j| \leq \frac{L_0}{2880} [\varphi(x_{2j+2}) - \varphi(x_{2j})][\omega_\varphi(2h)]^4,
\]  

(29)

where \( j = 0, 1, \ldots, n - 1. \)

On the strength of (12), (14) and (15) analogously we have the following estimate

\[
|Q_j - N_j| \leq \frac{L_1}{2880} [\psi(x_{2j+2}) - \psi(x_{2j})][\omega_\psi(2h)]^4,
\]  

(30)

where \( j = 0, 1, \ldots, n - 1. \)

Taking into account the notations (2) and (15) from (1) and (10) we obtain

\[
\begin{aligned}
I &= \int_a^b f(x) \, du(x) = \sum_{i=1}^n (P_j - Q_j) \\
A_n &= \sum_{i=1}^n (M_i - N_i).
\end{aligned}
\]  

(31)

Then on the strength (29) and (30) from (31) we have

\[
|I - A_n| \leq \sum_{i=1}^n [|P_i - M_i| + |Q_i - N_i|] \leq \frac{L_0}{2880} [\varphi(b) - \varphi(a)] [\omega_\varphi(2h)]^4 + \frac{L_1}{2880} [\psi(b) - \psi(a)][\omega_\psi(2h)]^4.
\]

The theorem 5 is proved.

**Corollary 1**: Let \( \varphi(x) \) is the strictly increasing continuous function on \([a, b], \psi(x) = 0 \) for all \( x \in [a, b] \) and \( f^{(IV)}(x) \in C[a, b] \). Then

\[
|I - A_n'| \leq \frac{L_0}{2880} [\varphi(b) - \varphi(a)][\omega(2h)]^4,
\]

where

\[
A_n' = \frac{1}{6} \sum_{j=1}^n [\varphi(x_{2j+2}) - \varphi(x_{2j})][f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})].
\]
Corollary 2: Let $\varphi(x)$ and $\psi(x)$ are the strictly increasing continuous functions on $[a, b]$, $f^{(IV)}(x) = f^{(IV)}(x)x \in \mathcal{C}[a, b]$, $\varphi(x)\in \mathcal{C}^\alpha[a, b]$, $0 < \alpha \leq 1$, $\psi(x)\in \mathcal{C}^\beta[a, b]$, $0 < \beta \leq 1$, i.e.

for all $x, y \in [a, b]$

$$|\varphi(x) - \varphi(y)| \leq c_0|x - y|^\alpha, |\psi(x) - \psi(y)| \leq c_1|x - y|^\beta, c_0 > 0, c_1 > 0.$$ 

Then

$$|I - A_n| \leq \frac{L_0c_0^4}{2880} (\varphi(b) - \varphi(a))(2h)^{4\alpha} + \frac{L_1c_1^4}{2880} (\psi(b) - \psi(a))(2h)^{4\beta}.$$

Corollary 3: Let $\varphi(x)$ is the strictly increasing continuous function on $[a, b]$, $\psi(x) = 0$ for all $x \in [a, b]$, $f^{(IV)}(x)x \in \mathcal{C}[a, b]$, $\varphi(x)\in \mathcal{C}^\alpha[a, b]$, $0 < \alpha \leq 1$.

Then

$$|I - A_n^1| \leq \frac{L_0c_0^4}{2880} (\varphi(b) - \varphi(a))(2h)^{4\alpha}.$$ 

Corollary 4: Let $\varphi(x) = x$ and $\psi(x) = 0$ for all $x \in [a, b]$, $f^{(IV)}(x)x \in \mathcal{C}[a, b]$. Then

$$|I - A_n^1| \leq \frac{L_0}{180} (b - a)h^4.$$ 

References


