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Functional Analysis and Energy Input/Output Modeling of a Real Pendulum

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Abstract

In a pendulum, if the weight of the rod and friction at bearings are considered, this is called a real pendulum. If a constant torque T acts on it which opposes the rotational motion, as in case of dynamic friction, it will dissipate energy as heat due to work done by friction and the amplitude of oscillation will gradually decrease. In this study, a real pendulum is taken up for analysis. Expressions are derived for angular velocity and angular acceleration with or without constant resisting torque acting on it. Further, the motion is analyzed to find gradually decreasing amplitudes in successive swings. As a next step, it is presumed that energy is manually supplied at one end after each one full swing of the pendulum so that the pendulum goes on oscillating between two extreme fixed deflection angles. The energy manually supplied at the end of each cycle is converted to work done by pendulum in the ensuing cycle.

Keywords: Real pendulum; swing; energy; angular velocity; angular acceleration.

1 INTRODUCTION

If the mass of the string / rod is not considered, it is a simple pendulum with time period of oscillation $T = 2\pi\sqrt{\frac{l}{g}}$. In the real pendulum the mass of the rod is also considered. In this study the mass of the rod is considered and a disc with the centre at the pivot point of the pendulum is also considered. Energy is manually imparted to the pendulum at one end of the swing and work output is obtained at the pin at the pivot point of the pendulum. Maintaining the proper energy balance, the manual energy input can be converted to work done at the end of the pivot pin. This energy output can be utilized in various ways such as running a compressor of an air-conditioner / refrigerator etc.

Nakamura et al. [1] proposed construction of a simulator for the real pendulum covering the upswing phase of the pendulum motion. Tarzo and Peranzoni [2] presented use of real time data acquisition with the aid of simple numerical simulations and included the analysis of both damping and large oscillations. Mathew et al. [3] elaborated on design of a controller for swinging up an inverted pendulum from downward equilibrium position to an upright unstable equilibrium position. Alvarez-Icaza [4] designed an IDA-PBC (Interconnection and damping assignment passivity based control) to stabilize an inverted pendulum in upright position. Okanouchi et al. [5] used a variable length pendulum with pivot movable in horizontal direction to control the dampening of oscillations. Xin Xin [6] worked on analysis of energy based swing up control of a cart – double pendulum system. Analysis of convergence of energy was used to show that in long run the energy of the system can be controlled to the energy at the up-right equilibrium position. Takhaashi et al. [7] showed that, by controlling the amplitude of sinusoidal signal provided at the pivot of a single link pendulum, the pendulum can be swung from the pendant position to the up-right position. Aoustin et al. [8] considered three unstable equilibrium positions of a two link pendulum. They contemplated stabilization of the system with a flywheel.

In this work, angular velocity and angular acceleration are calculated at different angles of the pendulum swing in all the four relevant sectors of the movement of the pendulum i.e. moving up and moving down on the right and on the left side of the equilibrium position. Considerations of the conservation of energy are applied and energy losses due to friction, available energy at the output point are calculated for utilization in useful work output.

2 SIMPLE PENDULUM

In a simple pendulum, the bob of mass M is connected to the pivot through a string or a mass less rod of length l (Fig.1). The forces acting on the bob, as shown in the figure are weight of the bob Mg and tension T in the rod. The restoring torque at an angle θ is provided by the component Mg $\sin\theta$ and the moment of inertia of the bob about the axis of rotation O is $M\lambda^2$.

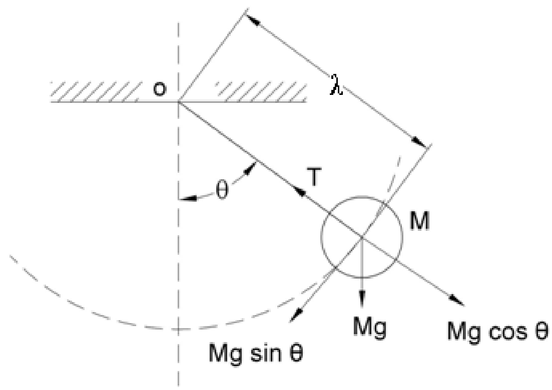

Fig.1: Simple pendulum

The governing equation for the system is:

$$I\alpha = -(Mg\sin\theta)\times\lambda$$

$$\alpha = -\frac{Mg\lambda}{M\lambda^2}.\theta \qquad (\theta \text{ is assumed small, thus } \sin\theta = \theta)$$

$$\alpha = -(\frac{g}{\lambda})\theta$$

This is angular simple harmonic motion with angular frequency $\omega = \sqrt{g/\lambda}$ and time period T $= 2\pi\sqrt{\lambda/g}$

3 REAL PENDULUM – WITHOUT RESISTING TORQUE

In case of a real pendulum, mass m of the rod is also considered and the restoring torque becomes

$$\tau = -Mg\lambda\sin\theta - mg\frac{\lambda}{2}\sin\theta$$

We may also consider a disc of mass m' and radius r attached to the rod and pivoted at O (Fig.2). Expressions will be derived for angular velocity and angular acceleration of the pendulum at a deflection θ based on principle of conservation of mechanical energy and linear relationship between torque and angular acceleration ($\tau = I\alpha$).

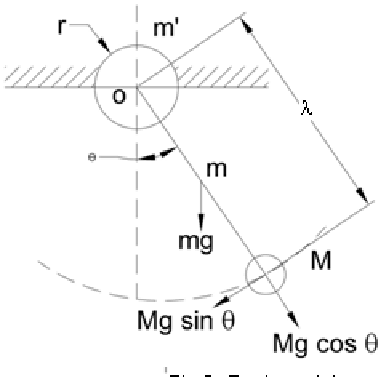

Fig.2: Real pendulum

3.1 Angular Velocity (ω)

To calculate angular velocity ω at an angular deflection θ, energy conservation principle can be applied as no non-conservative forces are acting on the system. The principle of conservation of mechanical energy envisages that

KE + PE = constant at any position

Considering that the pendulum was initially released from an angular deflection γ and taking lowest position of the bob as reference level with PE = 0, the above equation leads to (Fig.2)

$KE_1 + PE1 = KE_2 + PE_2$

$$O + Mg\lambda(1-\cos\gamma) + mg\frac{\lambda}{2}(1-\cos\gamma) =$$

$$\frac{1}{2}Mv^2 + \frac{1}{2}I'\omega^2 + \frac{1}{2}I''\omega^2 + Mg(1-\cos\theta) + mg\frac{\lambda}{2}(1-\cos\theta)$$

Where I', I" are moments of inertia of rod and disc respectively about the axis through 'O'. Rotational KE of bob about its own axis is neglected taking the size of the bob (radius) small.

Hence,

$$\left(M+\frac{m}{2}\right)g\lambda(cos\theta - cos\gamma) = \frac{1}{2}M\lambda^2\omega^2 + \frac{1}{2}\left(\frac{1}{3}m\lambda^2\right)\omega^2 + \frac{1}{2}(\frac{1}{2}m'r^2)\omega^2$$

$$= \left[\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m'r^2\right].\omega^2$$

or $\omega^2 = \frac{\left(M+\frac{m}{2}\right)g\lambda(cos\theta - cos\gamma)}{\frac{1}{2}M\lambda^2+\frac{1}{6}m\lambda+\frac{1}{4}m'r^2}$; or $\omega = [\eta\cos\theta - \mu]^{\frac{1}{2}}$ .... (1)

where $\eta = \frac{(M+\frac{m}{2})g\lambda}{\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m'r^2}$; $\mu = \frac{\left(M+\frac{m}{2}\right)g\lambda\cos\gamma}{\frac{1}{2}M\lambda^2+\frac{1}{6}m\lambda^2+\frac{1}{4}m'r^2}$

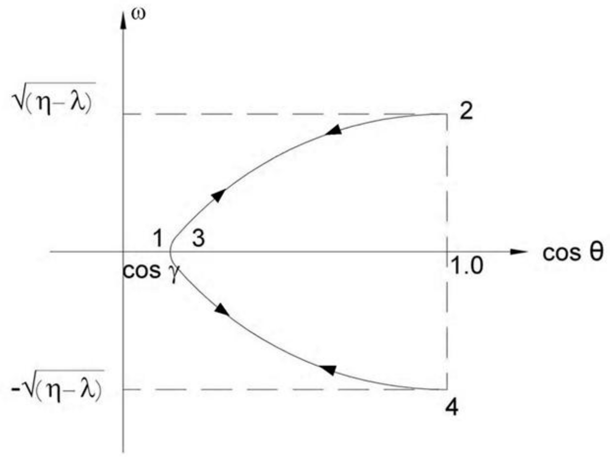

Fig 3(a): Angular velocity ω as a function of $\cos\theta$

The graph in Fig.3 (a) shows the variation of ω with $\cos\theta$ a parabolic curve. As the pendulum is released at position 1 with $\cos\theta = \cos\gamma$ and $\omega = 0$, the pendulum accelerates up to equilibrium position 2 with $\omega = \sqrt{\eta - \mu}$ following a parabolic curve. From 2 onwards θ increases in –ve direction and $\cos\theta$ decreases from 1.0 to position 3 where again $\omega = 0$ and $\cos\theta = \cos\gamma$. From 3 to 4, ω increases from 0 to $\omega = \sqrt{\eta - \mu}$ in opposite direction and $\cos\theta$ increases from cos γ to 1.0. From 4 to 1, θ increases and $\cos\theta$ decreases from 1.0 to cos γ and the value of ω becomes 0.

3.2 Angular Acceleration (α)

Angular acceleration α at an angular deflection θ is calculated from the formula $\tau = I\alpha$.

Restoring Torque = $Mg\lambda\sin\theta + mg\frac{\lambda}{2}\sin\theta$

$$\tau = -\left(M + \frac{m}{2}\right) g\lambda \sin\theta \qquad \text{(-ve sign is used as } \tau \text{ is opposite to } \theta\text{)}$$

Moment of Inertia, $I = M\lambda^2 + \frac{1}{3}m\lambda^2 + \frac{1}{2}m'r^2$; $\tau = I\alpha$

$$I\alpha = -\,(M + \frac{m}{2})g\lambda\sin\theta;$$

$$\alpha = -\frac{(M + \frac{m}{2})g\lambda\sin\theta}{M\lambda^2 + \frac{1}{3}m\lambda^2 + \frac{1}{2}m'r^2} \;\; ; \;\; \alpha = -\eta'\sin\theta \qquad \text{----------------------(2)}$$

Where $\eta' = \dfrac{(M + \frac{m}{2})g\lambda}{M\lambda^2 + \frac{1}{3}m\lambda^2 + \frac{1}{2}m'r^2}$

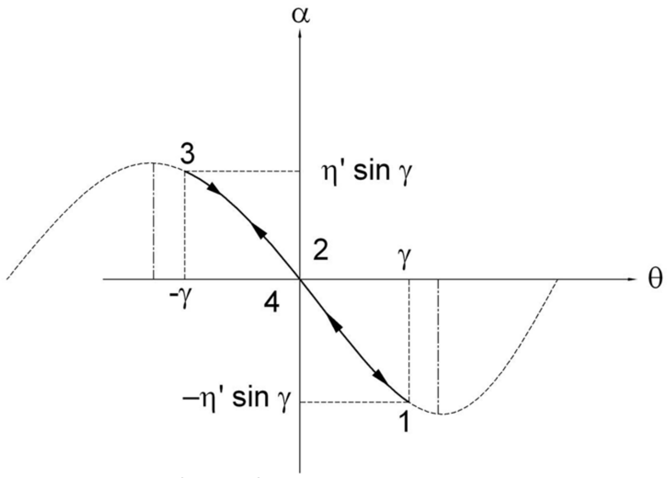

Fig 4: Angular acceleration α as a function of θ

The graph in figure 4 shows variation of angular acceleration α with θ.

The pendulum moves from position 1→2→3→4→1 as shown in figure 4 and value of α changes from $-\eta' \sin\gamma$ to 0 , 0 to $\eta' \sin\gamma$, $\eta' \sin\gamma$ to 0 and 0 to $-\eta' \sin\gamma$ as a sinusoidal function.

Example: Consider a real pendulum with following parameters.

Mass of bob (M) = 60 kg; Radius of disc(r) = 0.25 m; Mass of rod (m) = 0.5kg; Initial deflection (γ) = 80^0; Length of rod (λ) = 1m; Mass of disc (m') = 1kg

Angular velocity ω is given by the equation

$$\eta = \frac{\left(M + \frac{m}{2}\right) g\lambda}{\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{m'r^2}{4}} = \frac{60.25X9.81X1}{\frac{1}{2}X60X1 + \frac{1}{6}X0.5X1 + \frac{1X0.0625}{4}} = 19.64\ s^{-2}$$

$$\omega^2 = \eta \cos\vartheta - \mu$$

$$\mu = \frac{\left(M + \frac{m}{2}\right) g\lambda \cos\gamma}{\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m'r^2} = \frac{60.25X9.81X1Xcos\ 80}{\frac{1}{2}X60X1 + \frac{1}{6}X0.5X1 + \frac{1X0.0625}{4}} = 3.41\ s^{-2}$$

$$\omega^2 = 19.64\cos\theta - 3.411$$

For example, at $\theta = 30^0, \omega^2 = 19.64 \times 0.866 - 3.41 = 13.60$; $\omega = 3.69$ rad/s

Angular acceleration α is given by the equation $\alpha = -\eta' \sin\theta$.

$$\eta' = \frac{(M + \frac{m}{2})g\lambda}{M\lambda^2 + \frac{1}{3}m\lambda^2 + \frac{1}{2}m'r^2} = \frac{60.25 \times 9.81 \times 1}{60 \times 1 + \frac{1}{3} \times 0.5 \times 1 + \frac{1}{2} \times 1 \times 0.0625}$$

$$= 9.82\ s^{-2}$$

$$\alpha = -9.82\sin\theta$$

For example at $\theta = 30^0, \alpha = -9.82 \times \frac{1}{2}$; $\alpha = -4.91$ rad/s ^{2}

(-ve sign indicates that angular acceleration is in opposite direction to the deflection)

4 REAL PENDULUM WITH CONSTANT RESISTING TORQUE:

In this case it is assumed that a constant resisting torque T acts on the real pendulum described in section 3. This torque T may be due to dynamic friction or an external load at the pivot point. Angular velocity and angular acceleration for this model are derived below.

4.1 Angular Velocity (ω)

Since non-conservative forces (due to friction/external load) are also acting on the pendulum, we apply work energy theorem. The initial potential energy Mgl(1 – cos$Ø_0$) equals the new potential energy Mgl(1 – cos$Ø_1$) plus the energy lost due to the dry friction work C($Ø_{0\,+}$ $Ø_1$) (Torzo, Giacomo et al [3]).

ΔKE = work done on the system by all the forces

Sector 1 (Extreme Right to Equilibrium Position)

Movement of the pendulum is clockwise and resisting torque is counterclockwise. See figure 5

$$Workdone = -T(\gamma - \theta) + Mg\lambda[(1-\cos\gamma) - (1-\cos\theta)] + mg\frac{\lambda}{2}[(1-\cos\gamma) - (1-\cos\theta)]$$

$$W = -T(\gamma - \theta) + (M + \frac{m}{2})g\lambda(\cos\theta - \cos\gamma)$$

Change in K.E = K.E at angle θ = 0 = $\frac{1}{2}Mv^2 + \frac{1}{2}I^{'}\omega^2 + \frac{1}{2}I''\omega^2$

(I’ and I” are moments of inertia of the rod and the disc respectively about the axis through O)

ΔKE= $\frac{1}{2}M\lambda^2\omega^2 + \frac{1}{2}(\frac{1}{3}m\lambda^2)\ \omega^2 + \frac{1}{2}(\frac{1}{2}m' r^2)\omega^2 \ = [\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m' r^2]\omega^2$

Work done = ΔK.E

$$-T(\gamma - \theta) + (M + \frac{m}{2})g\lambda(\cos\theta - \cos\gamma) = [\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m' r^2]\omega^2$$

$$\omega_1 = [\eta\cos\theta + \zeta\theta - \mu]^{\frac{1}{2}} \qquad \text{---------------(3)}$$

Where $\eta = \dfrac{(M + \frac{m}{2})g\lambda}{\{\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m' r^2\}}$; $\varsigma = \dfrac{T}{\{\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m' r^2\}}$

$$\mu = \frac{T\gamma + \left(M + \frac{m}{2}\right)g\lambda\cos\gamma}{\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m' r^2}$$

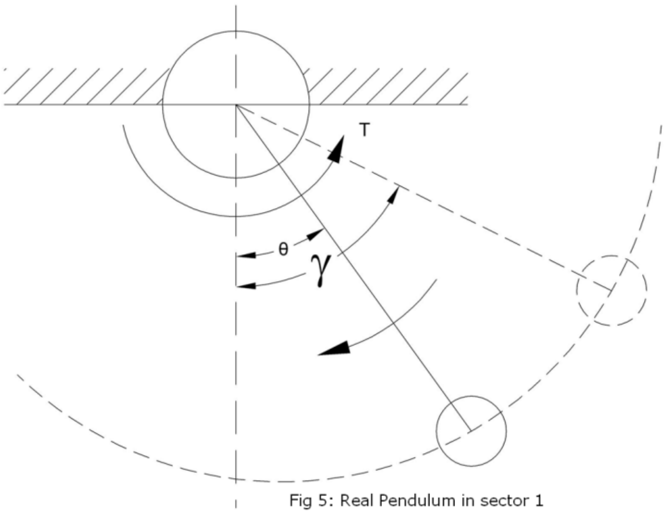

Fig 5: Real Pendulum in sector 1

Sector 2 (Equilibrium position to extreme left)

Movement of the pendulum is clockwise and torque T is counterclockwise. See figure 6.

Work done = - T $(\gamma+\theta)+Mg\lambda(\cos\theta-\cos\gamma)+mg\dfrac{\lambda}{2}(\cos\theta-\cos\gamma)$

K.E = $\dfrac{1}{2}Mv^2+\dfrac{1}{2}I'\omega^2+\dfrac{1}{2}I''\omega^2$ (I' and I" as explained earlier)

$$-T(\gamma+\theta)+(M+\frac{m}{2})g\lambda(\cos\theta-\cos\gamma)=[\frac{1}{2}M\lambda^2+\frac{1}{6}m\lambda^2+\frac{1}{4}m'r^2]\omega^2$$

$\omega_{2}=[\eta\cos\theta-\zeta\theta-\mu][\eta\cos\theta-\zeta\theta-\lambda]^{\frac{1}{2}}$ ---------------------(4)

(η, ς and μ are as explained in sector 1)

θ_1 : <u>Maximum angle in sector 2</u>

The pendulum stops at θ_1 angle in sector 2.

ΔKE= 0 (initial and final angular velocities are zero)

W = Mg $\lambda(\cos\theta_1-\cos\gamma)+mg(\lambda/2)(\cos\theta_1-\cos\gamma)-T(\gamma+\theta_1)$

$$(M+\frac{m}{2})g\lambda(\cos\theta_1-\cos\gamma)-T(\gamma+\theta_1)=0$$

--------------------(5)

$$[(M+\frac{m}{2})g\lambda]\cos\theta_1-(T)\theta_1-\{(M+\frac{m}{2})g\lambda\cos\gamma+T\gamma\}=0$$

θ_1 is calculated from the above equation by hit and trial method.

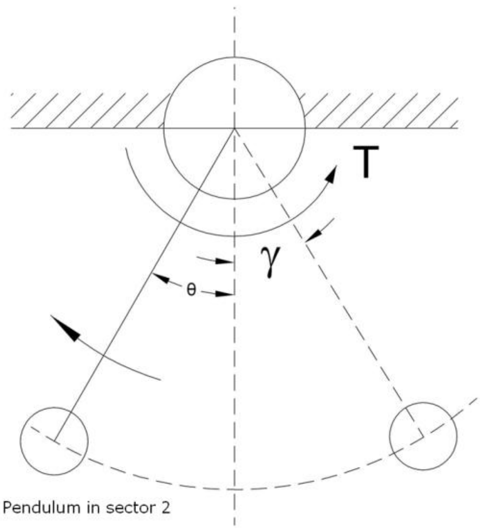

Fig 6: Real Pendulum in sector 2

Sector 3 (Extreme left to equilibrium position)

Movement of pendulum is counterclockwise and torque T is clockwise. See figure 7.

Work Done = $-T\{\gamma + \theta_1 + (\theta_1 - \theta)\} + Mg\lambda(\cos\theta - \cos\gamma) + mg\dfrac{\lambda}{2}(\cos\theta - \cos\gamma)$

$$= -T(\gamma + 2\theta_1 - \theta) + (M + \frac{m}{2})g\lambda(\cos\theta - \cos\gamma)$$

ΔKE = $[\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m'r^2]\omega^2$

Applying W = ΔKE

$\omega_3 = [\eta\cos\theta + \varsigma\theta - \lambda']^{\frac{1}{2}}$ ---------------------(6)

Where η and ς are as explained earlier and

$$\lambda' = \frac{T(\gamma + 2\theta_1) + (M + \frac{m}{2})g\lambda\cos\gamma}{\{\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m'r^2\}}$$

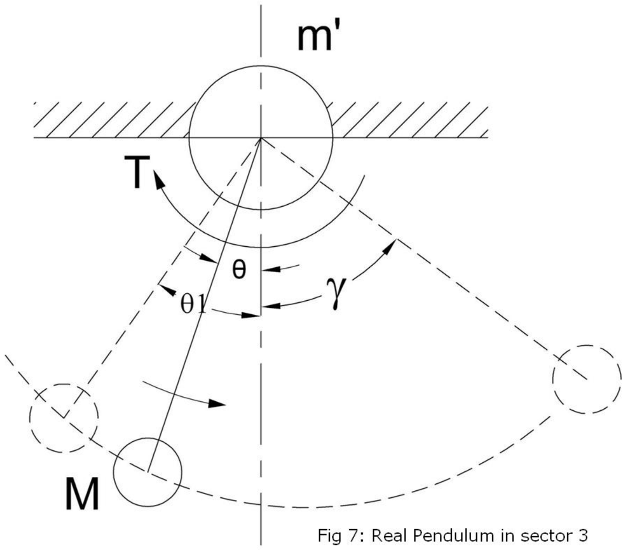

Fig 7: Real Pendulum in sector 3

Sector 4 (Equilibrium position to extreme right)

Movement of the pendulum is counterclockwise and torque T is clockwise (Ref.figure 8).

Work Done $= -T(\gamma + 2\theta_1 + \theta) + (M + \frac{m}{2})g\lambda(\cos\theta - \cos\gamma)$

K.E=$[\frac{1}{2}M\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m'r'^2]\omega^2$

$$\omega_4 = [\eta\cos\theta - \varsigma\theta - \lambda']^{\frac{1}{2}} \quad \text{------------(7)}$$

Where η, ς and λ' are as defined earlier

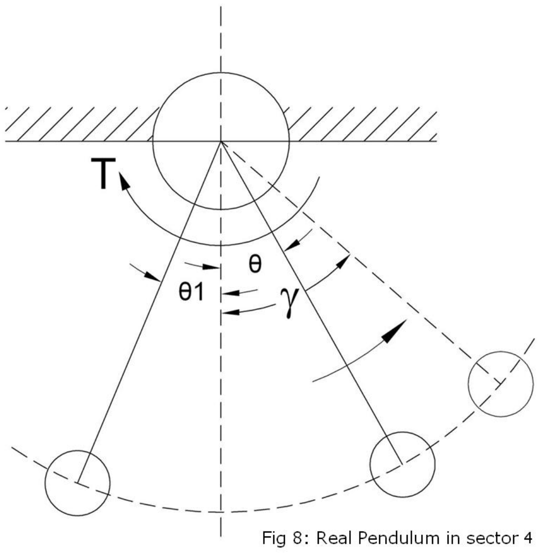

Fig 8: Real Pendulum in sector 4

(b) Angular Acceleration (α)

Sector 1: (Extreme right to equilibrium position). See figure 9.

Torque= $-[Mg\lambda\sin\theta + mg\frac{\lambda}{2}\sin\theta - T]$

I= $M\lambda^2 + \frac{1}{3}m\lambda^2 + \frac{1}{2}m'r^2$

$$I\alpha = \tau$$

$$\alpha_1 = \frac{\tau}{I} = -\frac{(M+\frac{m}{2})g\lambda\sin\theta - T}{[M\lambda^2 + \frac{1}{3}m\lambda^2 + \frac{1}{2}m'r^2]}$$

$$\alpha_1 = -\chi\sin\theta + \frac{T}{I} \qquad \text{-------------(8)}$$

Where $\chi = \frac{(M+\frac{m}{2})g\lambda}{I}$ and $I = M\lambda^2 + \frac{1}{3}m\lambda^2 + \frac{1}{2}m'r^2$

Sector 2 (Equilibrium position to extreme left

Torque $= Mg\lambda\sin\theta + mg\frac{\lambda}{2}\sin\theta + T$

$$\alpha_2 = \frac{(M+\frac{m}{2})g\lambda\sin\theta}{I} + \frac{T}{I}$$

$$\alpha_2 = \chi\sin\theta + \frac{T}{I} \qquad \text{--------------------(9)}$$

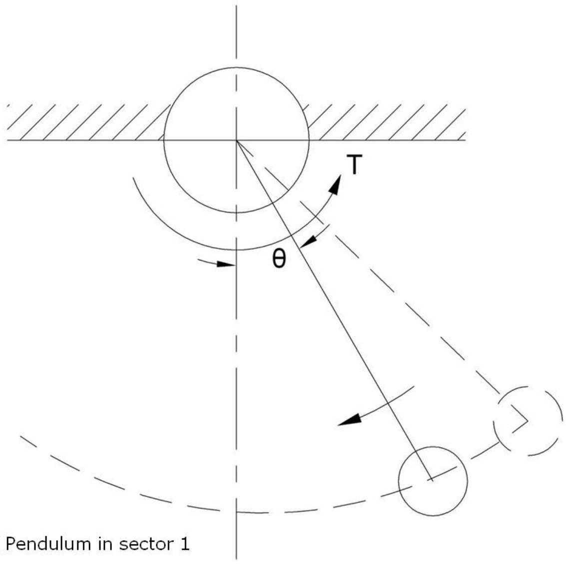

Fig 9: Real Pendulum in sector 1

Sector 3 (Extreme left to equilibrium position)

$$\text{Torque} = Mg\lambda \sin\theta + mg\frac{\lambda}{2}\sin\theta - T$$

$$\alpha_3 = \chi \sin\theta - \frac{T}{I} \qquad \text{--------------------(10)}$$

Sector 4 (Equilibrium to extreme right position)

$$\text{Torque} = -(M + \frac{m}{2})g\lambda \sin\theta - T$$

$$\alpha_4 = -\chi \sin\theta - \frac{T}{I} \qquad \text{----------(11)}$$

Example:

For the pendulum taken up for example, assuming torque applied is 8 Nm and initial deflection γ is 80^0 (1.396 radians)

For sector 1, $\omega_1 = [\eta \cos\theta + \zeta\theta - \mu]^{\frac{1}{2}}$

$\eta = 19.64 s^{-2}$; $\zeta = \frac{8}{30.099}$ =0.266 s^{-2}; µ=3.79 s^{-2}

$$\omega_1 = [19.64\cos\theta + 0.266\theta - 3.79]^{\frac{1}{2}}$$

For example, for $\theta = 30^0$ (0.523 radians), $\omega_1 = 3.65$ rad/s

For sector 2, $\omega_2 = [\eta \cos\theta - \varsigma\theta - \lambda]^{\frac{1}{2}}$

$$\omega_2 = [19.64\cos\theta - 0.266\theta - 3.79]^{\frac{1}{2}}$$

For example, for θ=30^0, $\omega_2 = 3.62$ rad/s

For Sector 3

$$\omega_3 = [\eta\cos\theta + \varsigma\theta - \lambda']^{\frac{1}{2}}$$

$$\lambda' = \frac{T(\gamma + 2\theta_1) + (M + \frac{m}{2})g\lambda\cos\gamma}{\frac{1}{2}m\lambda^2 + \frac{1}{6}m\lambda^2 + \frac{1}{4}m'r^2}$$

To find θ_1, we use the equation

$$[(M + \frac{m}{2})g\lambda]\cos\theta_1 - (T)\theta_1 - \{(M + \frac{m}{2})g\lambda\cos\gamma + T\gamma\} = 0$$

The solution gives $\theta_1 = 77.8 = 1.358 rad$

$$\lambda' = \frac{8(1.396 + 2\times 1.358) + 60.25\times 9.81\times 1\times \cos 80^0}{30.099} = 4.38\ \text{s}^{-2}$$

$$\omega_3 = [19.07\cos\theta + 0.258\theta - 4.38]^{\frac{1}{2}}$$

For example at $\theta = 30^0$ (0.523 rad.), $\omega_3 = 3.50 rad/s$

For Sector 4, $\omega_4 = [\eta\cos\theta - \varsigma\theta - \lambda']^{\frac{1}{2}}$

$$\omega_4 = [19.07\cos\theta - 0.258\theta - 4.38]^{\frac{1}{2}}$$

For example, for $\theta = 45^0$, $\omega_4 = 2.98$ rad/s

Angular Acceleration,

$\alpha_1 = -\chi\sin\theta + T/I$ (sector 1)

$$\chi = \frac{(M + \frac{m}{2})g\lambda}{I} \qquad I = M\lambda^2 + \frac{1}{3}m\lambda^2 + \frac{1}{2}m'r^2, \quad I = 60.198$$

$$\chi = \frac{60.25\times 9.8\times 1}{60.198} = 9.82$$

$$\frac{T}{I} = \frac{8}{60.198} = 0.133$$

For sector 1

$\alpha_1 = -9.82 \sin\theta + 0.133$

For example, for $\theta = 60^0$, $\alpha_1 = -8.37$ rad/s^2

At $\theta = \gamma = 80^0, \alpha_1 = -9.82 \sin 80^0 + .133 = -9.54 rad/s^2$

$\alpha_1 = -9.54 rad/s^2$ (max.)

For sector 2

$\alpha_2 = \chi \sin\theta + \frac{T}{I}$ or $\alpha_2 = 9.82 \sin\theta + 0.133$

For example for $\theta = 30^0$, $\alpha_2 = 8.64 rad/s^2$

$For \theta = \theta_1 = 77.8^0 (1.358 rad.)$, $\alpha_2 = 9.73 rad/s^2$ (max.)

For sector 3

$\alpha_3 = \chi \sin\theta - T/I = 9.82 \sin\theta - 0.133$

For example, for $\theta = 30^0, \alpha_3 = 4.78 rad/s^2$

For $\theta = \theta_1 = 77.8^0, \alpha_3 = 9.82 \times \sin 77.8^0 - 0.133 = 9.47 rad/s^2$

$\alpha_3 = 9.47 rad/s^2$ (max.)

For sector 4

$\alpha_4 = -x \sin\theta - T/I$, $\alpha_4 = -9.82 \sin\theta - 0.133$

For example, for $\theta = 45^0$, $\alpha_4 = -7.08 rad/s^2$

To find θ_2 (extreme angle on right after completing one full to and fro swing after release), we apply work energy theorem, $\Delta KE = 0$ (Initial and final speeds are zero).

Hence, total work done= $\Delta KE = 0$

$(M + \frac{m}{2}) g\lambda(\cos\theta_2 - \cos\gamma) - T(\gamma + 2\theta_1 + \theta_2) = 0$

$$[(M+\frac{m}{2})g\lambda]\cos\theta_2-(T)\theta_2-[(M+\frac{m}{2})g\lambda\cos\gamma+T\gamma+2T\theta_1]=0$$

$$591.05\cos\theta_2-8\theta_2-[60.25\times9.81\times0.174+8\times1.396+2\times8\times1.358]=0$$

Solution gives $\theta_2=75.7^0$

$At\ \theta_2=75.7^0;\ \alpha_4=-9.82\sin75.7^0-0.133$ = -9.65rad/s^2 (max.)

Graph showing variation of angular acceleration α with θ is given in figure 10.

α increases from -9.54 rad/s^2, at position 1 to +9.73 rad/s^2 at 3 and from 9.47 rad/s^2 from 3' (downward motion) to -9.65 rad/s^2 at position 5 (2 and 4 are equilibrium positions with θ = 0 and $\alpha=\pm T/I$.

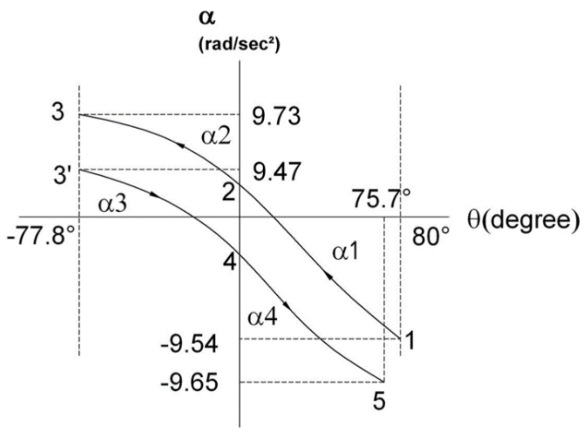

Fig 10: Variation of angular acceleration α with θ

5 EXTREME ANGULAR DEFLECTION WITH RESISTING TORQUE APPLICATION:

To calculate extreme angular deflection on either side during each swing while a constant resisting torque is applied, the work energy theorem is applied.

ΔKE = Work done by all forces.

Here the resisting torque T may represent torque due to frictional forces or torque due to an external load being driven by the pendulum.

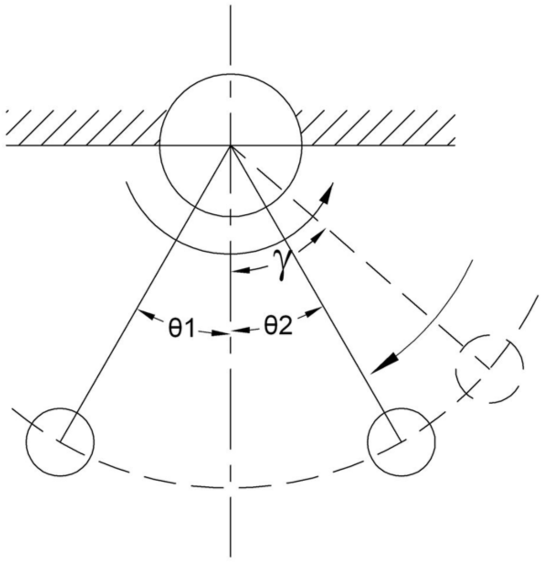

Fig 11:Extreme positions of Pendulum when released from deflection γ

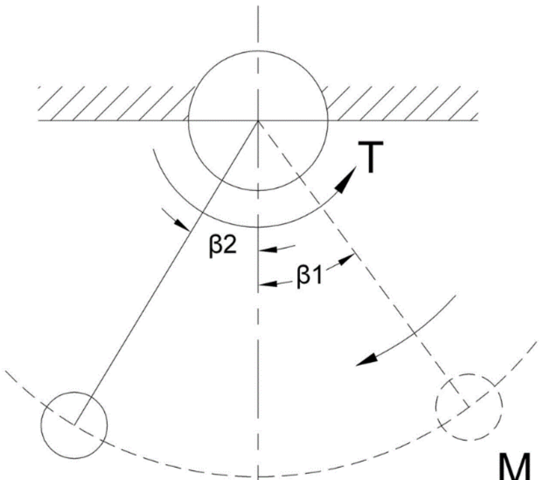

Fig 12: Pendulum swinging between two extreme deflections

The pendulum is released from an initial angular deflection γ. It goes up to an angle θ_1 on extreme left and subsequently up to an extreme angular deflection θ_2 on extreme right position (See figure 11). Successive maximum angular deflections at the end of swings will be $\theta_{3,}$ θ_4 and so on.

θ_1 (77.8$^{0)}$ and θ_2 (75.7$^{0)}$ have been calculated in preceding section.

If a pendulum swings from an initial known angle β_1 (released from rest) to an extreme angle β_2 (where it comes to a stop) on other side under a constant opposite torque (See figure 12), then β_2 can be calculated by applying work energy theorem

ΔKE = Total work done

$$0 = Mg\lambda(\cos\beta_2 - \cos\beta_1) + mg\frac{\lambda}{2}(\cos\beta_2 - \cos\beta_1) - T(\beta_1 + \beta_2)$$

$$[(M + \frac{m}{2})g\lambda]\cos\beta_2 - (T)\beta_2 - [(M + \frac{m}{2})g\lambda\cos\beta_1 + T\beta_1] = 0$$

For θ_1:

For initial angle $\gamma = 80^0, \theta_1$ is calculated from

$$60.25\times 9.81\times 1\cos\theta_1 - 8\theta_1 - [60.25\times 9.81\times 1\cos 80^0 + 8\times 1.396] = 0$$

which gives the solution $\theta_1 = 77.8^0$

For θ_2:

$$591.05\cos\theta_2 - 8\theta_2 - [591.05\cos 77.8^0 + 8\times {}^{77.8}\!/_{57.31}] = 0$$

which gives the solution $\theta_2 = 75.7^0$

For θ_3:

$$591.05\cos\theta_3 - 8\theta_3 - [591.05\cos 75.7^0 + 8\times\frac{75.7}{57.31}] = 0$$

which gives the solution $\theta_3 = 73.6^0$

For θ_4:

$$591.05\cos\theta_4 - 8\theta_4 - [591.05\cos 73.6^0 + 8\times\frac{73.6}{57.31}] = 0$$

which gives the solution $\theta_4 = 71.6^0$

For θ_5:

$$591.05\cos\theta_5 - 8\theta_5 - [591.05\cos 71.6^0 + 8\times\frac{71.6}{57.31}] = 0$$

which gives the solution $\theta_5 = 69.6^0$

For θ_6:

$$591.05\cos\theta_6 - 8\theta_6 - [591.05\cos 69.6^0 + 8\times\frac{69.6}{57.31}] = 0$$

which gives the solution $\theta_6 = 67.6^0$

Further diminishing extreme angles can be calculated on the same lines.

It is observed that during each swing, the maximum angular deflection reduces by approximately 2°; this is due to the negative work done by the constant torque T.

6 WORK DONE BY TORQUE

Work done by torque on a given mechanism is given by $W = \int_{\theta_1}^{\theta_2} \tau \, d\theta$

where under torque τ , the link rotates from angle θ_1 to θ_2 .

If the torque is constant, the above formula can be described as $W = \tau \ \ \Delta\theta$ where $\Delta\theta$ is total angle moved.

In case of the pendulum taken up in Example,

For angle γ to θ_1 : (θ_1: extreme angle on the other side)

$W_1 = -T(\gamma + \theta_1) = -8(\frac{80 + 77.8}{57.31})$ = -22.032 J

For angle θ_1 to θ_2 : (θ_2 on other side), $W_2 = -8 \times (\frac{77.8 + 75.7}{57.31})$ = -21.43 J

For angle θ_2 to θ_3 : (θ_3 on other side), $W_3 = -8 \times (\frac{75.7 + 73.6}{57.31})$ = -20.84 J

For angle θ_3 to θ_4 : (on other side), $W_4 = -8 \times (\frac{73.6 + 71.6}{57.31})$ = -20.27 J

For θ_4 to θ_5 : (on other side), $W_5 = -8 \times [\frac{71.6 + 69.6}{57.31}]$ =-19.71 J

For angle θ_5 to θ_6 : (on other side), $W_6 = -8 \times (\frac{69.6 + 67.6}{57.31})$ = -19.15 J

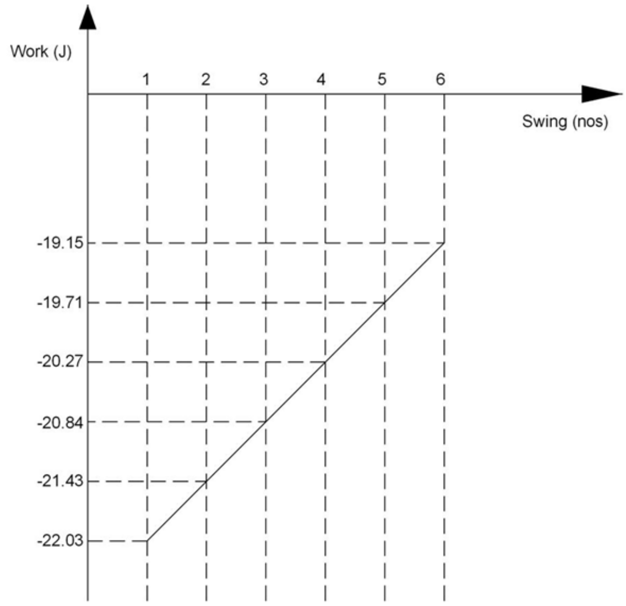

Fig 13: Graph showing work done in each swing

Graph showing work done against each swing is given in figure 13.

The relationship is observed to be linear as, in each swing, approximately 2º angular deflection is lost which is more or less constant.

7 ENERGY SUPPLIED TO COMPENSATE FOR WORK DONE

We can now consider a case in which energy is supplied manually to the pendulum at the end of each cycle which is equal to the work done by the pendulum (or the negative work done by the resistive torque T on the pendulum) in one cycle.

Suppose the pendulum is released from angle γ, energy is supplied to it during initial angular deflection α and it goes up to an extreme angle θ on the other side. In the return journey, it deflects to the same angle γ at which it had started.

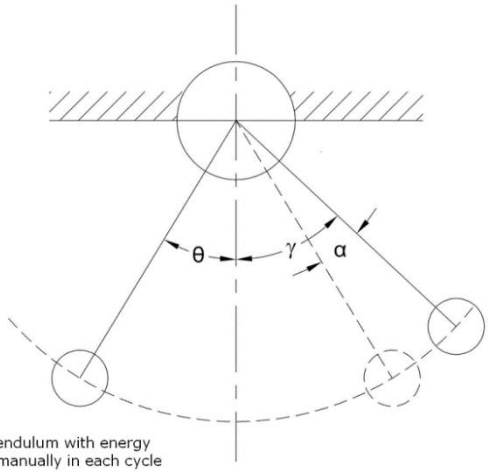

Fig 14: Pendulum with energy supplied manually in each cycle

Thus the pendulum will go on oscillating between extremes γ and θ. See figure 14.

The energy supplied to it between angles γ and $(\gamma-\alpha)$ is equal to the –ve work done by the constant resistive torque T. If F is the manually applied force while supplying energy during deflection through the angle α,

Work done by manual force = $\tau\ \alpha$ (in one cycle) $= F\ \lambda\alpha$

Work done by torque T = $-T(\theta+\gamma)$ (in one cycle)

Since initial and final speeds are zero and work done by gravity is zero in one cycle,

$$-T(\theta+\gamma)+F\lambda\alpha=0$$

The above equation gives relationship between F, T, α, θ and γ.

Example:

For the pendulum taken up in examples, assume that force applied by the man is 10 x 9.81 = 98.1 N. It is further assumed that this force is applied during deflection of 10^0 (0.174 radians), the work done by the man will be:

$$W = F\lambda\alpha = 98.1\times1\times0.174\ \text{J} \quad =17\,J$$

Hence the work done by the torque is

$$2T(\gamma+\theta)=17 \text{ or } 2\times8(\gamma+\theta)=17$$

$$\gamma+\theta=\frac{17}{16}=1.063\ \ radians\,(=1.063\times57.31=60.89^{0}\ ) \qquad \text{----------(9)}$$

To determine γ and θ separately, we consider the return journey in which the pendulum swings from deflection θ to γ under the constant resistive torque T and gravity.

Applying work energy theorem,

$\Delta KE = 0$ as initial and final speeds are zero

$$W = -T(\theta + \gamma) + (M + \frac{m}{2})g\lambda(\cos\gamma - \cos\theta) = 0$$

$$60.25 \times 9.81 \times (\cos\gamma - \cos\theta) = 8 \times 1.063 = 8.504$$

$$\cos\gamma - \cos\theta = \frac{8.504}{591.05} = 0.0144 \qquad \text{----------(10)}$$

Solving equation (9) and (10), $\gamma = 29.63^0$ and $\theta = 31.26^0$.

The pendulum is released initially at an angular deflection 29.63^0, energy 17J is supplied to it during 10^0 deflection at the beginning of the cycle, it swings up to a maximum deflection of 31.26^0 on the other side and comes back to the same starting point.

8 CONCLUSION

A simple pendulum, if deflected by a small angle, performs angular simple harmonic motion. When mass of the suspension rod is also significant, it is called a physical or real pendulum.

In the present study, motion of a real pendulum having significant masses of bob, rod and an integral disc is analyzed to find out angular velocity and angular acceleration at different angles of deflection when approached from two sides, towards or away from equilibrium position. Analysis is carried out in both conditions – with and without a constant resistive torque. Finally, work done by the torque is calculated for each sector of swing of the pendulum.

The device can be used for giving work output from the pendulum by providing manual energy input at the start of each cycle. It can be used for various purposes like driving a machine, achieving repetitive motion from a mechanism or storing energy to be utilized at an appropriate time.

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