International Journal of Theoretical and Applied Mechanics.
ISSN 0973-6085 Volume 13, Number 4 (2018) pp. 259-270

http://www.ripublication.com

Approximate Solution of Externally Pressurized Functionally Graded Thick Cylinder

Lakshman Sondhi[1, b], Shubhashis Sanyal[2, c] and Shubhankar Bhowmick[2, a]

[1]*Department of Mechanical Engineering, Shri Shankaracharya Group of Institutes, Bhilai, India.*

[2]*Department of Mechanical Engineering, National Institute of Technology, Raipur 492010, India.*

[a)] *Corresponding author*

Abstract

The present work employs variation principle to investigate the deformation and stresses of externally pressurized thick cylinder made of functionally graded material is carried out for different aspect ratio and different gradient index. The gradient index varies from 2 to -2 and aspect ratio considered as 0.3, 0.5 and 0.9. In the present study, the mechanical property of FG is assumed to be exponential functions. Deformation and stress distributions along the radial direction are studied. An approximate solution is obtained and the results are reported in normalized form for different values of gradient index and plotted graphically. The computed results are compared to those published in the literature show good agreement and superiority.

Keywords: Externally pressurized, functionally graded materials, Stress analysis, Numerical technique, Thick cylinder

NOMENCLATURE

a	Internal radius of the cylinder.
b	Outer radius of the cylinder
c_i	The vector of unknown coefficients
h	Thickness of the cylinder
n	Gradient Index
r,θ,z	Radial, tangential and axial directions
u	Displacement field of the cylinder (dimensionless form $\bar{u} = uE/b\, q_{bb}$)

E	Elasticity modulus
E_o	Constant Elasticity modulus
$\overline{E_r}$	Elasticity modulus of the cylinder material in the radial direction (Normalized form $\bar{E} = \frac{E(r)}{E}$)
U	Strain energy of the cylinder
V	Potential energies of the cylinder
Δ	The variational operator
μ	Poisson's ratio
n	Gradient Index
ε_r , ε_θ	Strains in radial and tangential direction respectively
σ_r,σ_θ	Stress in radial and tangential direction respectively
$\overline{\sigma_r}$, $\overline{\sigma_\theta}$	Normalized stress in radial and tangential direction respectively $\bar{\sigma} = \frac{\sigma(r)}{Pi}$
σ_o	Yield stress of the cylinder material
$\overline{\sigma_y}$	Normalized yield stress of the material
α_i	The set of orthogonal polynomials used as coordinate functions.
ξ	Normalized radial co-ordinate $({r-a}/{b-a})$
$\bar{r}$	(b-a)
q_b	Outer pressure

INTRODUCTION AND LITERATURE REVIEW

In functionally graded materials, the volume fraction of two materials, usually ceramic-metal mixture is varied continuously as a function of position from one point to another along certain dimension(s). The composition change makes it possible to get the desired change in material properties with variation of the volume fraction. In addition to superb heat properties, they are corrosion and erosion resistant and have high fracture resistance. The basic concept is to mix the ceramic and metal such that the material properties continuously vary from one constituent material to the other. Recently, investigation for FGMs has received considerable attention by many researchers. In the last two decades, FGMs have been widely used in engineering applications, particularly in high-temperature environment, microelectronic, power transmission equipment, etc. On the other hand, hollow cylinders are a kind of typical structures as vessels and pipes which are utilized in reserving or transferring chemical gas, oil, etc. Due to the importance of the structural integrity, the safe design of such hollow cylindrical structures, in particular for functionally graded hollow annuli or tubes have attracted considerable attention in recent years. The closed –form solution for stresses and displacement in functionally graded cylindrical and spherical vessels subjected to internal pressure alone are obtained using the infinitesimal theory of elasticity by Tununcu, N. et al 2001 .Stress analysis of multilayered pressure vessels possessing cylindrical anisotropy and under internal, external and inter laminar pressure is given by Verijenko, V.E. et al., 2001.The thermo mechanical states in a class of functionally graded cylinders under extension, torsion, shearing, pressuring and temperature changes in investigated by Tarn, J.Q., 2001.A general analysis of one- dimensional steady-state thermal stresses in a hollow thick cylinder made of functionally graded material is developed by Jabbari,M. et al., 2001. By using a multi-layered approach based on the theory of laminated composites, the solution of temperature, displacement, and thermal/mechanical stresses in a functionally graded circular hollow cylinder are investigated by Shao, Z.S., 2005..Two different kinds of heterogeneous elastic hollow cylinders. One is a multi-layered cylinder with different values in different layers for both elastic modulus and Poisson’s, the exact solutions of the multi-layered structure can be found based on Lame’s solution by Xiang, H.et al 2001. Power series solutions for stresses and displacement in functionally-graded cylindrical vessels subjected to internal pressure alone are obtained using the infinitesimal theory of elasticity by Tutuncu, N., 2007. An analytical procedure for evaluation of elastic stresses and strains in non-linear variable thickness disks, subjected to thermal load, and having a fictitious density variation along the radius is presented by Vullo,V. et al 2001. Novel method for analyzing steady thermal stresses in a functionally graded hollow cylinder is investigated by Peng, X.L. et al 2001.Elastic- Plastic deformation of a solid cylinder with fixed ends, made of functionally graded material (FGM) with inform internal heat generated based on Tresca’s yield criterion is investigated by Ozturk, A. et al 2001 .Analytical solution for heat conduction in a cylindrical multilayer composite laminate in which the fiber direction may vary between layers is presented by Kayhani, M.H. et al 2001. Analytical solution of the macro mechanical model useful for evaluating the stress and the strain distribution produced by the press fit operation in orthotropic cylinders is investigated by Croccolo, D., 2013.. An analytical approach on the nonlinear response of thick functionally graded circular cylindrical shells with temperature independent material property surrounded on elastic foundation subjected to mechanical and thermal loads is presented by Duc, N.D. et al 2001 . Variational principle to investigate the stress and deformation states and estimate the limit angular speed of functionally graded high-speed rotating disks is investigated by Sondhi, L. et al 2016 . The linear elastic analysis of variable thickness clamped boundary condition at the inner surface and the free boundary condition at the outer surface of rotating disks made of functionally graded materials by the finite element method is investigated by Thawait, A.K. et al 2017 .

The present work reports the elastic analysis of FG cylinder is carried out, change in stresses and displacement states of cylinder with functional variation in material properties. The problem is modeled by using variational principle, taking the radial displacement field as the unknown dependent variable. Assuming, in such cylinder, series approximation Following Galerkin's principle, the solution of the governing equation is obtained. The application of variational principle yields advantage over classical approaches in terms of simplicity, being an integral formulation, and ease, due to the resulting algebraic solution. The validation of the present numerical scheme is carried out with existing literature. The relevant results are reported in dimensionless form, and hence one can readily obtain the corresponding dimensional value for any geometry through appropriate normalizing parameters. Numerical results are given which are useful for engineer to design a cylinder made of functionally graded materials.

MATHEMATICAL FORMULATION

The mathematical formulation of a thick cylinder of inner radius, 'a', outer radius, 'b', length 'h_0'and having functionally graded material property is analyzed under plane strain assumption (ε_z=0).

It is subjected to external loading q_b on the outer circular boundary, thus producing radial and tangential strain field. Due to application of externally loading at the outer boundary, radial displacements will occur, the magnitude of which is governed by the boundary conditions of the cylinder

The expression of strain energy U is given by

$$U = \frac{1}{2}\int_{Vol} (\sigma\varepsilon)\, dv = \frac{1}{2}\int_{Vol} \left(\sigma_\theta \varepsilon_\theta + \sigma_r \varepsilon_r\right) dv \tag{1}$$

While external work, V is given by,

$$V = -(2\pi a h q_b)\, u\big|_b . \tag{2}$$

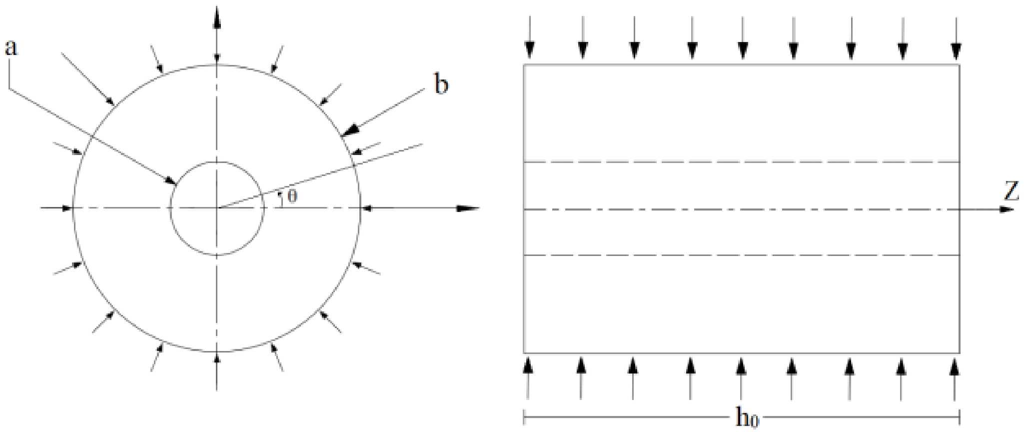

Fig.1 Externally pressurized thick Cylinder Geometry

In the present work mechanical property posssion ration μ = 0.3 and modulus of elasticity varies exponentially functions has a property [17].

$$
\begin{aligned}
&E(r) = E_O exp(n(r-a)/(b-a)) \\
&E(r)_{r=a} = E_o \\
&E(r)_{r=b} = E_o exp(n)
\end{aligned}
\tag{3}
$$

Eo is a constant and variation of material index n takes from -2 to 2 and represents the property of FGMs and is called gradient parameter

The strain-displacement relations are given by

$\varepsilon_r = du/dr$ and

$$\varepsilon_\theta = u/r \tag{4}$$

On substituting Esq.(4). in Eq.1, the expression for strain energy becomes,

$$U = \frac{\pi}{(1-2\mu)(1+\mu)} \int_a^b \left[(1-\mu)\left(E(r)\frac{u^2}{r} + E(r)r\left(\frac{du}{dr}\right)^2 \right) + 2\mu E(r)(u)\left(\frac{du}{dr}\right) \right] drh \tag{5}$$

Based on Minimum Potential Energy principle Eq. the solution for the displacement field is obtained.

$$\delta\ (U+V) = 0, \tag{6}$$

Substituting Eq. (2) and Eq. (5) in (Eq.) the governing equation takes the following form

$$\delta \left[\frac{\pi}{(1-2\mu)(1+\mu)} \int_a^b \left[(1-\mu)\left(E(r)\frac{u^2}{r} + E(r)\frac{2\mu(u)}{(1-\mu)}\left(\frac{du}{dr}\right) + rE(r)\left(\frac{du}{dr}\right)^2 \right) \right] dr - \int_a^b uq_b 2\pi r dr \right] = 0 \tag{7}$$

Eq.7 is expressed in normalized co-ordinate (ξ) and non-dimensional space $\xi = \frac{(r-a)}{(b-a)}$ to facilitate the numerical computation work and the governing equation is expressed as

$$\delta\left[\frac{\pi\bar{r}}{(1-2\mu)(1+\mu)} \int_0^1 \left\{ E(r)(1-\mu)\frac{u^2}{(\bar{r}\xi + a)} + E(r)\frac{2\mu\, u}{\bar{r}}\left(\frac{du}{d\xi}\right) + E(r)\frac{(\bar{r}\xi + a)}{\bar{r}^2}(1-\mu)\left(\frac{du}{d\xi}\right)^2 \right\} hd\xi \right.$$

$$\left. -2\pi ahq_b\, u\big|_0 \right] = 0 \tag{8}$$

The displacement functions $u(\xi)$) in Eq.8 can be approximated by a linear combination of sets of orthogonal coordinate functions as

$$u(\xi) = \sum c_i \alpha_i \quad , i = 1, 2\ldots n. \tag{9}$$

The set of orthogonal functions φi are developed through Gram-Schmidt scheme. The starting function α_i , necessary in the first hand, is so selected as to satisfy the boundary conditions. The boundary condition for externally pressurized thick cylinder take the following form ($\sigma_r\big|_{(r=b)}=-p_b$ and $\sigma_r\big|_{(r=a)}=0$) and expressed as expressed as follows (Srinath (17)).

$$\alpha_0(r)=\frac{q_b a^2(1+\mu)}{E(r)(b^2-a^2)}\left[(\frac{b^2}{r})+(1-2\mu)r\right] \tag{10}$$

The governing differential equation is obtained in matrix form by substituting Eq.9 in Eq 8.

$$\delta\left[\left(\frac{\bar{r}}{(1-2\mu)(1+\mu)}\right)\int_0^1\left\{E(r)(1-\mu)\frac{\left(\Sigma c_i\alpha_i\right)^2}{(\bar{r}\xi+a)}+E(r)\frac{2\mu}{\bar{r}}\left[\left(\Sigma c_i\alpha_i\right)\frac{d}{d\xi}\left(\Sigma c_i\alpha_i\right)\right]+E(r)\frac{(\bar{r}\xi+a)(1-\mu)}{\bar{r}^2}\left[\frac{d}{d\xi}\left(\Sigma c_i\alpha_i\right)\right]^2\right\}hd\xi \right. \tag{11}$$

$$\left. -2ahq_b c_i\alpha_i\big|_0\right]=0$$

In Eq.11 the operator 'δ' by $\partial/\partial c_j$, j = 1, 2, n, according to Galerkin's error minimization principle, we obtain

$$\left(\frac{\bar{r}}{(1-2\mu)(1+\mu)}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}c_i\int_0^1 E(r)\left\{\frac{\alpha_i\varphi_j}{(\bar{r}\xi+a)}+\frac{\mu}{\bar{r}}(\alpha_i'\alpha_j+\alpha_i\alpha'_j)+\frac{(\bar{r}\xi+a)}{\bar{r}^2}\alpha_i'\alpha'_j\right\}hd\xi = q_b a h_0\, c_i\alpha_i\big|_0 \tag{12}$$

Where $(\;)'$ indicates differentiation with respect to normalized coordinate ξ.

RESULT AND DISCUSSIONS

The numerical values of E_0, and σ_{yo} for the cylinder material at the outer surface is taken as 210 GPa, and 350 MPa respectively and the results are evaluated for outer pressure of 100 MPa. The results are plotted reported for grading index, n ranging 2 to -2. The proposed methodology is validated with Chen and Lin (2008) and very good agreement is reported. Following normalized variables are used.

$$\bar{r}=\frac{r-a}{b-a},\ \bar{E}=\frac{E(r)}{E_0},\ \ \bar{\sigma}=\frac{\sigma(r)}{p},\ \bar{\sigma}_y=\frac{\sigma_y(r)}{\sigma_{y0}},\ \ \bar{u}=\frac{u\,E(r)}{b\,p},\quad \bar{\sigma}=\frac{\sigma}{\sigma_{y0}}$$

In Fig.2, the variation of material properties modulus of elasticity for different grading index is plotted. It is evident from Eq.(3) and Fig.2, that positive n value results in cylinder with increasing modulus along the radius while negative n ensures that the cylinder is stronger at the inner radius. The results are calculated for externally pressurized cylinder for different aspect ratio (a/b) 0.3, 0.5 and 0.9 at different grading index, n

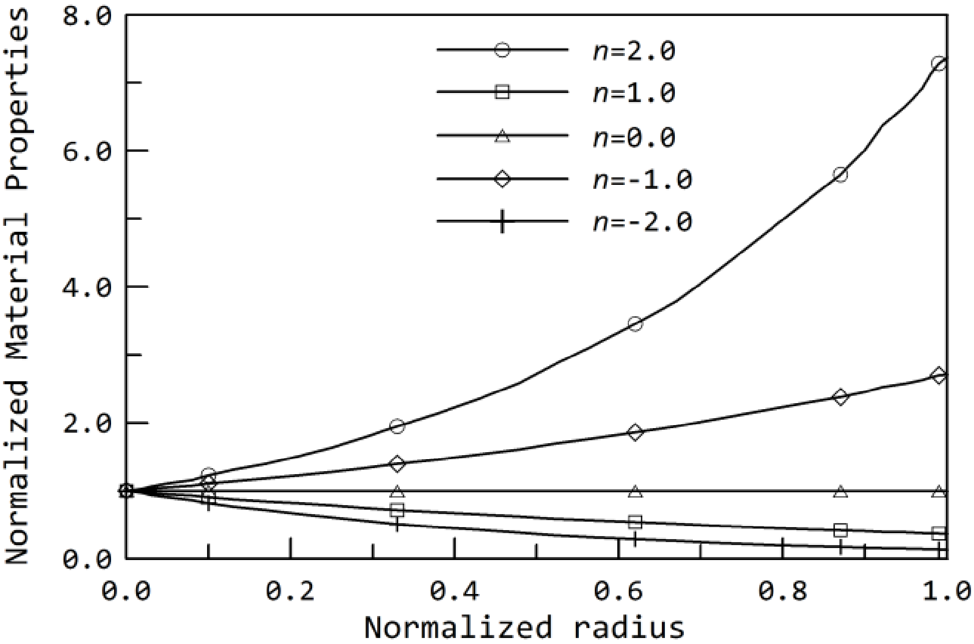

Fig. 2. Distribution of elasticity modulus at different power indices, *n*.

Fig.3 (a-c) the normalized displacement and stresses are plotted for externally pressurized cylinder with aspect ratio 0.5. It is observed from Fig.3a that for positive values of *n*, the displacement at inner radius is maximum and decreases at location away from the centre thus experiencing a pull in terms of material flow or displacement at the centre of cylinder. However for negative *n,* the displacement is minimum at the inner radius with decreasing magnitude at location near the tips. Thus in general would result into a push to the material flow or displacement. The normalized radial and tangential stresses are plotted in Fig. 3b and 3c respectively. The effect of pull / push resulting from radial variation of displacement is evident on the radial and tangential stresses. In Fig. 3(b-c), the result are compared with Chen and Lin (2008) establishing the validity of the present methodology. Interestingly, the variation of displacement and stresses with *n* plotted in Fig.3 points towards the possibility of attaining constant displacement irrespective of radial location within the cylinder by careful tailoring of material at a certain *n* value that would result in uniform tangential stresses throughout the radial coordinate. In Fig. 4 (a-c) and Fig.5 (a-c) similar results are plotted for cylinder with aspect ratio 0.3 and 0.9. The displacement and stresses of the cylinders gives a good insight into the effect of grading parameter n on the performance of the cylinder.

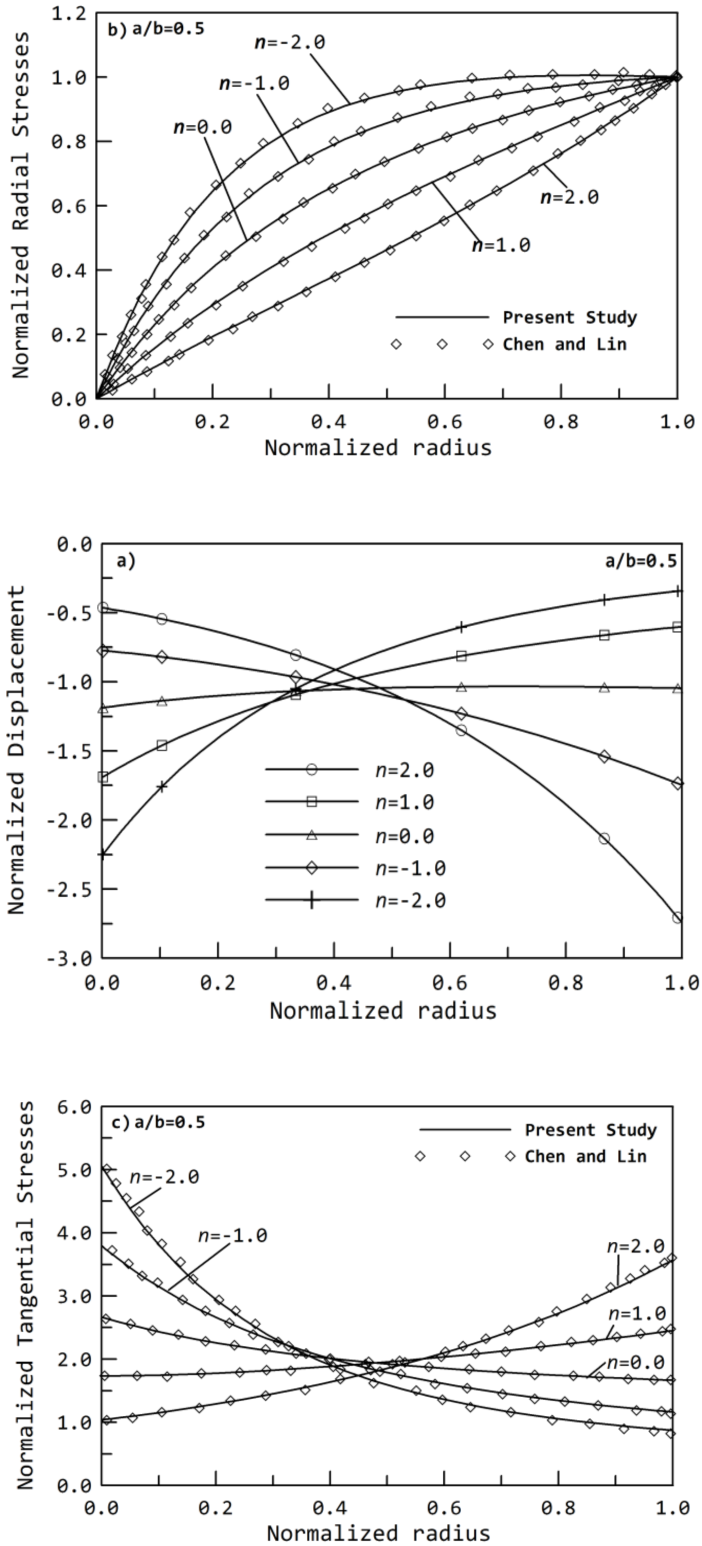

Fig.3. (a,b,c) Effect of grading index on, displacement, radial stress and tangential stress (a/b = 0.5, with an outer loading q_b).

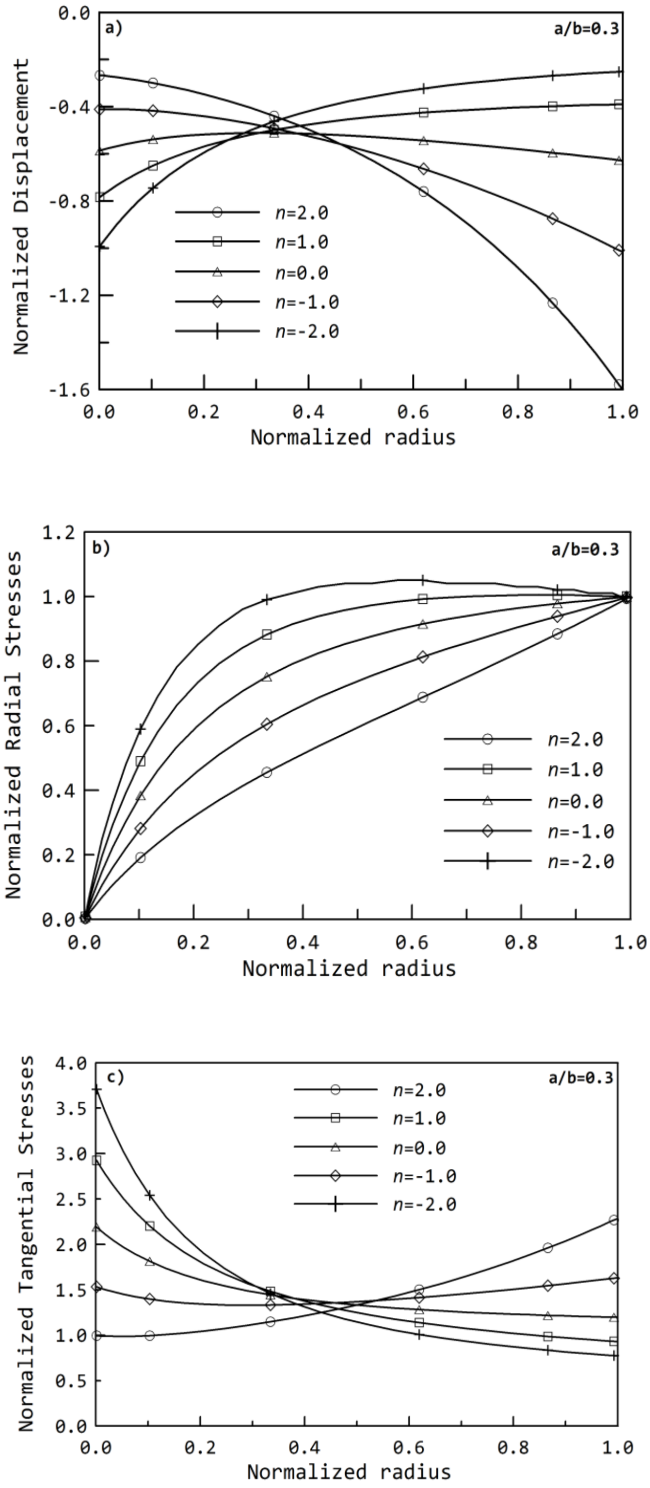

Fig. 4 (a,b,c) Effect of grading index on, normalized displacement, radial stress and tangential stress ($a/b = 0.3$, with an outer loading q_b).

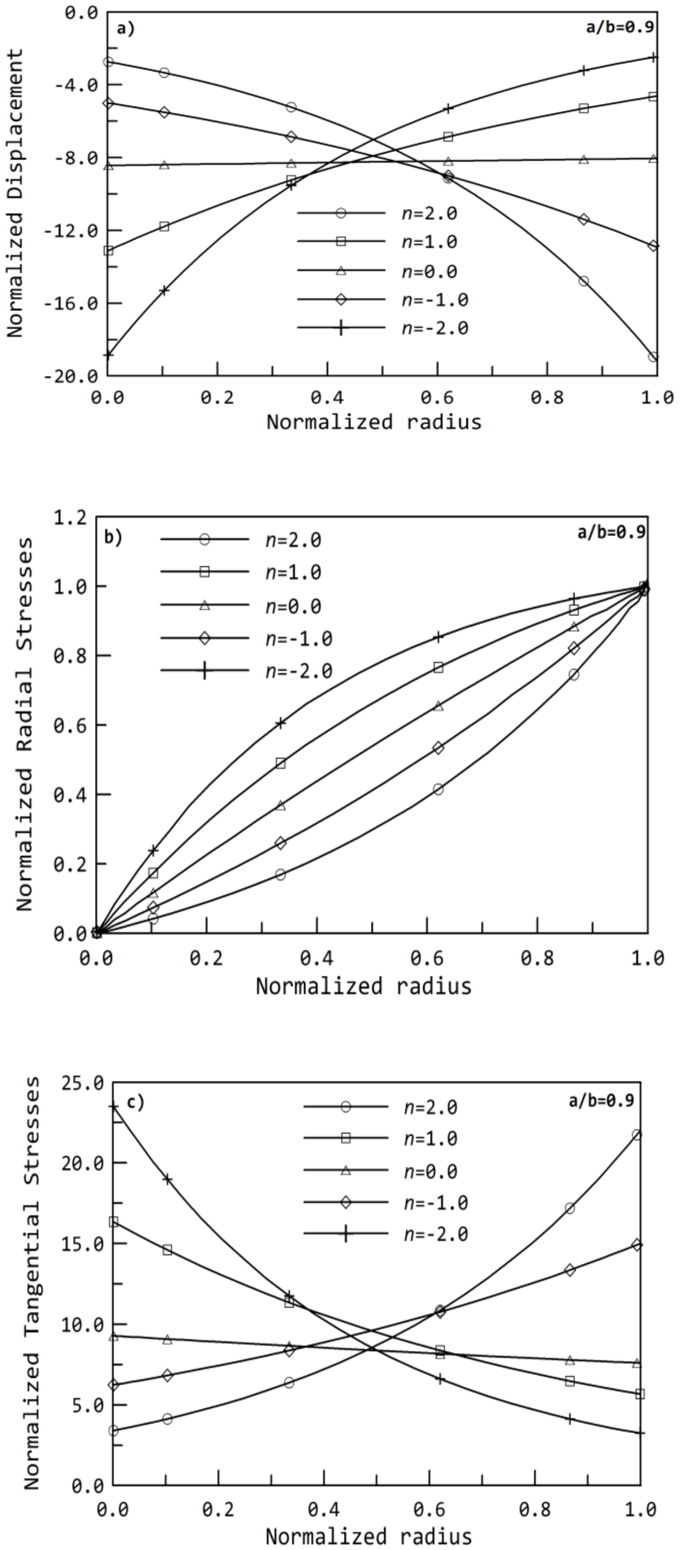

Fig.5.(a,b,c) Effect of grading index on, normalized displacement, radial stress and tangential stress (a/b = 0.9, with an outer loading q_b).

CONCLUSIONS

The present work employs a numerical scheme to obtain an approximate solution to of cylinder subjected to externally pressure made of functionally graded material. The validation of the present numerical scheme is carried out with existing literature and good agreement is reported. The effect of the gradient index n is studied on the displacement and stresses for different aspect ratio field of the cylinder. The results obtained present a good insight into the variation of stress and deformation with gradient index n and reveal the existence of grading parameter resulting into uniform strength cylinder. Result shows for positive values of *n*, the displacement at inner radius is maximum and increases at location away from the centre. Whereas, for negative *n,* the displacement is minimum at the inner radius with decreasing magnitude at location near the tips.

REFERENCES

[1] Tununcu, N., Ozturk, M., 2001. Exact solution for stresses in functionally graded pressure vessels.Composites Part B: Engineering,vol.32,no.8,p.683-686.

[2] Verijenko, V.E., Adali, S., Tabakov, P.Y., 2001. Stress distribution in continuously heterogeneous thick laminated pressure vessels. Composite Structures 54, 371–377.

[3] Tarn, J.-Q., 2001. Exact solutions for functionally graded anisotropic cylinders subjected to thermal and mechanical loads. International Journal of Solids and Structures 38, 8189–8206.

[4] Jabbari,M.,Sohrabpour,S.,Croccolo, Eslami,M.R.,2002.Mechanical and thermal stresses in a functionally graded hollow cylinder due to radially symmetric loads.

[5] Shao, Z.S., 2005. Mechanical and thermal stresses of a functionally graded circular hollow cylinder with finite length. International Journal of Pressure Vessels and Piping 82, 155–163. doi:10.1016/j.ijpvp.2004.09.007

[6] Xiang, H., Shi, Z., Zhang, T., 2006.Elastic analyses of heterogeneous hollow cylinders. Mechanics Research Communications 33, 681–691. doi:10.1016/j.mechrescom.2006.01.005

[7] Tutuncu, N., 2007.Stresses in thick-walled FGM cylinders with exponentially-varying properties. Engineering Structures 29, 2032–2035. doi:10.1016/j.engstruct.2006.12.003

[8] Vullo, V., Vivio, F., 2008. Elastic stress analysis of non-linear variable thickness rotating disks subjected to thermal load and having variable density along the radius.International Journal of Solids and Structures 45, 5337–5355. doi:10.1016/j.ijsolstr.2008.05.018

[9] Peng, X.L., Li, X.F., 2010. Thermoelastic analysis of a cylindrical vessel of functionally graded materials. International Journal of Pressure Vessels and Piping 87, 203–210. doi:10.1016/j.ijpvp.2010.03.024

[10] Ozturk, A., Gulgec, M., 2011. Elastic–plastic stress analysis in a long

functionally graded solid cylinder with fixed ends subjected to uniform heat generation. International Journal of Engineering Science 49, 1047–1061. doi:10.1016/j.ijengsci.2011.06.001
[11] Kayhani, M.H., Norouzi, M., Amiri Delouei, A., 2012. A general analytical solution for heat conduction in cylindrical multilayer composite laminates. International Journal of Thermal Sciences 52, 73–82. doi:10.1016/j.ijthermalsci.2011.09.002
[12] Croccolo, D., De Agostinis, M., 2013. Analytical solution of stress and strain distributions in press fitted orthotropic cylinders. International Journal of Mechanical Sciences 71, 21–29. doi:10.1016/j.ijmecsci.2013.03.002
[13] Duc, N.D., Thang, P.T., Dao, N.T., Tac, H.V., 2015. Nonlinear buckling of higher deformable S-FGM thick circular cylindrical shells with metal–ceramic– metal layers surrounded on elastic foundations in thermal environment. Composite Structures 121, 134–141. doi:10.1016/j.compstruct.2014.11.009
[14] Sondhi, L., Sanyal, S.,Saha.,K.N., Bhowmick, S., 2016. An approximate solution to the stress and deformation states of functionally graded rotating disks. AIP Conference Proceedings 1754, 030007 (2016); doi: http://dx.doi.org/10.1063/1.4958351
[15] Thawait, A.K., Sondhi, L., Sanyal, S., Bhowmick, S., 2017. An investigation of stresses and deformation states of clamped rotating functionally graded disk. Journal of Theoretical and Applied Mechanics 189. doi:10.15632/jtam-pl.55.1.189.
[16] Srinath L. Advanced strength of materials 2001 (Tata McGraw Hill, New Delhi, India).
[17] Chen, Y.Z., Lin, X.Y., 2008. Elastic analysis for thick cylinders and spherical pressure vessels made of functionally graded materials. Computational Materials Science 44, 581–587. doi:10.1016/j.commatsci.2008.04.018