On skew $q$-gaussian distribution

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Abstract

The skew $q$-gaussian distribution is defined by combining the skew distribution with the $q$-gaussian distribution. Recurrence formulae for the central moments are derived. The likelihood equation and the Fisher information matrix are calculated. Moreover, the extreme value distribution is derived.

Keywords: Skew distribution, $q$-gaussian distribution, maximum likelihood estimation, extreme value distribution.

1. Introduction

A skew distribution was introduced by Azzalini [2] and is a relatively new probability distribution. This distribution is obtained by distorting a symmetric probability distribution about the origin by a skew parameter, and it can represent more flexible shape by comparison with conventional probability distributions. Therefore, it is possible to fit data to distributions more accurately. In data analysis, we often assume the symmetry of a probability distribution, however, this assumption is not always satisfied for

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most real situations. By using the skew distribution, the estimation accuracy may increase relatively. On the other hand, the $q$-gaussian distribution is also well-known as a generalization of the gaussian, or the normal distribution. This distribution can also represent the heavy tailed distributions such as the $t$-distribution, and the distributions have a bounded support such as the semicircle distribution of Wigner. For these reasons, the $q$-gaussian distribution has been applied to problems in the fields of statistical mechanics, geology, finance, and machine learning.

In this paper, we propose a new probability distribution called the skew $q$-gaussian distribution which combines the skew distribution with the $q$-gaussian distribution. In Section 2, we review the family of skew distribution introduced by Azzalini [2]. As a special case of this family, we define the skew-normal distribution. In Section 3, we introduce the $q$-gaussian distribution. The $q$-gaussian distribution is a symmetric probability distribution about the origin and includes some important distributions such as the normal distribution and the $t$-distribution. This probability distribution is derived by maximizing the Tsallis entropy under some constraints (Furuichi [4]). In Section 4, we propose a new probability distribution called the skew $q$-gaussian distribution. The skew $q$-gaussian distribution has the advantages of both the skew distribution and the $q$-gaussian distribution. We show the $n$-th order moments about the origin for this probability distribution. In Section 5, we show some results for the maximum likelihood estimator and the Fisher information matrix of the skew $q$-gaussian distribution. In Section 6, we derive the extreme value distribution of the random variable of the skew $q$-gaussian distribution.

2. The skew distribution family

In this section, we introduce the skew distribution family.

**Proposition 2.1.** [Azzalini [3]] Let $f_0(\cdot)$ be a probability density function (p.d.f.) on $\mathbb{R}^d$, $G_0(\cdot)$ be a cumulative distribution function (c.d.f.) on $\mathbb{R}$ and $w(\cdot)$ be a real-valued function on $\mathbb{R}^d$, such that

\[
f_0(-x) = f_0(x), \quad G_0(-y) = 1 - G_0(y), \quad w(-x) = -w(x)
\]

for any $x$ on $\mathbb{R}^d$ and any $y$ on $\mathbb{R}$. Then

\[
f(x) = 2f_0(x)G_0(w(x))
\]

(2.1)

is a p.d.f. on $\mathbb{R}^d$.

For the proof, see Azzalini [3].

The distribution having p.d.f. (2.1) is called the skew distribution.

If we change $f_0$, $G_0$, and $w$ we can construct various types of skew distributions, for example, the skew normal distribution, the skew-$t$ distribution, the skew-logistic distribution, and so on. The skew distribution is obtained by distorting a symmetric probability
distribution about the origin. If we put \( w(x) = 0 \) in (2.1), the skew distribution is identical to the symmetric distribution. If we put \( f_0 = \varphi, G_0 = \Phi \) and \( w(x) = \alpha x \) for \( \alpha \in \mathbb{R} \) in (2.1), we have the skew normal distribution which has the following density function

\[
\varphi(x; \alpha) = 2\varphi(x)\Phi(\alpha x),
\]

where \( \varphi(\cdot) \) and \( \Phi(\cdot) \) are the p.d.f. and the c.d.f. of \( N(0, 1) \), respectively. We denote the skew normal distribution by \( SN(0, 1, \alpha) \), where \( \alpha \) is called the skew parameter which controls the skewness of \( \varphi(x; \alpha) \). We note that the skew-normal distribution for \( \alpha = 0 \) coincides with the standard normal distribution, that is, \( \varphi(x; 0) = \varphi(x) \). We also denote the probability distribution of \( Y = \mu + \sigma Z \) by \( SN(\mu, \sigma^2, \alpha) \), where \( Z \sim SN(0, 1, \alpha) \).

3. The \( q \)-gaussian distribution

As an example of symmetric distributions, in this section we review the \( q \)-gaussian distribution, or the \( q \)-normal distribution according to Furuichi [4] and Suyari [6].

**Definition 3.1.** [Furuichi [4] and Suyari [6]] We say that a random variable \( X \) has a \( q \)-gaussian distribution, written by \( qN(0, 1, q) \), if the density function is given by

\[
p_q(x) = \frac{1}{Z_q(1)} \left[ 1 - \frac{1 - q}{3 - q} x^2 \right]_+^{1/(1-q)}
\]

for \( q < 3 \) and \( q \neq 1 \), where \( [a]_+ = \max\{a, 0\} \) and

\[
Z_q(\sigma) = \int_{-\infty}^{\infty} \left[ 1 - \frac{1 - q}{3 - q} x^2 \right]_+^{1/(1-q)} dx
\]

\[
= \begin{cases} 
\left( \frac{3 - q}{q - 1} \right)^{1/2} F \left( \frac{3 - q}{2(q - 1)}, \frac{1}{2} \right) & (1 < q < 3), \\
\left( \frac{3 - q}{1 - q} \right)^{1/2} B \left( \frac{2 - q}{1 - q}, \frac{1}{2} \right) & (q < 1)
\end{cases}
\]

is the normalizing constant with the beta function \( B(a, b) = \int_0^1 t^{a-1}(1 - t)^{b-1} dt \) \((a, b > 0)\).

\(^2\)The p.d.f. of \( q \)-gaussian distribution is often represented as

\[
p_q(x; \beta) = \frac{\sqrt{\pi}}{C_q} \left[ 1 - (1 - q)\beta x^2 \right]_+^{1/(1-q)}.
\]

In this paper, we adopt a different form because of the relation with the \( t \)-distribution and the number of parameters under consideration.
For $Z \sim qN(0, 1, q)$, we represent the probability distribution of $Y = \mu + \sigma Z$ by $qN(\mu, \sigma^2, q)$. Then, the p.d.f. of $Y$ is given by

$$p_q\left(\frac{y - \mu}{\sigma}\right) = \frac{1}{Z_q(\sigma)} \left[1 - \frac{1 - q (y - \mu)^2}{3 - q \sigma^2}\right]_{+}^{\frac{1}{1-q}}.$$

If we change the value of parameter $q$, we can represent various types of distributions. As $q$ is approaching to $3$, the tail of the distribution becomes thicker. On the contrary, as $q$ is decreasing, the tail of the distribution becomes thinner, and the distribution has a bounded support for $q < 1$.

**Proposition 3.2.** [Suyari [6]] The p.d.f. $p_q(x)$ of $qN(0, 1, q)$, has the following properties.

1. When $q \to 1$, the $q$-gaussian distribution coincides with the standard normal distribution:
   $$p_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

2. When $q = 1 + \frac{2}{\nu + 1}$, the $q$-gaussian distribution coincides with the $t$-distribution with degree of freedom $\nu$:
   $$p_{1+\frac{2}{\nu+1}} = \frac{1}{\sqrt{\nu B\left(\frac{\nu}{2}, \frac{1}{2}\right)}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

3. When $q = -1$, the $q$-gaussian distribution coincides with the semicircle distribution of Wigner:
   $$p_{-1}(x) = \frac{1}{\sqrt{2} B\left(\frac{3}{2}, \frac{1}{2}\right)} \left[1 - \frac{x^2}{2}\right]_{+}^{\frac{1}{2}}.$$

4. **Definition of the skew $q$-gaussian distribution and its property**

   In this section, we propose the skew $q$-gaussian distribution that combines the skew distribution with the $q$-gaussian distribution. We also show the $n$-th order moments about the origin for the skew $q$-distribution.

**Definition 4.1.** We say that a random variable $X$ has a skew $q$-gaussian distribution, written by $SqN(0, 1, \alpha, q)$ if the density function is given by

$$p_q(x; \alpha) = 2p_q(x)P_q(\alpha x) \quad (q < 3, \ q \neq 1),$$

where $\alpha \in \mathbb{R}$ is the skew parameter and $p_q(\cdot)$ and $P_q(\cdot)$ are the p.d.f. and the c.d.f. of $qN(0, 1, q)$, respectively.
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Figure 1: The p.d.f.'s of $SqN(0, 1, 1, q)$ for $q = -1, 0, 1, 2$ (DotDashed, Dotted, Line, Dashed)

Note that (4.1) is derived by putting $f_0(x) = p_q(x)$ and $G_0(w(x)) = P_q(\alpha x) = \int_{-\infty}^{\alpha x} p_q(t)dt$ in (2.1). We denote the c.d.f. of $SqN(0, 1, q)$ by

$$P_q(x; \alpha) = \int_{-\infty}^{x} p_q(t; \alpha)dt.$$ 

Note that if we put $\alpha = 0$ in (4.1), the skew $q$-gaussian distribution coincides with the $q$-gaussian distribution. For $Z \sim SqN(0, 1, q)$, we represent the probability distribution of $Y = \mu + \sigma Z$ by $SqN(\mu, \sigma^2, \alpha, q)$. Then, the p.d.f. of $Y$ is given by

$$p_q\left(\frac{y - \mu}{\sigma}; \alpha\right) = 2p_q\left(\frac{y - \mu}{\sigma}\right) P_q\left(\alpha\frac{y - \mu}{\sigma}\right).$$

Figure 1 shows the p.d.f.’s of the $q$-gaussian distribution $SqN(0, 1, 1, q)$ for $q = -1, 0, 1, 2$. As $q$ is approaching to 3, the tail of distribution becomes thicker. On the contrary, as $q$ is decreasing, the tail of distribution becomes thicker, and the distribution has a boundend support. These properties are similar to the $q$-gaussian distribution.

As a property of the skew $q$-gaussian distribution, first, we consider the $n$-th order moments about the origin.

**Proposition 4.2.** If $X \sim SqN(0, 1, q)$, then the moments about the origin of $X$ satisfy the following recurrence formulae.

$$E\left[X^{2n}\right] = \frac{(2n - 1)(3 - q)}{2n - 1 - q(2n + 1)} E\left[X^{2(n-1)}\right],$$

$$E\left[X^{2n-1}\right] = \frac{(n - 1)(3 - q)}{n(1 - q) - 1} E\left[X^{2(n-1)-1}\right] + \frac{(3 - q)}{n(1 - q) - 1} f_n(\alpha, q).$$
Here, $f_n(\alpha, q)$ is expressed as follows.

\[
f_n(\alpha, q) = \begin{cases} 
\frac{\text{sgn}(\alpha)\alpha^2}{2B(\frac{3}{4}, 1 - q, \frac{1}{2})^2} \left(3 - q\right) n^{-\frac{1}{2}} \frac{\Gamma(n - \frac{1}{2})\Gamma\left(\frac{3}{2} - 2n\right)}{\Gamma(n + \frac{5 - 3q}{2})} F \left(\frac{1}{q - 1}, n - \frac{1}{2}, n + \frac{5 - 3q}{2}, (1 - q)\alpha^2\right) 
& \left(0 < q < 1, 0 < |\alpha| \leq 1\right), \\
\frac{\text{sgn}(\alpha)|\alpha|^{3 - 2n}}{2B(\frac{2}{1 - q}, \frac{1}{2})^2} \left(3 - q\right) n^{-\frac{1}{2}} \frac{\Gamma(n - \frac{1}{2})\Gamma\left(\frac{3 - q}{2} - 2n\right)}{\Gamma(n + \frac{3}{2})} F \left(\frac{2 - q}{q - 1}, n - \frac{1}{2}, n + \frac{3 - q}{2}, (1 - q)\alpha^2\right) 
& \left(0 < q < 1, 1 \leq |\alpha|\right), \\
\frac{\text{sgn}(\alpha)|\alpha|^{2\left(\frac{1}{2} - q\right)}}{2B(\frac{3}{2} - q, \frac{1}{2})^2} \left(3 - q\right) n^{-\frac{1}{2}} \frac{\Gamma(n - \frac{1}{2})\Gamma\left(\frac{3 - q}{2} - 2n\right)}{\Gamma(n + \frac{3}{2})} F \left(\frac{2 - q}{q - 1}, n - \frac{1}{2}, n + \frac{3 - q}{2}, (1 - q)\alpha^2\right) 
& \left(1 < q < 1 + \frac{4}{2n + 1}, 0 < |\alpha| < \sqrt{2}\right), \\
\frac{\text{sgn}(\alpha)|\alpha|^{2n - 1 + \frac{q}{2}}}{2B(\frac{3 - q}{2}, \frac{1}{2})^2} \left(3 - q\right) n^{-\frac{1}{2}} \frac{\Gamma(n - \frac{1}{2})\Gamma\left(\frac{3 - q}{2} - 2n\right)}{\Gamma(n + \frac{3}{2})} F \left(\frac{2 - q}{q - 1}, n - \frac{1}{2}, n + \frac{3 - q}{2}, (1 - q)\alpha^2\right) 
& \left(1 < q < 1 + \frac{4}{2n + 1}, 0 < |\alpha| < \sqrt{2}\right), \\
\end{cases}
\]

where $\text{sgn}(x)$ is the sign function defined as

\[
\text{sgn}(x) = \begin{cases} 
-1 & (x < 0), \\
0 & (x = 0), \\
1 & (x > 0), \\
\end{cases}
\]

and $F(\alpha, \beta, \gamma, z)$ is the hypergeometric function of Gauss given by

\[
F(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \ n!} \frac{z^n}{n!} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)z^n}{\Gamma(\gamma + n) \ n!} \quad (|z| < 1),
\]

which has the Euler’s integral representation

\[
F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta - 1}(1-t)^{\gamma - \beta - 1}(1-zt)^{-\alpha} \, dt \quad (\gamma > \beta > 0, \ |z| < 1)
\]
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Proof. The proper integral range can be defined as

$$x \in (-C, C), \quad C := \begin{cases} \sqrt{\frac{3 - q}{1 - q}} & (q < 1), \\ \infty & (1 < q < 3). \end{cases}$$

First, by differentiating $p_q(x)$, we have

$$2xp_q(x) = -(3-q) \left(1 - \frac{1-q}{3-q}x^2\right)p'_q(x).$$

For any $m \in \mathbb{N}$, integration by parts gives

$$E[X^m] = \int_{-C}^{C} x^m 2p_q(x)p_q(\alpha x)dx$$

$$= -(3-q) \int_{-C}^{C} \left(1 - \frac{1-q}{3-q}x^2\right)p'_q(x)x^{m-1}P_q(\alpha x)dx.$$ 

Therefore, we have

$$-\frac{1}{3-q} E[X^m] = \left[p_q(x) \left(1 - \frac{1-q}{3-q}x^2\right)x^{m-1}P_q(\alpha x)\right]_{-C}^{C}$$

$$- \frac{1-q}{3-q} \int_{-C}^{C} x^m p_q(x)p_q(\alpha x)dx$$

$$+ (m-1) \int_{-C}^{C} x^{m-2} \left(1 - \frac{1-q}{3-q}x^2\right)p_q(x)p_q(\alpha x)dx$$

$$+ \alpha \int_{-C}^{C} \left(1 - \frac{1-q}{3-q}x^2\right)x^{m-1}p_q(x)p_q(\alpha x)dx.$$ 

We calculate each term of the RHS above. Each term equals zero, $-\frac{1-q}{3-q} E[X^m]$, 

$$\frac{m-1}{2} \left\{ E[X^{m-2}] - \frac{1-q}{3-q} E[X^m] \right\},$$

and

$$\frac{\alpha}{2Z_q(1)Z_q(\frac{1}{q})} \int_{-C}^{C} x^{m-1} \left(1 - \frac{1-q}{3-q}x^2\right)^{\frac{2-q}{1-q}} \left(1 - \frac{1-q}{3-q}x^2\right)^{\frac{1}{1-q}} =: f_n(\alpha, q)$$ (say),

respectively. Then we have

$$E[X^m] = \frac{(m-1)(3-q)}{m-1-q(m+1)} E[X^{m-2}] + \frac{2(3-q)}{m-1-q(m+1)} f_n(\alpha, q).$$
If $m = 2n (n \in \mathbb{N})$, then $f_n(\alpha, q) = 0$ since the integrand of $f_n(\alpha, q)$ is an odd function of $x$. Therefore we have the desired equation.

If $m = 2n - 1 (n \in \mathbb{N})$, since the integrand of $f_n(\alpha, q)$ is an even function of $x$, we have

$$f_n(\alpha, q) = \frac{\alpha}{Z_q(1)Z_q(\frac{1}{2})} \int_0^C t^{2(n-1)} \left( 1 - \frac{1-q}{3q} t^2 \right)^{\frac{2-q}{1-q}} \left( 1 - \frac{1-q}{3q} \alpha^2 x^2 \right)^{\frac{1}{1-q}} dx.$$ 

We calculate $f_n(\alpha, q)$ by considering four cases, (i) $q < 1, 0 < |\alpha| \leq 1$, (ii) $q < 1, |\alpha| \geq 1$, (iii) $1 < q < 3, |\alpha| > 1/\sqrt{2}$, and (iv) $1 < q < 3, |\alpha| < 1/\sqrt{2}$.

[(i) $q < 1, 0 < |\alpha| \leq 1$]

The substitution $t = \frac{1-q}{3-q} x^2$ yields, since $x = \left( \frac{3-q}{1-q} \right)^{1/2} t^{1/2}$ and

$$dx = \frac{1}{2} \left( \frac{3-q}{1-q} \right)^{1/2} t^{-1/2} dt,$$

$$f_n(\alpha, q)$$

$$= \frac{\alpha}{2Z_q(1)Z_q(\frac{1}{2})} \int_0^1 \left( \frac{3-q}{1-q} \right)^{n-1} t^{n-1} (1-t)^{\frac{2-q}{1-q}} (1-\alpha^2 t)^{\frac{1}{1-q}} \frac{1}{2} \left( \frac{3-q}{1-q} \right)^{1/2} t^{-1/2} dt$$

$$= \frac{\alpha}{2Z_q(1)Z_q(\frac{1}{2})} \left( \frac{3-q}{1-q} \right)^{n-\frac{1}{2}} \Gamma \left( n \frac{1}{2} \right) \Gamma \left( \frac{3-2q}{1-q} \right) F \left( \frac{1}{q-1}, n - \frac{1}{2}, n + \frac{5-3q}{2(1-q)}, \alpha^2 \right)$$

from the Euler’s integral representation. Here, it is necessary to be $n + \frac{5-3q}{2(1-q)} > n - \frac{1}{2} > 0$, and this is true in all of $n$ and $q$.

[(ii) $q < 1, |\alpha| \geq 1$]

The substitution $t = \frac{1-q}{3-q} \alpha^2 x^2$ yields a similar equation to the case (i) since $x = \left( \frac{3-q}{1-q} \right)^{1/2} \left| \alpha \right| t^{1/2}$ and

$$dx = \frac{1}{2} \left( \frac{3-q}{1-q} \right)^{1/2} t^{-1/2} dt.$$ Then we have

$$f_n(\alpha, q)$$

$$= \frac{\text{sgn}(\alpha)|\alpha|^{3-2n}}{2B \left( \frac{2-q}{1-q}, \frac{1}{2} \right)^2} \left( \frac{3-q}{1-q} \right)^{n-\frac{1}{2}} \Gamma \left( n \frac{1}{2} \right) \Gamma \left( \frac{2-q}{1-q} \right) F \left( \frac{2-q}{q-1}, n - \frac{1}{2}, n + \frac{3-q}{2(1-q)}, \frac{1}{\alpha^2} \right).$$
Here it is necessary to be \( n + \frac{3 - q}{2(1 - q)} > n - \frac{1}{2} > 0 \), and this is true in all of \( n \) and \( q \).

[(iii) \( 1 < q < 3, |\alpha| > 1/\sqrt{2} \)]

The substitution \( \frac{1}{w} = \frac{1 - q}{3 - q} x^2 \) yields, since \( x = \left( -\frac{3 - q}{1 - q} \right)^{1/2} w^{-1/2} (1 - w)^{1/2} \) and \( dx = -\frac{1}{2} \left( -\frac{3 - q}{1 - q} \right)^{1/2} w^{-3/2} (1 - w)^{-1/2} dw \),

\[
fn(\alpha, q) = \frac{\alpha}{Z_q(1)Z_q \left( \frac{1}{q} \right)} \int_0^1 \left( -\frac{3 - q}{1 - q} \right)^{n-1} w^{-n+1} (1 - w)^{n-1} w^{-\frac{3-q}{1-q}} \left( 1 - \alpha^2 + \alpha^2 w^{-1} \right)^{\frac{1}{2}}
\]

\[
\times \frac{1}{2} \left( -\frac{3 - q}{1 - q} \right)^{1/2} w^{-3/2} (1 - w)^{-1/2} dw
\]

\[
= \frac{\text{sgn}(\alpha)|\alpha|^{\frac{3-q}{1-q}}}{2Z_q(1)Z_q \left( \frac{1}{q} \right)} \left( -\frac{3 - q}{1 - q} \right)^{n-\frac{1}{2}}
\]

\[
\times \int_0^1 w^{-\frac{5-q}{2(1-q)}} (1 - w)^{-\frac{5-q}{2(1-q)}} \left( 1 - \alpha^2 + \alpha^2 w^{-1} \right)^{-\frac{1}{2}} dw
\]

\[
= \frac{\text{sgn}(\alpha)|\alpha|^{\frac{3-q}{1-q}}}{2B \left( \frac{3-q}{2(1-q)}, \frac{1}{2} \right)^2} \left( \frac{3 - q}{q - 1} \right)^{n-\frac{1}{2}} \frac{\Gamma(n - \frac{1}{2}) \Gamma \left( -n - \frac{5-q}{2(1-q)} \right)}{\Gamma \left( \frac{3-q}{q-1} \right)}
\]

\[
\times F \left( \frac{1}{q-1}, -n - \frac{5-q}{2(1-q)}; \frac{3-q}{q-1}, \frac{3-q}{\alpha^2} - 1 \right)
\]

from the Euler's integral representation. Here it is necessary to be \( \frac{3 - q}{q - 1} > -n - \frac{5-q}{2(1-q)} > 0 \), i.e., \( n < -\frac{5-q}{2(1-q)} \). This is also equivalent to \( 1 < q < 1 + \frac{4}{2n + 1} \).

[(iv) \( 1 < q < 3, |\alpha| < 1/\sqrt{2} \)]

The substitution \( \frac{1}{w} = \frac{1 - q}{3 - q} \alpha^2 x^2 \) yields a similar equation to the case (iii) since \( x = \left( -\frac{3 - q}{1 - q} \right)^{1/2} w^{-\frac{1}{2}} (1 - w)^{1/2} \) and \( dx = -\frac{1}{2|\alpha|} \left( -\frac{3 - q}{1 - q} \right)^{1/2} w^{-\frac{3}{2}} (1 - w)^{-1/2} dw \). Then we have

\[
f_n(\alpha, q) = \frac{\text{sgn}(\alpha)|\alpha|^{-2n+\frac{q}{1-q}}}{2B \left( \frac{3-q}{2(1-q)}, \frac{1}{2} \right)^2} \left( \frac{3 - q}{q - 1} \right)^{n-\frac{1}{2}} \frac{\Gamma(n - \frac{1}{2}) \Gamma \left( -n - \frac{5-q}{2(1-q)} \right)}{\Gamma \left( \frac{3-q}{q-1} \right)}
\]

\[
\times F \left( \frac{2-q}{q-1}, -n - \frac{5-q}{2(1-q)}; \frac{3-q}{q-1}, 1 - \alpha^2 \right).
\]
Here it is necessary to be \( \frac{3 - q}{q - 1} > -n - \frac{5 - q}{2(1 - q)} > 0 \), i.e., \( n < -\frac{5 - q}{2(1 - q)} \). This is also equivalent to \( 1 < q < 1 + \frac{4}{2n + 1} \).

5. Estimation of skew \( q \)-gaussian distribution

In this section, we consider the log-likelihood function, the maximum likelihood estimator and the Fisher information matrix for the skew \( q \)-gaussian distribution.

Let \( Y_1, \ldots, Y_n \) be an i.i.d. random sample from \( SqN(\mu, \sigma^2, \alpha, q) \). Denote \( Y = (Y_1, Y_2, \ldots, Y_n) \) and \( \theta = (\mu, \sigma, \alpha, q) \). Then, the likelihood function of \( \theta \) is

\[
L(\theta, y) = \frac{2^n}{Z_q^{n}(\sigma)} \prod_{i=1}^{n} \left\{ 1 - \frac{1 - q}{3 - q} \left( \frac{y_i - \mu}{\sigma} \right)^2 \right\}^{\frac{1}{1-q}} P_q \left( \alpha \frac{y_i - \mu}{\sigma} \right). 
\]

Therefore, the log-likelihood function \( l(\theta, z) \) is

\[
l(\theta, z) = n \log 2 - n \log Z_q(\sigma) + \frac{1}{1 - q} \sum_{i=1}^{n} \log \left( 1 - \frac{1 - q}{3 - q} z_i^2 \right) + \sum_{i=1}^{n} \log P_q(\alpha z_i),
\]

where \( z_i = (y_i - \mu)/\sigma \) and \( z = (z_1, z_2, \ldots, z_n) \). Under the condition of \( 1 - (1 - q)/(3 - q) > 0 \), the partial derivatives with respect to \( \theta \) are

\[
\frac{\partial l}{\partial \mu} = \frac{2}{\sigma(3 - q)} \sum_{i=1}^{n} \frac{z_i}{1 - \frac{1 - q}{3 - q} z_i^2} - \frac{\alpha}{\sigma} \sum_{i=1}^{n} \zeta_1(\alpha z_i),
\]

\[
\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{2}{\sigma(3 - q)} \sum_{i=1}^{n} \frac{z_i^2}{1 - \frac{1 - q}{3 - q} z_i^2} - \frac{\alpha}{\sigma} \sum_{i=1}^{n} z_i \zeta_1(\alpha z_i),
\]

\[
\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{n} z_i \zeta_1(\alpha z_i),
\]

\[
\frac{\partial l}{\partial q} = -n A_q + \frac{1}{1 - q^2} \sum_{i=1}^{n} \log \left( 1 - \frac{1 - q}{3 - q} z_i^2 \right) + \frac{2}{(1 - q)(3 - q)^2} \sum_{i=1}^{n} \frac{z_i}{1 - \frac{1 - q}{3 - q} z_i^2}
\]

\[+ \sum_{i=1}^{n} g_q(\alpha z_i),
\]

where \( \zeta_0(x) = \log P_q(x) \), \( \zeta_i(x) = (\partial^i / \partial x^i) \zeta_0(x) \) \((i = 1, 2, \cdots)\),

\[
A_q = \frac{\partial}{\partial q} \frac{Z_q(\sigma)}{Z_q(\sigma)} = \begin{cases} \frac{1}{(q - 1)^2} \left[ \frac{q - 1}{3 - q} + \psi \left( \frac{1}{q - 1} \right) - \psi \left( \frac{3 - q}{2q - 2} \right) \right] & (1 < q < 3), \\ \frac{1}{(1 - q)^2} \left[ \frac{1 - q}{3 - q} + \psi \left( \frac{2 - q}{1 - q} \right) - \psi \left( \frac{5 - 3q}{2(1 - q)} \right) \right] & (q < 1) \end{cases}
\]
with the digamma function \( \psi(x) = (\partial/\partial x) \log \Gamma(x) = \Gamma'(x)/\Gamma(x) \), and \( g_q(x) = \partial/\partial q p_q(x)/P_q(x) \). Setting these partial derivatives equal to 0 and solving yields the maximum likelihood estimator of \( \theta \).

The second-order partial derivatives of the log-likelihood function are

\[
\frac{\partial^2 l}{\partial \mu^2} = -\frac{2}{\sigma^2(3-q)} \sum_{i=1}^{n} \left( 1 + \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2} \right)^2 + \frac{\alpha^2}{\sigma^2} \sum_{i=1}^{n} \xi_2(\alpha z_i),
\]

\[
\frac{\partial^2 l}{\partial \mu \partial \sigma} = -\frac{4}{\sigma^2(3-q)} \sum_{i=1}^{n} \frac{z_i}{\gamma_i} \left( 1 - \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2} \right) + \frac{\alpha\sigma^2}{\sigma^2} \sum_{i=1}^{n} \xi_1(\alpha z_i) + \frac{\alpha^2}{\sigma^2} \sum_{i=1}^{n} z_i \xi_2(\alpha z_i),
\]

\[
\frac{\partial^2 l}{\partial \mu \partial \alpha} = -\frac{1}{\sigma} \sum_{i=1}^{n} \xi_1(\alpha z_i) - \frac{\alpha}{\sigma} \sum_{i=1}^{n} z_i \xi_2(\alpha z_i),
\]

\[
\frac{\partial^2 l}{\partial \mu \partial q} = \frac{2}{\sigma(3-q)} \sum_{i=1}^{n} \frac{z_i(1-z_i^2)}{(1 - \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2})^2} - \frac{\alpha}{\sigma} g_q'(\alpha z_i),
\]

\[
\frac{\partial^2 l}{\partial \sigma^2} = -\frac{2}{\sigma^2(3-q)} \sum_{i=1}^{n} \frac{z_i^2}{(1 - \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2})^2} + \frac{2\alpha\sigma^2}{\sigma^2} \sum_{i=1}^{n} z_i \xi_1(\alpha z_i) + \frac{\alpha^2}{\sigma^2} \sum_{i=1}^{n} z_i^2 \xi_2(\alpha z_i),
\]

\[
\frac{\partial^2 l}{\partial \sigma \partial \alpha} = -\frac{1}{\sigma} \sum_{i=1}^{n} z_i \xi_1(\alpha z_i) - \frac{\alpha}{\sigma} \sum_{i=1}^{n} z_i \xi_2(\alpha z_i),
\]

\[
\frac{\partial^2 l}{\partial \sigma \partial q} = \frac{2}{\sigma(3-q)} \sum_{i=1}^{n} \frac{z_i^2}{(1 - \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2})^2} - \frac{\alpha}{\sigma} \sum_{i=1}^{n} z_i g_q'(\alpha z_i),
\]

\[
\frac{\partial^2 l}{\partial \alpha^2} = \sum_{i=1}^{n} z_i^2 \xi_2(\alpha z_i), \quad \frac{\partial^2 l}{\partial \alpha \partial q} = \sum_{i=1}^{n} z_i g_q'(\alpha z_i),
\]

\[
\frac{\partial^2 l}{\partial q^2} = -\frac{\partial}{\partial q} A_q + \frac{2}{(1-q)^3} \sum_{i=1}^{n} \log \left( 1 - \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2} \right)
\]

\[
+ \frac{2}{(1-q)^2(3-q)} \sum_{i=1}^{n} \frac{z_i^2}{(1 - \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2})^2}
\]

\[
+ \frac{2(5-3q)}{(1-q)^2(3-q)^3} \sum_{i=1}^{n} \frac{z_i^2}{1 - \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2}} + \frac{4}{(1-q)(3-q)^4} \sum_{i=1}^{n} \frac{z_i^4}{(1 - \frac{1-q}{3-q} \frac{z_i^2}{\gamma_i^2})^2}
\]

\[
+ \sum_{i=1}^{n} \frac{\partial}{\partial q} g_q'(\alpha z_i).
\]

By taking the expectations for these values, the elements of Fisher information matrix
Theorem 6.1. Let $k \leq 1$ and $q < M_n$.

It is necessary to consider the cases of $q > 1$ separately since the support of the distribution depends on the value of $q$.

\[
i_{\mu\mu} = \frac{2d_{q,\alpha} + (3 - q)\alpha a_{0,2}}{\sigma^2 (3 - q)}, \quad i_{\mu\sigma} = i_{\sigma\mu} = n \frac{4c_{q,\alpha} + (3 - q)\alpha \{a_{0,2} + \alpha (b_{1,q} - a_{1,2})\}}{\sigma^2},
\]

\[
i_{\mu\mu} = \frac{n a_{0,1} + \alpha (b_{1,q} - a_{1,2})}{\sigma}, \quad i_{\mu q} = i_{q\mu} = -\frac{5 - q}{2} nc_{q,\alpha} + \frac{\alpha}{\sigma} ne_{q,\alpha},
\]

\[
i_{\sigma\sigma} = \frac{n (6 - q) + \alpha^2 q a_{2,2}}{q \sigma^2}, \quad i_{\sigma q} = i_{q\sigma} = n \frac{a_{0,1} - \alpha a_{2,2}}{\sigma},
\]

\[
i_{\sigma q} = i_{q\sigma} = \alpha n h_{q,\alpha}, \quad i_{\alpha q} = i_{\alpha q} = n a_{2,2},
\]

\[
i_{\alpha q} = -nh_{q,\alpha}, \quad i_{qq} = n \frac{\partial}{\partial q} A_q - \frac{2n}{(1 - q)^2} j_{q,\alpha} - \frac{2n(5 - 3q)}{(1 - q)^2 (3 - q)^2} - nk_{q,\alpha},
\]

where

\[
a_{k,1} = E \left[ Z^{2k} \xi_1^2 (\alpha Z) \right], \quad a_{k,2} = E \left[ Z^k \xi_1^2 (\alpha Z) \right], \quad b_{k,q} = E \left[ Z^{2k-1} \xi_2 (\alpha Z) \right] + a_{k,2},
\]

\[
c_{q,\alpha} = E \left[ \frac{Z}{(1 - \frac{1}{1-q} Z^2)^2} \right], \quad d_{q,\alpha} = E \left[ \frac{1 + \frac{1}{1-q} Z^2}{(1 - \frac{1}{1-q} Z^2)^2} \right], \quad e_{q,\alpha} = E \left[ g_q^\prime (\alpha Z) \right],
\]

\[
h_{q,\alpha} = E \left[ Z g_q^\prime (\alpha Z) \right], \quad j_{q,\alpha} = E \left[ \log \left( 1 - \frac{1}{\frac{3}{1-q} Z^2} \right) \right], \quad k_{q,\alpha} = E \left[ \frac{Z^2}{(1 - \frac{1}{1-q} Z^2)^2} \right]
\]

for $k \in \mathbb{N}$.

## 6. Extreme value distribution of the skew $q$-gaussian distribution

In this section, we consider the extreme value distribution of the skew $q$-gaussian distribution. It is necessary to consider the cases of $q < 1$ and $q > 1$ separately since the support of the distribution depends on the value of $q$.

**Theorem 6.1.** Let $X_1, X_2, \ldots, X_n$ be an i.i.d. random sample from $SqN(0, 1, \alpha, q)$. Let $M_n := \max_{1 \leq i \leq n} X_i$. Then we have, for $q < 1$ and $\alpha > -1$,

\[
P \left( \frac{M_n - C}{b_n} \leq x \right) \rightarrow G_{2,(2-q)/(1-q)}(x) = \begin{cases} \exp \left\{ -(-x)^{-(2-q)/(1-q)} \right\} & (x < 0), \\ 1 & (x \geq 0), \end{cases}
\]

where $C = \sqrt{(3 - q)/(1 - q)}$ and $b_n = \frac{C}{2} \left\{ \frac{(2 - q) B(\frac{2-q}{1-q}, \frac{1}{2})}{n(1 - q) P_q(\alpha C)} \right\}^{(1-q)/(2-q)}$, and for $q < 1$ and $\alpha \leq -1$,

\[
P \left( \frac{M_n + C}{\alpha} \leq x \right) \rightarrow G_{2,(3-2q)/(1-q)}(x) = \begin{cases} \exp \left\{ -(-x)^{-(3-2q)/(1-q)} \right\} & (x < 0), \\ 1 & (x \geq 0), \end{cases}
\]
On skew $q$-gaussian distribution

where $b_n$ is given by

\[ b_n = \frac{2^{(3-2q)/(2-q)}}{-\alpha} \left(1 - \frac{1}{\alpha^2}\right)^{1/(3-2q)} \left\{C^2 B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)^2 (2 - q)(3 - 2q)\right\}^{(1-q)/(2q-3)} n(q - 1)^2 \]  

For $1 < q < 3$ and $\alpha > 0$,

\[ P\left(\frac{M_n}{b_n} \leq x\right) \rightarrow G_{1, (q-1)/(3-q)}(x) = \begin{cases} \exp \left\{-x^{-(3-q)/(q-1)}\right\} & (x > 0) \\ 0 & (x \leq 0), \end{cases} \]

where $C' = \sqrt{(3 - q)/(q - 1)}$, $b_n = C'\left\{B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)\right\}^{(1-q)/(3-q)}$. For $1 < q < 3$ and $\alpha < 0$,

\[ P\left(\frac{M_n}{b_n} \leq x\right) \rightarrow G_{1,2, (q-1)/(3-q)}(x) = \begin{cases} \exp \left\{-x^{2(3-q)/(q-1)}\right\} & (x > 0) \\ 0 & (x \leq 0), \end{cases} \]

where $b_n = \sqrt{-\alpha} \left\{B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)\right\}^{(1-q)/(2-q)} n$.  

Remark 6.2. $G_{1, q}$ and $G_{2, q}$ are known as the Fréchet distribution and the Weibull distribution, respectively (Galambos[5]).

Proof. At first, we consider the case of $q < 1$ and $\alpha > -1$. The distribution has the upper boundary $\sup \{x : P_q(x; \alpha) < 1\} = \sqrt{(3 - q)/(1 - q)} = C$. Since the p.d.f. of $SqN(0, 1, \alpha, q)$ is

\[ p_q(x; \alpha) = \frac{2}{Z_q(1)} \left(1 - \frac{x^2}{C^2}\right)^{1/(1-q)} P_q(\alpha x), \]

the asymptote of $p_q(x; \alpha)$ as $x \rightarrow C$ is given by

\[ p_q(x; \alpha) \sim \frac{2P_q(\alpha)}{C(3-q)/(1-q) B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)} (C^2 - x^2)^{1/(1-q)} \]

\[ = \frac{2P_q(\alpha C)}{C(3-q)/(1-q) B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)} (C + x)^{1/(1-q)} (C - x)^{1/(1-q)} \]

\[ \sim \frac{2P_q(\alpha C)}{C(3-q)/(1-q) B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)} (2C)^{1/(1-q)} (C - x)^{1/(1-q)} \]

\[ = \frac{2^{(2-q)/(1-q)} P_q(\alpha C)}{C(2-q)/(1-q) B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)} (C - x)^{1/(1-q)}. \]
Then,
\[ 1 - P_q(x; \alpha) \sim \int_C^x \frac{2(2-q)/(1-q)}{C(2-q)/(1-q)} P_q(\alpha C) \left( C - t \right)^{1/(1-q)} dt \]
\[ = \frac{(1 - q)2^{(2-q)/(1-q)} P_q(\alpha C)}{(2 - q)C(2-q)/(1-q)} \left( C - x \right)^{(2-q)/(1-q)}. \]

Since the coefficient is constant and slowly varying, and \( (2 - q)/(1 - q) > 0 \), therefore
\[ P \left( \frac{M_n - C}{b_n} \leq x \right) = P_q(C + b_n x)^n \to G_{2,(2-q)/(1-q)}(x) \]
from Theorem 2.1.2 in Galambos [5]. Here, the normalizing constant \( b_n \) can be chosen
\[ 1 - P_q(C - b_n; \alpha) \sim \frac{(1 - q)2^{(2-q)/(1-q)} P_q(\alpha C)}{(2 - q)C(2-q)/(1-q)} b_n^{(2-q)/(1-q)} = \frac{1}{n}, \]
that is,
\[ b_n = \frac{C}{2} \left\{ \frac{(2 - q)B(\frac{2-q}{1-q}, \frac{1}{2})}{n(1 - q)P_q(\alpha C)} \right\}^{(1-q)/(2-q)}. \]

Next, we consider the case of \( q < 1 \) and \( \alpha \leq -1 \). Note that the upper boundary is
\[ \sup \{ x : P_q(x; \alpha) < 1 \} = -C/\alpha \] in this case. Since
\[ P_q(x) \sim \frac{1}{Z_q(1)} 2^{1/(1-q)} \int_{-C}^x \left( 1 + \frac{t}{C} \right)^{1/(1-q)} dt \]
\[ = \frac{2^{1/(1-q)}(q - 1)}{C(2-q)/(1-q)} B(\frac{2-q}{1-q}, \frac{1}{2})(2 - q) (C + x)^{(2-q)/(1-q)} \]
as \( x \downarrow -C \), it holds
\[ P_q(\alpha x) \sim \frac{2^{1/(1-q)}(q - 1)}{C(2-q)/(1-q)} B(\frac{2-q}{1-q}, \frac{1}{2})(2 - q) (C + \alpha x)^{(2-q)/(1-q)} \]
as \( x \uparrow -C/\alpha \). Thus \( p_q(x; \alpha) \) is approximated as
\[ p_q(x; \alpha) = \frac{2^{(2-q)/(1-q)}(q - 1)}{C^2 B(\frac{2-q}{1-q}, \frac{1}{2})^2(2 - q)} \left( 1 - \frac{1}{\alpha^2} \right)^{1/(1-q)} (C + \alpha x)^{(2-q)/(1-q)} \]
as \( x \uparrow -C/\alpha \), therefore
\[ 1 - P_q(x; \alpha) \sim \int_{-C/\alpha}^x \frac{2^{(2-q)/(1-q)}(q - 1)}{C^2 B(\frac{2-q}{1-q}, \frac{1}{2})^2(2 - q)} \left( 1 - \frac{1}{\alpha^2} \right)^{1/(1-q)} (C + \alpha t)^{(2-q)/(1-q)} dt \]
\[ = \frac{2^{(2-q)/(1-q)}(q - 1)^2}{C^2 B(\frac{2-q}{1-q}, \frac{1}{2})^2(2 - q)(3 - 2q)} \left( 1 - \frac{1}{\alpha^2} \right)^{1/(1-q)} (C + \alpha x)^{(3-2q)/(1-q)} \]
as \( x \uparrow -C/\alpha \). From \((3-2q)/(1-q) > 0\), therefore

\[
P\left( \frac{M_n + \frac{C}{\alpha}}{b_n} \leq x \right) = P_q(C + b_n x)^n \to G_{2,(3-2q)/(1-q)}(x)
\]

from Theorem 2.1.2 in Galambos [5]. Here, the normalizing constant \( b_n \) can be chosen

\[
1 - P_q \left( \frac{-C}{\alpha} - b_n; \alpha \right) \sim \frac{2^{(2-q)/(1-q)}(q-1)^2}{C^2 B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)(2-q)(3-2q)} \left( 1 - \frac{1}{\alpha^2} \right)^{1/(1-q)} (-\alpha b_n)^{(3-2q)/(1-q)} \]

that is, we may take

\[
b_n = \frac{2^{(3-2q)/(2-q)}}{-\alpha} \left( 1 - \frac{1}{\alpha^2} \right)^{1/(3-2q)} \left\{ \frac{C^2 B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)^2(2-q)(3-2q)}{n(q-1)^2} \right\}^{(1-q)/(3-2q)}
\]

Next, we consider the case of \( 1 < q < 3 \). Note that the upper boundary of the distribution is \( \infty \). At first, suppose \( \alpha > 0^3 \). Since, for a sufficiently large \( x \),

\[
1 - P_q(x; \alpha) = \int_{-\infty}^{\infty} p_q(t; \alpha) dt = \int_{-\infty}^{\infty} \frac{2}{Z_q(1)} \left( 1 + \frac{t^2}{C^2} \right)^{1/(1-q)} P_q(\alpha t) dt
\]

\[
\sim \frac{2}{Z_q(1)} \int_{-\infty}^{\infty} C^{2/(1-q)} t^{2/(1-q)} dt
\]

\[
= \frac{2}{C^{(5-3q)/(1-q)} B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)} x^{(3-q)/(1-q)} ,
\]

where \( C' = \sqrt{(3-q)/(q-1)} \). thus

\[
1 - P_q(tx; \alpha) \sim \frac{(tx)^{(3-q)/(1-q)}}{t^{(3-q)/(1-q)}} = x^{(3-q)/(1-q)},
\]

and \((3-q)/(1-q) < 0\). Therefore, the limiting distribution of \( M_n \) is

\[
P\left( \frac{M_n}{b_n} \leq x \right) = P_q(b_n x)^n \to G_{1,(3-q)/(q-1)}(x)
\]

from Theorem 2.1.1 in Galambos [5]. Here, the normalizing constant \( b_n \) can be chosen as

\[
1 - P_q(b_n; \alpha) \sim \frac{2}{C^{(5-3q)/(1-q)} B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)} b_n^{(3-q)/(1-q)} = \frac{1}{n}.
\]

\footnote{Essentially, \( \alpha = 0 \) is included in this case. In fact, the limiting distribution is still Fréchet from (6.1), but the coefficient \( b_n \) is slightly different.}
that is, we may take

\[ b_n = C' \frac{(5 - 3q)/2n}{(3 - q)} \left\{ \frac{B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)}{2n} \right\} \left(1 - q\right)^{3(q - 1)/2(q - 1)}. \]

Next, we consider the case of \( \alpha < 0 \). In this case, for a sufficiently large \( t \), \( P_q(\alpha t) \) is approximated as

\[ P_q(\alpha t) = 1 - P_q(-\alpha t) = \int_{-\alpha t}^{\infty} p_q(t) \, dt \sim \frac{1}{C'(5 - 3q)/(1 - q) B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)} (-\alpha t)^{3(q - 1)/2(q - 1)} \]

from (1). Then

\[
1 - P_q(x; \alpha) = \int_x^{\infty} \frac{2}{Z_q(1)} \left( 1 + t^2 \right)^{1/(1-q)} \frac{P_q(\alpha t) \, dt}{C^2} \sim \frac{2(-\alpha)^{3(q - 1)/2(q - 1)}}{C^4(2-q)/(1-q) B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)^2} \left(3(q - 1)/2(q - 1)\right) x^{2(q - 1)/(1-q)},
\]

thus

\[
\frac{1 - P_q(x; \alpha)}{1 - P_q(t; \alpha)} \sim x^{2(q - 1)/(1-q)},
\]

and \( 2(3 - q)/(1 - q) < 0 \). Therefore, the limiting distribution of \( M_n \) is

\[ P \left( \frac{M_n}{b_n} \leq x \right) = P_q(b_n x)^n \rightarrow G_{1,2(3-q)/(q-1)}(x) \]

from Theorem 2.1.1 in Galambos [5]. Here, the normalizing constant \( b_n \) can be chosen as

\[ 1 - P_q(b_n; \alpha) \sim \frac{(-\alpha)^{3(q - 1)/(1-q)}}{C^2(5-3q)/(1-q) B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)^2} n \left(\frac{3-q}{2(q-1)}\right)^{2(q - 1)/(1-q)} = \frac{1}{n}, \]

that is, we may take

\[ b_n = \frac{C(5 - 3q)/(1-q)}{\sqrt{-\alpha}} \left\{ \frac{B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)}{2n} \right\}^{2(q - 1)/(1-q)}/n \quad \square \]
References


