Construction of some variance balanced block design for Nearest neighbor design

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Abstract

There are two methods for the construction of variance balanced block design for nearest neighbor design. The Method one is discuss the construction of variance balanced block design using nearest neighbor design for circular balanced repeated measurement design and the method two is discusses the construction of variance balanced block design using nearest neighbor design for series of block design and also optimality was checked for the construction of variance balanced block design which is proved that the constructed design is universal optimum.

Keywords: Circular block design, variance balanced block design, nearest neighbor design, universal optimal

1. INTRODUCTION

The class of designs to be considered here is the class $d$ for which $v$ treatments are to be tested in $b$ blocks of at most $k$ plots per block. The optimal subclass $d^*$ of $d$ for uncorrelated, equi-variance observations is kiefer(1958), (i) $k < v$ the balanced incomplete block designs with $k$ plot per block, Whenever they exists, (ii) $k = v$ the designs for which every treatments occurs once in each block, that is, complete block design

A block design for estimating direct (neighbor) effects in a competition effect model is variance balanced if $C_{\tau}(C_{\delta}, C_{\rho})$ has all its diagonal elements equal and all off-
diagonal elements equal. That is

$$C_\tau = aI_v + bE_{uv}$$

Where $E_{uv}$ is a matrix of all unities.

A block design is said to be balanced if every elementary treatment contrast is estimated with the same variance Rao(1958). In this sense, this design is also called variance balance design. It is well known that block design is a variance balanced if and only if it has

$$C_\tau = \theta \left[ I_\nu - \frac{1}{\nu} J_\nu J'_\nu \right]$$

Rees(1967) who introduced the concept and name of neighbor designs used these designs in serology. For the complete block case, i.e., $v=k$ Rees constructed neighbor designs for every odd $v$. For incomplete block case, i.e., $v>k$, Rees constructed neighbor designs for every $v$ up to $v=41$ when ever $k$ is not greater than 10, some by using Galois field theory.

Azais et al(1993) obtained are two main series of design. One for complete blocks in $t-1$ replications and the other for balanced incomplete blocks of size $t-1$, where $t$ is the number of treatment.

Bailey(2003) studied design for one-sided neighbour effects. Mendelsohn triple systems and perfect Mendelsohn designs are presented table in different block sizes. Ai et al(2007) constructed all order neighbour balanced design for $v=2m+1$(odd prime) in $2m$(where $m$ is minimum number) circular blocks. Kedia and Misra(2008) have constructed some new series of generalized neighbour designs. Ahmed and Akhtar(2008) studied some new construction algorithms have been developed to generate nearest neighbour design.

The model given by Seema Jaggi and Gupta(2007) in one-dimensional nearest neighbour design. The designs considered here are assumed to be in linear blocks, with neighbor effects only in the direction of the blocks (say left-neighbor or right neighbor or both). The following model with differentiated two-sided neighbour effects has been considered for analyzing a design with competition effects:

$$y_{ij} = \mu + \tau_{(i,j)} + \beta_j + \delta_{(i-1,j)} + \rho_{(i+1,j)} + e_{ij}$$

where $y_{ij}$ is the response from the $i^{th}$ plot in the $j^{th}$ block ($i=1,2,\ldots,k, j=1,2,\ldots,b$), $\mu$ is the general mean, $\tau_{(i,j)}$ is the direct effect of the treatment in the $i^{th}$ plot of $j^{th}$ block, $\beta_j$ is the effect of the $j^{th}$ block, $\delta_{(i-1,j)}$ is the left neighbour effect due to the
treatment in the \((i-1)^{th}\) plot of \(j^{th}\) block. \(\rho_{(i+1,j)}\) is the right neighbour effect due to the treatment in \((i+1)^{th}\) plot in \(j^{th}\) block. \(\varepsilon_{ij}\) are error terms independently and normally distributed with mean zero and variance \(\sigma^2\).

2. METHOD FOR CONSTRUCTION OF NEAREST NEIGHOR DESIGN

Here we discuss the construction of variance balanced block design in nearest neighbour design. First, the Hamiltonians cycles in uniform repeated measurement design with minimum number of experimental units exists, whenever \(v\) is an odd number. Next, we discuss a series of block designs balanced for direct effect with parameters \(v = b, r = k = 2v^2, \lambda = 2v\), can be obtained by writing the \(i^{th}\) block (modulo \(v\)) of the design as follows and then considering border plot at left end of each block.

Lemma 2.1

Let \(v\) be odd number. Then there exists a complete block design with \(b = v(v - 1)\) such that a circular balanced uniform repeated measurement design with minimum number of experimental units exists.

Proof: By constructions. Let \(v = 2s + 1\) and label the \(v\) treatments by \(0, 1, 2, ..., 2s\). Now consider the \(v\) treatments as \(v\) vertices in \(S^*_i\), can be covered by the following \(2s\) disjoint directed Hamiltonian cycles. \(H = \left( H^*_1, H^*_2, ..., H^*_s \right) \) Where,

\[
H^*_j = (0, j, j+1, j-1, j+2, j-2, ..., j+(s-1), j-(s-1), j+s)
\]

\[
H^-_j = (0, j+s, j-(s-1), j+(s-1), ..., j-2, j+2, j-1, j+1, j)
\]

for \(1 \leq i \leq s\). All the elements except 0 are taken as the positive integers 1, 2, ..., \(2s\mod 2s\). Each first \(v\) vertices of each Hamiltonian cycle is the initial column of a \(v \times v\) latin square which can be constructed by permutating the initial column. The method of cycling permuting of initial column for \(H^+_j\) is given in table 1.1.
The method of cycling permuting of initial column for $H_j^-$ is similar to that for $H_j^+$ and is given in table 1.2.

\[
\begin{pmatrix}
0 & j & j+1 & j-1 & j+2 & j-2 & \ldots & j+(s-1) & j-(s-1) & j+s \\
j & j+1 & j-1 & j+2 & j-2 & j+3 & \ldots & j-(s-1) & j+s & 0 \\
j+1 & j-1 & j+2 & j-2 & j+3 & j-3 & \ldots & j+k & 0 & j \\
j-1 & j+2 & j-2 & j+3 & j-3 & j+4 & \ldots & 0 & j & j+1 \\
j+2 & j-2 & j+3 & j-3 & j+4 & j-4 & \ldots & j & j+1 & j-1 \\
j-2 & j+3 & j-3 & j+4 & j-4 & j+5 & \ldots & j+1 & j-1 & j+2 \\
0 & j & j+1 & j-1 & j+2 & \ldots & j-(s-1) & j+(s-2) & j-(s-2) & j-1 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
j+(s-1) & i-(s-1) & j+s & 0 & j & j+1 & \ldots & j-(s-3) & j+(s-2) & j-(s-2) \\
j-(s-1) & i+s & 0 & j & j+1 & j-1 & \ldots & j+(s-2) & j-(s-2) & j+(s-1) \\
j+s & 0 & j & j+i & j-1 & j+2 & \ldots & j-(s-2) & j+(s-1) & j-(s-1)
\end{pmatrix}
\]

Table 1.1

Now the 2s Latin square together constitute a circular balanced uniform repeated measurement design with minimum number of experimental units exists whenever $v$ is an odd number.

**Theorem 2.1**

A design $d^* \in D_1(v, b, k)$ is variance balanced design for the estimation of direct effects if $G_1 = G_2 = G_3 = v(J_v - I_v)$ and $M_1 = M_2 = M_3 = J_{v^2}$ with parameters $v = 2s + 1, b = v(v-1), k = v$ when left and right neighbor effects are same.

**Proof:** Afsarinejad (1984) constructed circular balanced uniform repeated measurement design. The design of $d^*$ is using in two opposite directed Hamiltonian Cycles, the number of treatments $v$ is an odd number is given in lemma 2.1. The following given in the equation (1.1).

Let $G_1, G_2, and G_3$ are the $v \times v$ incidence matrices pertaining to direct versus left treatments, direct versus right treatments and left versus right treatments respectively.
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\( M_1, M_2 \) and \( M_3 \) are \( v \times b \) incidence matrices pertaining to direct treatment versus blocks, left treatments versus blocks and right treatments versus blocks respectively. Further \( K = \text{diag} (k_1, k_2, \ldots, k_b) \), \( R_v = \text{diag} (r_1, r_2, \ldots, r_v) \); \( R_\rho = \text{diag} (r_{21}, r_{22}, \ldots, r_{2v}) \),

\( r_i (i = 1, 2, \ldots, v) \) being the number of times the \( i^{th} \) treatment appears in the design, \( r_{11}(r_{21}) \) being the number of times the treatments in the design has \( l^{th} \) treatment as left (right) neighbour. The \( 3v \times 3v \) matrix \( C \) is symmetric, non negative definite with zero row and column sums.

\[
C = \begin{bmatrix}
R_v - \frac{1}{K} M_1 M_1' & G_1 - \frac{1}{K} M_1 M_2' & G_2 - \frac{1}{K} M_1 M_3' \\
G_1' - \frac{1}{K} M_2 M_1' & R_\delta - \frac{1}{K} M_2 M_2' & G_3 - \frac{1}{K} M_2 M_3' \\
G_2' - \frac{1}{K} M_3 M_1' & G_3' - \frac{1}{K} M_3 M_2' & R_\rho - \frac{1}{K} M_3 M_3'
\end{bmatrix}
\]

(2.2)

The information matrix for estimating the direct effects of treatments obtained from (2.2) is as follows:

\[
C_v = U_{11} - U_{12} U_{22}^{-1} U_{21}
\]

(2.3)

Where, \( U_{11} = R_v - \frac{1}{K} M_1 M_1' \), \( U_{12} = \begin{bmatrix} G_1 - \frac{1}{K} M_1 M_2' & G_2 - \frac{1}{K} M_1 M_3' \end{bmatrix} \), \( U_{22} = \begin{bmatrix} R_\delta - \frac{1}{K} M_2 M_2' & G_3 - \frac{1}{K} M_2 M_3' \\
G_3' - \frac{1}{K} M_3 M_2' & R_\rho - \frac{1}{K} M_3 M_3' \end{bmatrix} \)

We have to show that \( C_v \) is completely symmetric and variance balanced design.

Under the conditions of \( D_1(v, b, k) \), it is seen that \( R_v = R_\delta = R_\rho = v I_v \), \( K = k I_b \), \( G_1 = G_2 = G_3 = v (J_v - I_v) \) and \( M_1 = M_2 = M_3 = J_{vb} \), \( v = 2s + 1, b = v(v - 1), k = v \),

\[
A = v(k - 1) \left( I - \frac{J_v}{v} \right), \quad B = \left[ -v \left( I - \frac{J_v}{v} \right) - v \left( I - \frac{J_v}{v} \right) \right] \\
D = \left[ \begin{array}{cc}
v(k - 1) \left( I - \frac{J_v}{v} \right) & -v \left( I - \frac{J_v}{v} \right) \\
-v \left( I - \frac{J_v}{v} \right) & v(k - 1) \left( I - \frac{J_v}{v} \right) \end{array} \right]
\]
\[ C = \begin{pmatrix} v(k-1)\left( I - \frac{J}{v} \right) & -v\left( I - \frac{J}{v} \right) & -v \left( I - \frac{J}{v} \right) \\ -v\left( I - \frac{J}{v} \right) & v(k-1)\left( I - \frac{J}{v} \right) & -v \left( I - \frac{J}{v} \right) \\ -v\left( I - \frac{J}{v} \right) & -v \left( I - \frac{J}{v} \right) & v(k-1)\left( I - \frac{J}{v} \right) \end{pmatrix} \]

\[
C_r = \begin{pmatrix} a & b & b & b & \ldots & b \\ b & a & b & b & \ldots & b \\ b & b & a & b & \ldots & b \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & b & \ldots & a \end{pmatrix}_{v \times v}
\]

\[
= (a - b)I_v - bI_v
\]

Where \( a = r - \sum_j \left[ \frac{n_j^2}{k_j^3} \right] \) and \( b = -\sum_j \left[ \frac{n_j n_{j'}}{k_j} \right] \).

\[
C_r = \frac{vk(k-3)}{k-2} \left( I - \frac{J}{v} \right)
\]

The eigenvalues of the above \( C_r \) matrix are

\[
\theta = a\left[ \frac{v}{v-1} \right]
\]

With multiplicity \( v-1 \) and zero with multiplicity 1, where \( v \) is the number of treatments. Hence the design is variance balanced.

### 2.1 Numerical illustrations

Let \( v=5, b=20, r=20, k=5 \). Then the four Hamiltonian Cycles are

\[
C_1^+ = (0 \ 1 \ 2 \ 4 \ 3) \quad C_1^- = (0 \ 3 \ 4 \ 2 \ 1)
\]

\[
C_2^+ = (0 \ 2 \ 3 \ 1 \ 4) \quad C_2^- = (0 \ 4 \ 1 \ 3 \ 2)
\]
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\[
\begin{array}{cccccc|cccccc}
3 & 0 & 1 & 2 & 4 & 3 & 0 & 1 & 2 & 4 & 3 & 0 \\
0 & 1 & 2 & 4 & 3 & 0 & 1 & 2 & 4 & 3 & 0 & 1 \\
1 & 2 & 4 & 3 & 0 & 1 & 2 & 4 & 3 & 0 & 1 & 2 \\
2 & 4 & 3 & 0 & 1 & 2 & 4 & 3 & 0 & 1 & 2 & 4 \\
4 & 3 & 0 & 1 & 2 & 4 & 3 & 0 & 1 & 2 & 4 & 3 \\
4 & 0 & 2 & 3 & 1 & 4 & 0 & 2 & 3 & 1 & 4 & 0 \\
0 & 2 & 3 & 1 & 4 & 0 & 2 & 3 & 1 & 4 & 0 & 2 \\
2 & 3 & 1 & 4 & 0 & 2 & 3 & 1 & 4 & 0 & 2 & 3 \\
3 & 1 & 4 & 0 & 2 & 3 & 1 & 4 & 0 & 2 & 3 & 1 \\
1 & 4 & 0 & 2 & 3 & 1 & 4 & 0 & 2 & 3 & 1 & 4 \\
\end{array}
\]

\[
C_\tau = \frac{10}{3} \begin{pmatrix}
4 & -1 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
-1 & -1 & -1 & -1 & 4
\end{pmatrix}
\]

\[
= \frac{50}{3} \left[ I_4 - \frac{1}{5} E_{vv} \right] = \frac{50}{3} * I_4 - \frac{50}{3 \times 5} E_{vv}
\]

Characteristic roots are \( \left( \frac{50}{3} \right) \) and \( \left( \frac{50}{3} \right) + \left( \frac{50}{15} \right) * 5 = 0 \) with multiplicities 3 and 1, respectively. Hence the block design is variance balanced.

**Theorem 2.2**

A design \( d^* \in D_2(v, b, k) \) is variance balanced design for the estimation of direct effects if \( G_1 = G_2 = M_1 = N_2 = N_3 = \lambda J \), with parameter \( v = b, r = k = 2v^2, \lambda = 2v \)

**Proof:** For given \( v \), a series of block designs balanced for direct effect with parameters \( v = b, r = k = 2v^2, \lambda = 2v \), Can be obtained by writing the \( i^{th} \) block (modulo \( v \)) of the design as follows and then considering border plot at left end of each block. Where \( i = 1, 2, 3, ..., v \)

\[
\begin{array}{cccccc|cccccc}
i & 1 & i & 2 & ... & i & v & i+1 & 1 & i+1 & 2 & ... & i+1 & v & ... & i+v & 1 & i+v & 2 & ... & i+v & v \\
i+1 & i+1 & 2 & ... & i+1 & v & i+1 & 1 & i+v & 2 & ... & i+v & v & ... & i+v & 1 & i+2 & 2 & ... & i+2 & v \\
i+v & 1 & i+v & 2 & ... & i+v & v & i & 1 & i & 2 & ... & i & v & ... & i+1 & 1 & i+2 & 2 & ... & i+2 & v \\
\end{array}
\]
The following given in the equation (1.1). Let $G_1, G_2, and G_3$ are the $v \times v$ incidence matrices pertaining to direct versus left treatments, direct versus right treatments and left versus right treatments respectively. $M_1, M_2, and M_3$ are $v \times b$ incidence matrices pertaining to direct treatment versus blocks, left treatments versus blocks and right treatments versus blocks respectively. Further $K = \text{diag}(k_1, k_2, ..., k_b)$, $R_r = \text{diag}(r_{11}, r_{12}, ..., r_{1v})$, $R_g = \text{diag}(r_{21}, r_{22}, ..., r_{2v})$, $R_r = \text{diag}(r_{31}, r_{32}, ..., r_{3v})$, $r_1 (l = 1, 2, ..., v)$ being the number of times the $1^{\text{st}}$ treatment appears in the design, $r_{11} (r_{21})$ being the number of times the treatments in the design has $1^{\text{st}}$ treatment as left (right) neighbour. The $3v \times 3v$ matrix $C$ is symmetric, non negative definite with zero row and column sums.

\[
C = \begin{bmatrix}
R_r - \frac{1}{K} M_1 M_1' & G_1 - \frac{1}{K} M_1 M_2' & G_2 - \frac{1}{K} M_1 M_3' \\
G_1' - \frac{1}{K} M_2 M_1' & R_g - \frac{1}{K} M_2 M_2' & G_3 - \frac{1}{K} M_2 M_3' \\
G_2' - \frac{1}{K} M_3 M_1' & G_3' - \frac{1}{K} M_3 M_2' & R_r - \frac{1}{K} M_3 M_3'
\end{bmatrix}
\]

(2.4)

The information matrix for estimating the direct effects of treatments obtained from (2.4) is as follows:

\[
C_r = U_{11} - U_{12} U_{22}^{-1} U_{21}
\]

(2.5)

where, $U_{11} = R_r - \frac{1}{K} M_1 M_1'$, $U_{12} = \begin{bmatrix} G_1 - \frac{1}{K} M_1 M_2' & G_2 - \frac{1}{K} M_2 M_3' \end{bmatrix}$, $U_{22} = \begin{bmatrix} R_g - \frac{1}{K} M_2 M_2' & G_3 - \frac{1}{K} M_3 M_3' \\
G_3' - \frac{1}{K} M_3 M_2' & R_r - \frac{1}{K} M_3 M_3'
\end{bmatrix}$

We have to show that $C_r$ is completely symmetric and variance balanced design.

Under the conditions of $D_2(v, b, k)$, it is seen that the parameter $v = b, r = k = 2v^2, \lambda = 2v$

$$R_r = R_g = R_r = rI_v, \ K = kI_b, \ G_1 = G_2 = M_1 = N_2 = N_3 = \lambda J_v$$

Then $A = 2v \left( I - \frac{J}{v} \right)$
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\[ D = \begin{pmatrix}
2v \left( I - \frac{J}{v} \right) & N_4 - 2vJ_v \\
N_4' - 2vJ_v & 2v \left( I - \frac{J}{v} \right)
\end{pmatrix}, \]
\[ C = \begin{pmatrix}
2v \left( I - \frac{J}{v} \right) & 0 & 0 \\
0 & 2v \left( I - \frac{J}{v} \right) & N_4 - 2vJ_v \\
0 & N_4' - 2vJ_v & 2v \left( I - \frac{J}{v} \right)
\end{pmatrix} \]

\[ C_v = \begin{pmatrix}
a & b & b & b & \ldots & b \\
b & a & b & b & \ldots & b \\
b & b & a & b & \ldots & b \\
b & b & b & a & \ldots & b \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
b & b & b & b & \ldots & a
\end{pmatrix}_{v \times v} \]

\[ = (a-b)I_v - bJ_v \]

Where, \( a = r_j - \sum_j \left[ \frac{n_j^2}{k_j} \right] \) and \( b = -\sum_j \left[ \frac{n_j n_{i,j}}{k_j} \right] \).

\[ C_v = 2v \left( I - \frac{J}{v} \right) \]

The eigenvalues of the above \( C_v \) matrix are

\[ \theta = a \left[ \frac{v}{v-1} \right] \]

With multiplicity \( v-1 \) and zero with multiplicity 1, where \( v \) is the number of treatments. Hence the design is variance balanced

2.2 Numerical illustrations
For given \( v \), a series of block designs balanced for direct effect with parameters

\[ v = b, \ r = k = 2v^2, \lambda = 2v \], Can be obtained by writing the \( i^{th} \) block (modulo \( v \)) of the design as follows and then considering border plot at left end of each block

\[
\begin{align*}
41 & \ 1 \ 1 \ 2 \ 1 \ 3 \ 1 \ 4 \ 2 \ 12 \ 2 \ 2 \ 3 \ 2 \ 4 \ 3 \ 1 \ 3 \ 23 \ 3 \ 3 \ 3 \ 4 \ 4 \ 1 \ 4 \ 2 \ 4 \ 3 \ 4 \ 4 \ 41 \\
42 & \ 1 \ 2 \ 2 \ 2 \ 3 \ 2 \ 4 \ 3 \ 13 \ 2 \ 3 \ 3 \ 3 \ 4 \ 4 \ 1 \ 24 \ 3 \ 4 \ 4 \ 1 \ 1 \ 1 \ 2 \ 3 \ 1 \ 4 \ 2 \ 4 \ 3 \ 4 \ 42 \\
43 & \ 1 \ 3 \ 2 \ 3 \ 3 \ 3 \ 4 \ 4 \ 14 \ 2 \ 4 \ 3 \ 4 \ 4 \ 1 \ 1 \ 1 \ 21 \ 3 \ 1 \ 4 \ 2 \ 1 \ 2 \ 2 \ 3 \ 2 \ 3 \ 4 \ 43 \\
44 & \ 1 \ 4 \ 2 \ 4 \ 3 \ 4 \ 4 \ 1 \ 11 \ 2 \ 1 \ 3 \ 3 \ 4 \ 1 \ 4 \ 2 \ 1 \ 2 \ 22 \ 3 \ 2 \ 4 \ 3 \ 1 \ 3 \ 2 \ 3 \ 3 \ 4 \ 44
\end{align*}
\]
The parameters of design are, \( v = 4, b = 4, r = 32, k = 32, \lambda = 8 \)

\[ G_1 = G_2 = M_1 = N_2 = N_3 = \lambda J_v = 8J_v \]

\[
C_r = 8 \begin{pmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3
\end{pmatrix}
\]

\[ = 32 \left[ I_4 - \frac{1}{4} E_{vv} \right] = 32*I_4 - \frac{32}{4} E_{vv} \]

Characteristic roots are 32 and \( 32 + \left( \frac{32}{4} \right) * 4 = 0 \) with multiplicities 3 and 1, respectively. Hence the block design is variance balanced.

3. OPTIMAL VARIANCE BALANCED BLOCK DESIGN

Let \( d \) belong \( D(v, b, r, k, \lambda) \) with \( C_d \) matrix. Where, \( C_d = R_d - N_d K^{-1} N_d^T \).

Design \( d^* \) will be A-optimal if it maximizes \( \text{tr}(C_d) \),

\[
\text{tr}(C_d) = \text{tr}(R_d - N_d K^{-1} N_d^T) = \text{tr}(R_d) - \left( N_d K^{-1} N_d^T \right)
\]

For a design \( d \), it can be shown that the sums of the variance of the estimates of all elementary constraint are proportional to the sum of the reciprocals of the non-zero eigenvalues of \( C \). Let \( \{\theta_1, \theta_2, \theta_3, \ldots, \theta_{(v-1)}\} \) be non-zero eigenvalues. As we know that for variance balanced design there will be only one non-zero eigenvalue with multiplicities \( (v-1) \) of \( C_d \) matrix of design \( d \). That is, \( \theta_1 = \theta_2 = \ldots = \theta_{(v-1)} = \theta \) as \( C \)-matrix is positive semi-definite. Finally, we can say that the design is A-optimal if

\[
\sum_{i=1}^{(v-1)} \frac{1}{\theta_i} \geq \frac{(v-1)^2}{\text{tr}(C_d)}
\]

D-Optimality: Let \( \theta_1, \theta_2, \theta_3, \ldots, \theta_{(v-1)} \) be non-zero eigenvalues with multiplicities \( (v-1) \) of \( C_d \) matrix of design \( d \). A design is D-optimal if,

\[
\prod_{i=1}^{(v-1)} \frac{1}{\theta_i} \leq \prod_{i=1}^{(v-1)} \left( \frac{1}{\theta_i} \right)
\]

E-optimality: Let \( \theta_1, \theta_2, \theta_3, \ldots, \theta_{(v-1)} \) be non-zero eigenvalues with multiplicities \( (v-1) \) of \( C_d \) matrix of design \( d \). A design is E-optimal if
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\[ \text{Min}(\theta_i) \leq \frac{\text{tr}(C_d)}{(v - 1)} \]

3. To prove optimality of nearest neighbor design

Consider the VB block design obtained in numerical illustration 2.1 with parameters \( v = 5, b = 20 = r, k = 5, s = 2 \). The trace of C-matrix of VB design comes out as \( \left( \begin{array}{c} 50 \\ 3 \end{array} \right) \) and non-zero eigenvalue of C-matrix is \( \theta = \left( \begin{array}{c} 50 \\ 3 \end{array} \right) \) Multiplicity 3. Here the inequality

\[ \sum_{i=1}^{(v-1)} \frac{1}{\theta_i} \geq \frac{(v - 1)^2}{\text{tr}(C_d)}, \quad \sum_{i=1}^{(v-1)} \frac{1}{\theta_i} \geq \frac{(v - 1)^2}{\text{tr}(C_d)}, \quad \frac{4}{150} = \frac{4}{150} \]

Holds true which is required condition for the design \( d \) to be A-optimal. Hence VB block design constructed in numerical illustration 2.1 is an A-optimal.

\[ \prod_{i=1}^{(v-1)} \frac{1}{\theta_i} \leq \prod_{i=1}^{(v-1)} \left( \frac{1}{(v - 1)} \right) \left( \frac{1}{506250000} \right) = \left( \frac{1}{506250000} \right) \]

Holds true and hence the VB design constructed in numerical illustration 2.1 is D-optimal.

Also, the inequality

\[ \text{Min}(\theta_i) \leq \frac{\text{tr}(C_d)}{(v - 1)}, \quad \left( \begin{array}{c} 50 \\ 3 \end{array} \right) = \left( \begin{array}{c} 50 \\ 3 \end{array} \right) \]

Holds true and hence the VB block design constructed in numerical illustration 2.1 is E-optimal. Since the constructed VB is A-optimal, D-optimal as well as E-optimal, hence the design is universal optimal.

Consider the VB block design obtained in numerical illustration 2.2 with parameters \( v = b, r = k = 2v^2, \lambda = 2v \). The trace of C-matrix of VB design comes out as 9 and non-zero eigenvalue of C-matrix is \( \theta = 32 \) Multiplicity 3. Here the inequality

\[ \sum_{i=1}^{(v-1)} \frac{1}{\theta_i} \geq \frac{(v - 1)^2}{\text{tr}(C_d)}, \quad \left( \begin{array}{c} 3 \\ 32 \end{array} \right) = \left( \begin{array}{c} 3 \\ 32 \end{array} \right) \]

Holds true which is required condition for the design \( d \) to be A-optimal. Hence VB block design constructed in numerical illustration 2.1 is an A-optimal.
Holds true and hence the VB design constructed in numerical illustration 2.1 is D-optimal.

Also, the inequality, \( \text{Min}(\theta_i) \leq \frac{\text{tr}(C_d)}{(v-1)} \), \( v=32 \)

Holds true and hence the VB block design constructed in numerical illustration 2.1 is E-optimal. Since the constructed VB is A-optimal, D-optimal as well as E-optimal, hence the design is universal optimal.

REFERENCE


