The Deficient Discrete Quartic Spline Interpolation

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Abstract
The object of the paper is to investigate precise error estimate existence and uniqueness of deficient discrete quartic spline interpolation.

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1. INTRODUCTION
Deficient Splines are more useful than usual spline because they require less continuity requirement. Dubey, Rana and Dubey [10] have obtained precise error estimate concerning deficient discrete Quartic spline which interpolates given functional values at one intermediate points see also [1]. Rana and Dubey [6] have obtained local behavior of deficient discrete cubic spline which is some time used to smooth histogram. Best summation formula for discrete cubic spline given by Mangasarian and Schumaker [3, 4]. For some constructive aspect for discrete spline see Jia [7], Niyazi Ari and Savaş Tuylu [2], Astor and Duris [5].

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2. EXISTENCE AND UNIQUENESS

Let us consider a mesh point on [0, 1] defined by

\[ P: 0 = x_0 < x_1 < \ldots \ldots \ldots x_n = 1 \]

Such that

\[ x_i - x_{i-1} = P_i \quad \text{for} \quad i = 1, 2, \ldots \ldots n \]

Throughout \( h \) will represent a given position real number. The class \( D(m, r, p, h) \) of deficient discrete spline of degree-m with deficiency \( r \) is the set of all continuous function \( S(x, h) \) such that for \( i = 0, 1, \ldots \ldots, n-1 \) the restriction \( S_i \) of \( S(x, h) \) on \( [x_i, x_{i+1}] \) is a polynomial of degree \( m \) or less and

\[ D^{(1)}_h s_i(x_i, h) = D^{(0)}_h s_{i-1}(x_i, h), \quad j = 0, 1, \ldots \ldots, m-r-1. \quad (2.1) \]

Where the difference operator \( D^{(i)}_h \) for a function \( f \) is defined by

\[ D^{(0)}_h f(x) = f(x) \]
\[ D^{(1)}_h f(x) = \frac{f(x+h) - f(x-h)}{2h} \quad \text{for} \quad i = 0, 1 \]

Taking \( m=4 \) and \( r=1 \) in (2.1), the class of all such deficient discrete quartic with deficiency 1 satisfies the boundary condition

\[ s(x_0, h) = f(x_0, h) \]
\[ s(x_n, h) = f(x_n, h) \]

is denoted by \( D^* (4, 1, P, h) \)

**Problem 1.1:** Given \( h > 0 \) for what restriction on \( P \) does there exist an unique \( s(x, h) \in D^*(4, 1, P, h) \) which satisfies the following interpolatory condition.

\[ s(\alpha_i) = f(\alpha_i) \quad \alpha_i = x_i + \frac{1}{3} P_i \quad (2.2) \]
\[ s(\beta_i) = f(\beta_i) \quad \beta_i = x_i + \frac{1}{2} P_i \quad (2.3) \]

\[ D^{(1)}_h s\{\gamma_i\} = D^{(1)}_h s\{\gamma_i\} \quad \text{where} \quad \gamma_i = x_i + \theta P_i \quad 0 < \theta < 1 \quad (2.4) \]

Let \( P(t) \) be the quartic polynomial on [0, 1] then we can show that

\[ P(t) = P \left( \frac{1}{3} \right) Q_1(t) + P \left( \frac{1}{2} \right) Q_2(t) + P(\theta)Q_3(t) + P(0)Q_4(t) + P(1)Q_5(t) \quad (2.5) \]
Where

\[
Q_1(t) = \left\{ 36 \theta^3 - \frac{81}{2} \theta^2 + 9\theta + \left( 36\theta - \frac{27}{2} \right) h^2 \right\} z \\
+ \left\{ -108 \theta^3 - \frac{189 \theta^2}{2} - \frac{9}{2} + \left( -108\theta + \frac{63}{2} \right) h^2 \right\} z^2 \\
+ \left\{ 72\theta^3 - 63\theta + \frac{27}{2} + 72\theta h^2 \right\} z^3 \\
+ \left\{ 54\theta - 9 - 54\theta^2 - 18h^2 \right\} z^4 / A
\]

\[
Q_2(t) = \left\{ -\frac{64\theta^3}{3} + 64\theta^2 - \frac{32\theta}{9} + \left( -\frac{64\theta}{3} + \frac{64}{9} \right) h^2 \right\} z \\
+ z^2 \left\{ \frac{256}{3}\theta^3 - \frac{48}{27} + \left( \frac{256\theta}{3} - \frac{208}{9} \right) h^2 - \frac{208}{3} \theta^2 \right\} \\
+ z^3 \left\{ -64\theta^3 + \frac{416\theta}{9} - \frac{64}{9} - 64\theta h^2 \right\} \\
+ z^4 \left\{ 48\theta^2 + 16h^2 - \frac{128\theta}{3} + \frac{16}{3} \right\} / A
\]

\[
Q_3(f) = \left\{ -\frac{z}{9} + \frac{2z^2}{3} - \frac{11z^3}{9} + \frac{2z^4}{3} \right\} / A
\]

\[
Q_4(t) = \left\{ 1 + \left\{ -16\theta^3 + 20\theta^2 - \frac{50\theta}{9} + \left( -16\theta + \frac{20}{3} \right) h^2 \right\} z \\
+ \left\{ \left( \frac{88\theta}{3} - \frac{85}{9} \right) h^2 + \left( \frac{88\theta^3}{3} - \frac{85\theta}{3} \theta^2 \right) + \frac{25}{9} \right\} z^2 \\
+ \left\{ -16\theta^3 + \frac{170}{9} \theta - \frac{20}{3} - 16\theta \cdot h^2 \right\} z^3 \\
+ \left\{ 4 - \frac{44}{3} \theta + 12\theta^2 + 4h^2 \right\} z^4 \right\} / A
\]

\[
Q_5(z) = \left\{ \frac{4}{3} \theta^3 - \frac{5}{6} \theta^2 + \frac{\theta}{9} + \left( \frac{4}{3} \theta - \frac{5}{18} \right) h^2 \right\} z \\
+ \left\{ -\frac{20}{3} \theta^3 + \frac{19}{6} \theta^2 - \frac{1}{18} - \left( \frac{20}{3} \theta - \frac{19}{18} \right) h^2 \right\} z^2 \\
+ \left\{ 8\theta^3 + \frac{5}{18} - \frac{38\theta}{18} + 8\theta h^2 \right\} z^3 + \left\{ \frac{10}{3} \theta - 6\theta^2 - 2h^2 - \frac{1}{3} \right\} z^4 \right\} / A
\]
Where

\[ A = \left[ (8\theta^3 - \frac{4}{3}\theta^2 - \frac{4}{3}\theta - \frac{1}{9} + \left(\frac{8}{3}\theta - \frac{11}{9}\right)h^2 \right] \]

Now we are set to answer the problem 1.1 in the following

**Theorem 2.1**: for any \( h > 0 \) there exist an unique deficient discrete quartic spline \( s(x, h) \in D^*(4, 1, P, h) \) which satisfies the condition (2.2) - (2.4)

Proof :- Denoting \((x-x_i)/P_i \) by \( t \), \( 0 < t < 1 \) we can write (2.5) in the form of the restriction \( s_i(x, h) \) of the deficient discrete quartic spline \( s(x, h) \) on \([x_i, x_{i+1}]\) as follows

\[
s_i(x, h) = f(\alpha_i)Q_1(t) + f(\beta_i)Q_2(t) + P_i(\gamma_i)Q_3(t) + s_i(x_i)Q_4(t) + s_i(x_{i+1})Q_5(t) \tag{2.6}
\]

From equation (2.6) we can easily verified that \( s_i(x, h) \) is quartic on \([x_i, x_{i+1}]\) for \( i = 0, 1, \ldots, n-1 \) satisfying (2.2) - (2.4), we apply the continuity of first difference of \( s_i(x, h) \) at \( x_i \) in (2.1) to see that

\[
p_i^3 \left[ \left( \frac{16}{3}\theta^3 - \frac{34}{3}\theta^2 + \frac{68}{9}\theta - \frac{14}{9} + h^2 \left( \frac{16}{3}\theta - \frac{34}{9} \right) \right) \right] p_{i-1}^2
\]

\[ + h^2 \left[ 16\theta^3 - 48\theta^2 + \frac{358\theta}{9} + (16\theta - 16)h^2 + \frac{28}{3} \right] s_{i-1} \]

\[ + p_{i-1}^3 \left[ -12\theta^3 + \frac{37\theta^2}{2} - \frac{64\theta}{9} + \frac{11}{18} + h^2 \left( \frac{37}{6} - 12\theta \right) \right] p_{i-1}^2
\]

\[ + \left( 8\theta^2 + \frac{19}{18} - \frac{101\theta}{9} + 24\theta^2 - (8\theta - 8)h^2 \right) h^2 \]

\[ + p_{i-1}^3 \left[ \left( -16\theta^3 + 20\theta^2 - \frac{50\theta}{9} + \left( -16\theta + \frac{20}{3} \right)h^2 \right) \right] p_{i-1}^2
\]

\[ + \left( -16\theta^3 + \frac{170}{9}\theta - \frac{20}{3} - 16\theta h^2 \right) h^2 \]

\[ + p_{i-1}^3 s_{i+1} \left[ \left( -128\theta^3 + \frac{224}{3}\theta^2 + \frac{28}{9} + \left( \frac{4}{3} - \frac{5}{18} \right) h^2 \right) \right] p_{i-1}^2
\]

\[ + \left( 8\theta^3 + \frac{5}{18} - \frac{38\theta}{18} + 8\theta h^2 \right) h^2 = F_i \]

Where,

\[
F_i = p_i^3 \left\{ 36\theta^3 + 36\theta - \frac{9}{2} - \frac{135}{2}\theta^2 + \left( 36\theta + \frac{45}{2} \right) h^2 \right\} p_{i-1}^2
\]

\[ + h \left\{ 72\theta^3 - 153\theta - \frac{45}{2} - 216\theta^2 + 72(\theta - 1)h^2 \right\} f(\alpha_{i-1}) - p_{i-1}^3 \left\{ 36\theta^3 - \frac{81\theta^2}{2} + 9\theta + \left( 36\theta - \frac{27}{2} \right) h^2 \right\} p_{i-1}^2
\]

\[ + \left\{ 72\theta^3 - 63\theta + \frac{27}{2} + 72\theta h^2 \right\} h^2 f(\alpha_i) + p_i^3 \left\{ \left( \frac{128}{3}\theta^3 + \frac{224}{3}\theta^2 - \frac{320\theta}{9} + \frac{32}{9} \right) + \right\}
\]
3. ERROR BOUNDS

It may be observed that system of equation (2.7) may be written as

\[ A(h)M(h) = F \]  \hspace{1cm} (3.1)

Where \( A(h) \) is coefficient matrix and \( M(h) = s_i(x, h) \). However as already shown in proof of theorem 2.1 \( A(h) \) is invertible. Denoting the inverse of \( A(h) \) by \( A^{-1}(h) \) we note that row max norm \( \| A^{-1}(h) \| \) satisfies the following inequality

\[ \| A^{-1}(h) \| \leq y(h) \]  \hspace{1cm} (3.2)

Where,

\[ y(h) = \max \{ C_i(h) \}^{-1} \]

For convenience we assume in this section that 1=Nh, where N is a positive integer. It is also assumed that the mesh points \( \{ x_i \} \) are such that \( x_i \in [0,1] \) for \( i=1, 2, \ldots, n \) where discrete interval \([0, 1]n\) is the set of points \( \{0, h, 2h, \ldots, Nh\} \). For a function \( F \) and two distinct points \( x_1, x_2 \) in its domain the divided difference is defined by

\[ [x_1, x_2]f = \frac{f(x_1) - f(x_2)}{(x_1 - x_2)}. \]

For convenience we write \( f^{(1)} \) for \( D_h^{(1)} f \) and \( w(f, p) \) is the modulus of continuity of \( f \). The discrete norm of a function \( f \) over interval \([0, 1]n\) is defined by

\[ \|f\| = \max_{x \in [0,1]} |f(x)| \]  \hspace{1cm} (3.3)

Without assuming any smoothness condition on data \( f \), we shall obtain in the following bounds of error function

\[ e(x) = s(x, h) - f(x) \] over the discrete interval \([0, 1]h\)
**Theorem:** Suppose \( s(x, h) \) is the deficient discrete quartic spline interpolation of theorem 2.1 then

\[
|| e(x) || \leq y(h) k(P, h) w(f, P) \tag{3.4}
\]

\[
|| e(x) || \leq k^*(P, h) w(f, P) \tag{3.5}
\]

and

\[
|| e_{1}(x) || \leq k^{**}(P, h) w(f, P) \tag{3.6}
\]

Where \( k(P, h) \), \( k^*(P, h) \) and \( k^{**}(P, h) \) are some positive function of \( p \) and \( h \)

**Proof:** writing \( f(x_i) = f_i \) we notice that the equation (3.1) may be written as

\[
A(h) \cdot e(x_i) = F_i(h) - A(h) f_i = L_i \quad \text{(say)} \tag{3.7}
\]

Put \( e(x) = s(x, h) - f_i(x) \)

We need the following result due to Lyche [8, 9] to estimate R.H.S. of (3.7)

**Lemma 3.1** - Let \( \{a_i\}_{i=1}^m \) and \( \{b_j\}_{j=1}^n \) be given of non negative real numbers such that

\[
Σa_i = Σb_j
\]

Then for any real valued function \( f \) defined on discrete interval \([0, 1]_h\) we have

\[
Σ_{i=1}^m a_i [x_{i,0}, x_{i,1}, \ldots, x_{ik}] f - Σ_{j=1}^n b_j [y_{j0}, y_{j1}, \ldots, y_{jk}] f | < w[f^{(k)}, 1 - kh|Σa_i|k|] \tag{3.8}
\]

Where \( x_{ik}, y_{jk} \in [0, 1] \) for relevant values of i, j and k. It may be observed that R.H.S. of (3.7) is written as

\[
(L_i) = |Σ_{i=1}^5 a_i [x_{i0}, x_{i1}] f - Σ_{j=1}^3 b_j [y_{j0}, y_{j1}] f |	ag{3.9}
\]

Where,

\[
a_1 = p_i^3 p_{i-1} \left\{ \frac{20}{3} \theta^3 - \frac{2 \theta}{9} + \frac{17}{36} - \frac{43}{12} \theta^2 - \left( \frac{10}{3} \theta + \frac{43}{36} \right) h^2 \right\} p_{i-1}^2
\]

\[
- h^2 \left\{ 4 \theta^3 + \frac{257 \theta}{18} - \frac{149}{36} - 12 \theta^2 + 4(\theta - 1)h^2 \right\}
\]

\[
a_2 = p_i^3 p_{i-1} \left\{ \frac{16}{3} \theta^3 - \frac{34}{3} \theta^2 + \frac{68 \theta}{9} - \frac{14}{9} + h^2 \left( \frac{16}{3} \theta - \frac{34}{9} \right) \right\} p_{i-1}^2
\]

\[
+ h^2 \left\{ 16 \theta^3 - 48 \theta^2 + \frac{20}{3} + \frac{358 \theta}{9} + 16(\theta - 1)h^2 \right\}
\]

\[
a_3 = p_i^3 p_{i-1} \left\{ \frac{2}{9} p_{i-1}^2 + \frac{13}{9} h^2 \right\}
\]
\[ a_4 = p_i^3 p_i \left[ \left\{ \frac{64}{18} \theta^3 - \frac{64}{18} \theta^2 + \frac{32 \theta}{54} + \left( \frac{64}{18} \theta - \frac{64}{54} \right) h^2 \right\} p_i^2 \\
+ \left\{ \frac{64}{18} \theta^3 - \frac{416}{54} \theta + \frac{64}{6} \theta h^2 \right\} h^2 \right] \]

\[ a_5 = p_i^3 p_{i-1} \left[ \left\{ \frac{-8}{9} \theta^3 + \frac{10}{18} \theta^2 - \frac{2 \theta}{27} - \left( \frac{8 \theta}{9} - \frac{10}{54} \right) h^2 \right\} p_i^2 \\
+ \left\{ \frac{-16}{3} \theta^3 - \frac{10}{54} + \frac{76}{18} \theta - \frac{16}{3} \theta h^2 \right\} h^2 \right] \]

\[ b_1 = p_i^3 p_{i-1} \left[ \left\{ \frac{63 + 6 \theta - \frac{3}{4} \frac{135}{12} \theta^2 + \left( 6 \theta - \frac{45}{12} \right) h^2 \right\} p_{i-1}^2 \\
+ h^2 \left\{ \frac{12 \theta^3 + \frac{153 \theta}{6} - \frac{45}{12} - 36 \theta^2 + (12 \theta - 12) h^2 \right\} \right] \]

\[ b_2 = p_i^3 p_{i-1}^3 \left[ \left\{ \frac{8}{3} \theta^3 - \frac{11}{3} \theta^2 + \frac{4}{3} \theta + \left( \frac{8 \theta}{3} - \frac{11}{3} \right) h^2 \right\} \right] \]

\[ b_3 = p_i p_{i-1}^3 \left[ \left\{ \frac{8}{3} \theta^3 - 3 \theta^2 + \frac{14}{27} \theta + \left( \frac{8 \theta}{3} - 3 \right) h^2 \right\} p_i^2 \\
+ \left\{ \frac{16}{3} \theta^3 - \frac{170 \theta}{27} + \frac{20}{9} + \left( \frac{16}{3} \theta h^2 \right) \right\} \right] \]

\[ b_4 = p_i^3 p_{i-1} \left[ -\frac{1}{9} \right] \]

\[ b_5 = p_i^3 p_{i-1}^3 \left( \frac{11}{9} \right) h^2 \]

and

\[ x_{10} = p_{i-1}, \quad x_{11} = x_i = x_{20} = x_{21} \]

\[ x_{20} = x_{i-1}, \quad x_{30} = y_{i-1}-h, \quad x_{31} = y_{i-1}+h \]

\[ x_{40} = \alpha_i, \quad x_{41} = \beta_i, \quad x_{50} = \alpha_i, \quad x_{51} = \alpha_{i-1} \]

\[ y_{10} = \alpha_{i-1}, \quad y_{11} = \beta_{i-1}, \quad y_{20} = x_i, \quad y_{21} = \alpha_i \]

\[ y_{30} = x_i, \quad y_{31} = \alpha_i, \quad y_{40} = \gamma_i-h=y_{50}, \quad y_{41} = \gamma_i+h=51 \]
\[
\sum_{i=1}^{5} a_i = \sum_{j=1}^{5} b_j
\]
\[
= p_i^3 p_{i-1}^3 \left[ \left( \frac{26}{3} \theta^3 - \frac{189}{12} \theta^2 + \frac{22}{3} \theta - \frac{1}{9} + \left( \frac{10\theta}{3} - \frac{1}{12} \right) h^2 \right) p_{i-1}^2 
+ h^2 \left\{ 12\theta^3 + \frac{153\theta}{6} - \frac{45}{12} - 36\theta^2 + 12(\theta - 1) h^2 \right\} 
+ p_i p_{i-1}^3 \left\{ \frac{8}{3} \theta^3 - 3\theta^2 + \frac{14}{27} \theta + \left( \frac{8\theta}{3} - 3 \right) h^2 \right\} p_i^2 
+ \left\{ \frac{16}{3} \theta^3 - \frac{170\theta}{27} + 1 + \frac{16}{3} \theta h^2 \right\} h^2 \right] 
\]
\[
= k(P,h) \text{ (Say)} \quad (3.10)
\]
Thus apply Lemma 3.1 suitable in (3.10) for \(m=n=5\) and \(k=1\) to see that
\[
| L_i | \leq k(P,h) \text{ w } (f^{(i)}) | I-P |
\]  \quad (3.11)
Now using the equation (3.2) and (3.11) in (3.7) we get
\[
|| e(x_i) || \leq y(h) k(P,h) \text{ w } (f^{(1)}) | I-P |
\]  \quad (3.12)
Thus in equality (3.4) of theorem (3.1) To obtain the bound of \(e(x)\) we replace \(s_i(x,h)\) by \(e(x_i)\) in equality (2.6) to get
\[
e(x) = [e(x_i) Q_1(t) + e(x_{i+1}) Q_2(t)] + M_i(f)
\]  \quad (3.13)
Where,
\[
M_i(f) = [f(\alpha_i) Q_1(t) + f(\beta_i) Q_2(t) + P_{i-1} f^{(i)}(\gamma_i) Q_3(t) + f(x_{i-1}) Q_4(t) + f(x_i) Q_5(t) - f(x)]
\]
A Little computation shows that \(M_i(f)\) in (3.13) may be rewritten in the form of divided difference as follows
\[
|M_i(f)| = \left| \sum_{i=1}^{3} a_i [x_{i0}, x_{i1}] f - \sum_{j=1}^{2} b_j [y_{i0}, y_{i1}] f \right| 
\]  \quad (3.14)
Where,
\[
a_1 = \left[ t(-6\theta^3 + \frac{81}{12} \theta^2 - \frac{3}{2} \theta - (6\theta - \frac{27}{12}) h^2 
+ t^2 \left\{ \frac{108}{6} \theta^3 - \frac{189}{12} \theta^2 + \frac{3}{4} - h^2 \left( -18\theta + \frac{21}{4} \right) \right\} 
+ t^3 \left\{ -12\theta^3 - \frac{21\theta}{2} \theta^2 - \frac{9}{4} - 12\theta h^2 \right\} + t^4 \left\{ -9\theta + \frac{3}{2} + 9\theta^2 + 3h^2 \right\} \right]
\]
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\[ a_2 = P_i \left[ \left\{ t(8\theta^3 - 10\theta^2 + \frac{25}{9} \theta + \left(8\theta - \frac{10}{3}\right) h^2) \right\} \\
+ \left\{ \frac{-44}{3} \theta^3 + \frac{85}{6} \theta^2 + \frac{25}{18} \left(\frac{44}{3} \theta - \frac{85}{18}\right) h^2 \right\} t^2 \\
+ \left\{ 8\theta^3 - \frac{170}{18} \theta + \frac{10}{3} + 8\theta h^2 \right\} t^3 + \left\{ -2 + \frac{29}{3} \theta - 6\theta^2 - 2h^2 \right\} t^4 \right] \]

\[ b_1 = P_i \left[ t\left\{ \frac{-2}{3} \theta^3 + \frac{5}{12} \theta^2 - \frac{\theta}{18} + \left(\frac{2}{3} \theta + \frac{5}{36}\right) h^2 \right\} \\
+ \left\{ \frac{10}{3} \theta^3 - \frac{19}{12} \theta^2 + \frac{1}{36} + \left(\frac{10}{3} \theta + \frac{19}{36}\right) h^2 \right\} t^2 \\
- \left\{ 4\theta^3 + \frac{5}{36} \theta - \frac{19}{36} + 4\theta h^2 \right\} t^3 - \left(\frac{5}{3} \theta - 3\theta^2 - h^2 - \frac{1}{6}\right) t^4 \right] \]

\[ b_2 = P_i \left( \frac{8}{3} \theta^3 - \frac{11}{3} \theta^2 + \frac{4\theta}{3} - \frac{1}{9} + \left(\frac{8}{3} \theta - \frac{11}{9}\right) h^2 \right) \]

\[ a_3 = P_i \left( \frac{1}{9} t + \frac{2}{3} t^2 - \frac{11}{9} t^3 + \frac{2}{3} t^4 \right) \]

And

\[ x_{10} = \alpha_i \quad x_{11} = \beta_i = x_{21} \]
\[ x_{20} = x_{i-1} \quad x_{30} = \gamma_i - h \quad x_{3i} = \gamma_i + h \]
\[ y_{10} = \beta_i \quad y_{11} = x_i \quad y_{20} = x_{i-1} \quad y_{21} = x \]

Clearly

\[ \sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j \]

\[ = \left[ \left\{ 2\theta^3 - \frac{39}{12} \theta^2 + \frac{23}{18} \theta - \frac{1}{9} + \left(2\theta - \frac{13}{12}\right) h^2 \right\} t \\
+ t^2 \left\{ \frac{20}{6} \theta^3 - \frac{19}{12} \theta^2 + \frac{1}{36} + \left(\frac{10}{3} \theta - \frac{19}{36}\right) h^2 \right\} \\
+ \left( -4\theta^3 + \frac{19}{18} \theta - \frac{5}{36} - 4\theta h^2 \right) t^3 + \left( -\frac{5}{6} + \frac{2}{3} \theta + 3\theta^2 + h^2 \right) t^4 \right] \]

\[ = K^* (P, h) \]

Therefore applying lemma (3.1) for m=3, n=2 and k=1 we get

\[ | M_i(f) | < K^*(P, h) w (f^{(1)}, P) \quad (3.15) \]
Finally applying bounds of (3.12) and (3.15) in (3.13) we get inequality (3.5) when proceed to obtain an upper bound for $e^{i\ell}(x)$ for this we use first difference operator in (2.6) and get

$$P_i D_h^{i\ell} s_i(x, h) = f(\alpha_i) Q_i^{i\ell}(t) + f(\beta_i) Q_2^{i\ell}(t) + P_{f^{i\ell}}(\gamma) Q_3^{i\ell}(t) + s_{i-1}(x) Q_4^{i\ell}(t) + s_i(x) Q_5^{i\ell}(t)$$

Replace $s_i(x)$ by $e(x)$ we get

$$P_i e^{i\ell}(x) = e_{i-1} Q_4^{i\ell}(t) + e_i Q_5^{i\ell}(t) + U_i(f)$$

(3.16)

Where,

$$U_i(f) = f(\alpha_i) Q_i^{i\ell}(t) + f(\beta_i) Q_2^{i\ell}(t) + \beta f^{i\ell} (\gamma) Q_3^{i\ell}(t) + f_{i-1} Q_4^{i\ell}(t) + f_i Q_5^{i\ell}(t) - P_{f^{i\ell}}(x)$$

(3.17)

Now rewriting $U_i(f)$ in terms of Divided difference we have

$$|U_i(f)| = \left| \sum_{i=1}^{3} a_i[x_{i0}, x_{i1}]_f - \sum_{j=1}^{2} b_j[y_{j0}, y_{j1}]_f \right|$$

Where,

$$a_1 = P_i \left\{ 8\theta^3 - 10\theta^2 + \frac{25}{9} \theta + h^2 \left( -8\theta + \frac{10}{3} \right) \right\}$$

$$+ t \left\{ -\frac{88}{3} \theta^3 + \frac{85}{3} \theta^2 - \frac{25}{9} - \frac{88}{3} \theta - \frac{85}{9} h^2 \right\}$$

$$+ (3t^2 + h^2) \left\{ 8\theta^3 - \frac{85}{3} \theta + \frac{10}{3} + 8\theta h^2 \right\}$$

$$+ \left\{ -2 + \frac{22}{3} \theta - 6\theta^2 - 2h^2 \right\} (4t)(t^2 + h^2)$$

$$a_2 = P_i \left\{ \frac{2}{3} \theta^3 - \frac{5}{12} \theta^2 + \frac{\theta}{18} + h^2 \left( \frac{2}{3} \theta - \frac{5}{36} \right) \right\}$$

$$+ \left\{ -\frac{20}{3} \theta^3 + \frac{19}{6} \theta^2 - \frac{1}{18} - \frac{20}{3} \theta - \frac{19}{18} h^2 \right\} 2t$$

$$+ \left\{ 4\theta^3 + \frac{5}{36} - \frac{19}{18} \theta + 4\theta h^2 \right\} (t^2 + h^2)$$

$$+ \left\{ \frac{5}{3} \theta - 3\theta^2 - h^2 - \frac{1}{6} \right\} 4t(t^2 + h^2)$$

$$a_3 = P_i \left\{ -\frac{1}{9} + \frac{4}{3} t - \frac{11}{9} (3t^2 + h^2) + \frac{8}{3} t(t^2 + h^2) \right\}$$
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\[ b_1 = p_i \left[ 6\theta^3 - \frac{81}{12} \theta^2 + \frac{3}{2} \theta + \left( 6\theta - \frac{9}{4} \right) h^2 \right. \]
\[ + t \left\{ -36\theta^3 + \frac{63}{2} \theta^2 - \frac{3}{2} + \left( -36\theta + \frac{21}{2} \right) h^2 \right\} \]
\[ + \left\{ 12\theta^3 - \frac{21}{2} \theta + \frac{9}{4} + 12\theta h^2 \right\} \left( 3t^2 + h^2 \right) \]
\[ + \left( 9\theta - \frac{3}{2} - 9\theta^2 - 3h^2 \right) 4t (t^2 + h^2) \]
\[ b_2 = p_i \left[ \frac{8}{3} \theta^3 - \frac{11}{3} \theta^2 + \frac{4}{3} \theta - \frac{1}{9} + \left( \frac{8}{3} \theta - \frac{11}{9} \right) h^2 \right] \]

It can easily seen that

\[ \sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j \]

\[ = p_i \left[ \frac{26}{3} \theta^3 - \frac{125}{2} \theta^2 + \frac{17}{6} \theta \left( \frac{26}{3} \theta - \frac{125}{36} \right) h^2 \right. \]
\[ + t \left\{ -36\theta^3 + \frac{63}{2} \theta^5 - \frac{3}{2} + \left( -36\theta + \frac{21}{2} \right) h^2 \right\} \]
\[ + \left\{ 12\theta^3 - \frac{21}{2} \theta + \frac{9}{4} + 12\theta h^2 \right\} \]
\[ + \left( 9\theta - \frac{3}{2} - 9\theta^2 - 3h^2 \right) 4t (t^2 + h^2) \]

and

\[ y_{10} = \alpha_i \quad y_{11} = \beta_i \]
\[ y_{20} = x+h \quad y_2 = x-h \quad x_{10} = \beta_i \quad x_{11} = x_i \]
\[ x_{30} = \gamma-h \quad x_{20} = \beta_i \quad x_{21} = x_{i+1} \quad x_{31} = \gamma+h \]

This complete proof of inequality (3.6)

REFERENCES


