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An Inventory Model for Gompertz Distribution Deterioration Rate with Ramp Type Demand Rate and Shortages

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Abstract

In this paper, an inventory model is developed for deteriorating items which follows the Gompertz distribution deterioration rate. In inventory model, generally it was assumed that the demand is constant but practically demand varies with time. Thus, we consider the demand rate as a ramp type function of time. Shortages are allowed and completely backlogged. In the present paper, we determine the optimal replenishment time and the optimal order quantity in order to get minimum total inventory cost. A numerical example is given to demonstrate the model. The sensitivity of the solution is discussed with the changes in parameters.

Keywords: - Inventory, Gompertz distribution, Deterioration, Ramp type demand, Shortages.

1. INTRODUCTION

The inventory system is an important part of cost controlling in business. For smooth running of any business, there is always a need for business enterprises to maintain the inventory. For last decades, the researchers in this area have extended investigations into various models with considerations of item shortages, item deterioration, demand pattern and their combination.

Most goods will deteriorate, spoil or exceeded their expiration date during inventory and they will affect the total cost. So the topic of inventory system for deteriorating items became more popular in the field of research and business. Ghare and Schrader[1] was the first researcher who studied the concept of deterioration in to the inventory model. In this model the deteriorating rate was assumed as a constant. An order level inventory model for deteriorating items at a constant rate was presented by Shah and Jaiswal[2], Dave and Patel[3]. Inventory Models with a time dependent deterioration rate were discussed by Covert and Philip[4], Mishra[5], Deb and Chaudhri[6]. Some of recent work in this field has been established by Chung and Ting[7], Hariga[8], Skourie et. al[9], Jalan and Mishra et. al[10], Mandal[11] and Verma et. al[12].

Usually, demand rate is assumed to be constant but reality demand may be time dependent, stock dependent and price dependent demand. Demand plays an important role in inventory model. Also, when the shortages occur, it is assumed that it is either completely backlogged or completely lost. But in real life some customers are willing to wait for backorder and others to buy other. Uthya and Geetha[13] developed a replenishment policy for non-instantaneous deteriorating inventory model with partial backlogging. In the present paper, we assume the demand rate as ramp type function of time.

In the present paper, we developed an inventory model by considering deterioration rate as Gompertz distribution with ramp type demand and shortages which are completely backlogged. Here we minimize the total inventory cost based on inventory holding cost, shortage cost and deterioration cost. Finally, a numerical example is considered to demonstrate the model and also we studied the sensitive analysis of solution.

2. ASSUMPTIONS AND NOTATIONS

2.1. Assumptions

The following assumptions are made in developing the proposed model:

(i) The inventory model is supposed with single item only.
(ii) The demand rate is variable with respect to time. More precisely, the demand rate is assumed to be ramp-type function of time.
(iii) The replenishment is instantaneous and its size is finite.
(iv) Lead time is zero.
(v) The inventory system is considered over a finite time horizon.
(vi) The deterioration rate is taken as Gompertz distribution function of time.

2.2. Notations

The following notations are used for the proposed model:

T: The fixed length of each cycle

C_1: The holding cost per unit item per unit time

C_2: The shortage cost per unit item per unit time

C_3: The cost of each deteriorated unit

S: Initial inventory i.e. inventory at t=0

D: Total number of deteriorated items

$I(t)$: The level of inventory at any time t

$h(t)$: The deterioration rate which is taken as log-gamma distribution function of time with two parameters given by

$$h(t) = \theta e^{\alpha t}, \qquad 0 < \theta < 1, \alpha > 0$$

$R(t)$: The demand rate which is taken as ramp-type function of time given by

$$R(t) = D_0\left[t - (t-\mu)H(t-\mu)\right]$$

where D_0 is initial demand, μ is the time point of ramp type function of time and $H(t-\mu)$ is a Heaviside's function which is defined as follows:

$$H(t-\mu) = \begin{cases} 0 & t \le \mu \\ 1 & t \ge \mu \end{cases}$$

t_1 : Time when the inventory level becomes zero

t_1^* : Optimal reorder time.

Q: Total production quantity.

Q^*: Total optimal production quantity.

3. MATHEMATICAL FORMULATION

By fulfilling backlogged items, we get amount S as initial inventory and it is depleted due to deterioration and demand of item. At time t_1 the inventory becomes zero and shortages start occurring 0 to t_1. The total number of backlogged items is replaced by the next replenishment.

The rate of change of inventory I (t) at any time t during the stock period (0, t_1) and shortage period (t_1, T) is described by the following differential equations:

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -R(t), \quad 0 \le t \le t_1 \qquad (1)$$

and $$\frac{dI(t)}{dt} = -R(t), \quad t_1 \le t \le T \qquad (2)$$

where the deterioration rate $h(t)$ follows Gompertz distribution given by

$$h(t) = \theta e^{\alpha t}, 0 < \theta < 1 \, and \; \alpha > 0 \qquad (3)$$

The boundary conditions are $I(t_1) = 0 \quad and \; I(0) = S$ $\qquad (4)$

Here, we assume $\mu < t_1$ and therefore, the equations (1) and (2) become

$$\frac{dI(t)}{dt} + \theta e^{\alpha t} I(t) = -D_0 t, \quad 0 \le t \le \mu \qquad (5)$$

$$\frac{dI(t)}{dt} + \theta e^{\alpha t} I(t) = -D_0 \mu, \quad \mu \le t \le t_1 \qquad (6)$$

and $$\frac{dI(t)}{dt} = -D_0 \mu, \quad t_1 \le t \le T \qquad (7)$$

The above equations $(5)-(7)$ are linear differential equations. While solving equations (5) and (6), we find the integrating factor (I.F.) as follows:

$$I.F. = e^{\int \theta e^{\alpha t} dt} = \exp\left\{(\theta/\alpha) e^{\alpha t}\right\} = 1 + \frac{(\theta/\alpha)}{1!} e^{\alpha t} + \frac{(\theta/\alpha)^2}{2!} e^{\alpha t} + . \quad . \quad .$$

Now, when $0 < \theta < 1$, we may ignore the terms of $O\left(\theta^2\right)$ and therefore, the solutions of (5) and (6) using condition (4) are given by

$$I(t) = -D_0\left(\frac{t^2}{2} + \frac{\theta t}{\alpha^2} e^{\alpha t} - \frac{\theta}{\alpha^3} e^{\alpha t} - \frac{\theta t^2}{2\alpha} e^{\alpha t}\right) + \left\{S\left(1 + \frac{\theta}{\alpha}\right) - \frac{D_0 \theta}{\alpha^3}\right\}\left(1 - \frac{\theta}{\alpha} e^{\alpha t}\right)$$

$$0 \le \mu \le t \qquad (8)$$

$$I(t) = D_0\left(\frac{\mu^2}{2} + \frac{\theta}{\alpha^3} e^{\alpha \mu} - \frac{\theta \mu^2}{2\alpha} e^{\alpha t}\right) - D_0 \mu\left(t + \frac{\theta}{\alpha^2} e^{\alpha t} - \frac{\theta}{\alpha} t e^{\alpha t}\right) + \left\{S\left(1 + \frac{\theta}{\alpha}\right) - \frac{D_0 \theta}{\alpha^3}\right\}\left(1 - \frac{\theta}{\alpha} e^{\alpha t}\right),$$

$$\mu \le t \le t_1 \qquad (9)$$

The solution of equation (7) is given by

$$I(t) = -D_0 \mu (t - t_1), \quad t_1 \le t \le T \qquad (10)$$

Using condition $I(t_1) = 0$ in (9) and neglecting second and higher order terms of θ, we get

$$S = D_0\mu\left\{t_1\left(1-\frac{\theta}{\alpha}\right)+\frac{\theta}{\alpha^2}e^{\alpha t_1}\right\}-D_0\left\{\frac{\mu^2}{2}\left(1-\frac{\theta}{\alpha}\right)+\frac{\theta}{\alpha^3}e^{\alpha\mu}-\frac{\theta}{\alpha^3}\right\} \quad (11)$$

The total amount of items which is deteriorated in the interval $(0,t_1)$ is given by

$$D = S-\int_0^{t_1}R(t)dt = S-\left(\int_0^{\mu}D_0 t\,dt+\int_{\mu}^{t_1}D_0\mu\,dt\right)$$

Integrating the above integrals and then using equation (11), we get

$$D = D_0\mu\left(\frac{\theta}{\alpha^2}e^{\alpha t_1}-\frac{\theta t_1}{\alpha}+\frac{\theta\mu}{2\alpha}-\frac{\theta}{\alpha^3\mu}e^{\alpha\mu}+\frac{\theta}{\alpha^3\mu}\right) \quad (12)$$

The average total cost per unit time is given by

$$C(S,t_1)=\frac{C_3D}{T}+\frac{C_1}{T}\int_0^{t_1}I(t)dt-\frac{C_2}{T}\int_{t_1}^{T}I(t)dt$$

$$or\ C(S,t_1)=\frac{C_3D}{T}+\frac{C_1}{T}\left\{\int_0^{\mu}I(t)\,dt+\int_{\mu}^{t_1}I(t)\,dt\right\}-\frac{C_2}{T}\int_{t_1}^{T}I(t)dt \quad (13)$$

Substituting the value of $I(t)$ given by $(8)-(10)$ and value of D from (12) in (13) and then integrating, we get

$$C(t_1)=C_1\frac{D_0\mu}{T}\left(\frac{t_1^2}{2}-\frac{\mu^2}{6}+\frac{3\theta}{\alpha^4\mu}e^{\alpha\mu}-\frac{3\theta}{\alpha^4\mu}+\frac{\theta t_1}{\alpha^2}-\frac{\theta\mu}{2\alpha^2}-\frac{2\theta}{\alpha^3}e^{\alpha t_1}+\frac{\theta t_1}{\alpha^2}e^{\alpha t_1}-\frac{\theta}{\alpha^3}e^{\alpha\mu}\right)$$

$$+C_2\frac{D_0\mu}{2T}(T-t_1)^2+C_3\frac{D_0\mu}{T}\left(\frac{\theta}{\alpha^2}e^{\alpha t_1}-\frac{\theta t_1}{\alpha}+\frac{\theta\mu}{2\alpha}-\frac{\theta}{\alpha^3\mu}e^{\alpha\mu}+\frac{\theta}{\alpha^3\mu}\right) \quad (14)$$

Condition for the minimization of $C(t_1)$ is $\frac{dC(t_1)}{dt_1}=0,$ which gives

$$C_1\left(t_1+\frac{\theta}{\alpha^2}-\frac{\theta}{\alpha^2}e^{\alpha t_1}+\frac{\theta t_1}{\alpha}e^{\alpha t_1}\right)+C_2(t_1-T)+C_3\left(\frac{\theta}{\alpha}e^{\alpha t_1}-\frac{\theta}{\alpha}\right)=0 \quad (15)$$

Now, we solve (15) for t_1 as follows:

Let $g(t_1)=C_1\left(t_1+\frac{\theta}{\alpha^2}-\frac{\theta}{\alpha^2}e^{\alpha t_1}+\frac{\theta t_1}{\alpha}e^{\alpha t_1}\right)+C_2(t_1-T)+C_3\left(\frac{\theta}{\alpha}e^{\alpha t_1}-\frac{\theta}{\alpha}\right) \quad (16)$

Therefore, we have $g(0)=-C_2T<0$ *and* $g(T)>0$ so that $g(0)g(T)<0$.

Also, we find $g'(t_1)=C_1\left(1+\theta t_1e^{\alpha t_1}\right)+C_2+C_3\theta e^{\alpha t_1}>0 \quad (17)$

It implies that $g(t_1)$ is a strictly monotonic increasing function and equation (15) has unique solution at t_1^* $for\ t_1^* \in (0,T)$.

Again, we have $$\frac{d^2C(t_1)}{dt_1^2}\Big|_{t_1=t_1^*} = \frac{D_0\mu}{T}\left\{C_3\theta e^{\alpha t_1^*} + C_1\left(1+\theta t_1^* e^{\alpha t_1^*}\right) + C_2\right\} > 0$$

which is the sufficient condition for minimization of $C(t_1)$.

Putting $t_1 = t_1^*$ in the equation (11), we find the optimum value of S as follows:

$$S^* = D_0\mu\left\{t_1^*\left(1-\frac{\theta}{\alpha}\right) + \frac{\theta}{\alpha^2}e^{\alpha t_1^*}\right\} - D_0\left\{\frac{\mu^2}{2}\left(1-\frac{\theta}{\alpha}\right) + \frac{\theta}{\alpha^3}e^{\alpha\mu} - \frac{\theta}{\alpha^3}\right\} \qquad (18)$$

Again, the total amount of backorder at the end of the cycle is given by

$$\int_{t_1}^{T} D_0\mu dt = D_0\mu(T-t_1)$$

Therefore, the optimum value of Q is given by

$$Q^* = S^* + D_0\mu\left(T - t_1^*\right)$$

$$or \quad Q^* = D_0\mu\left\{\frac{\theta}{\alpha^2}e^{\alpha t_1^*} - \frac{\theta t_1^*}{\alpha} - \frac{\mu}{2}\left(1-\frac{\theta}{\alpha}\right) - \frac{\theta}{\alpha^3\mu}e^{\alpha\mu} + \frac{\theta}{\alpha^3\mu} + T\right\} \qquad (19)$$

The minimum value of the total average cost $C(t_1^*)$ is given by

$$C(t_1^*) = C_1\frac{D_0\mu}{T}\left(\frac{t_1^{*^2}}{2} - \frac{\mu^2}{6} + \frac{3\theta}{\alpha^4\mu}e^{\alpha\mu} - \frac{3\theta}{\alpha^4\mu} + \frac{\theta t_1^*}{\alpha^2} - \frac{\theta\mu}{2\alpha^2} - \frac{2\theta}{\alpha^3}e^{\alpha t_1} + \frac{\theta t_1^*}{\alpha^2}e^{\alpha t_1^*} - \frac{\theta}{\alpha^3}e^{\alpha\mu}\right)$$

$$+C_2\frac{D_0\mu}{2T}\left(T-t_1^*\right)^2 + C_3\frac{D_0\mu}{T}\left(\frac{\theta}{\alpha^2}e^{\alpha t_1^*} - \frac{\theta t_1^*}{\alpha} + \frac{\theta\mu}{2\alpha} - \frac{\theta}{\alpha^3\mu}e^{\alpha\mu} + \frac{\theta}{\alpha^3\mu}\right) \qquad (20)$$

4. CASE OF ABSENCE OF DETERIORATION

In the absence of deterioration *i.e when* $\theta = 0$ the equation (15) takes the following form

$$(C_1 + C_2)t_1 - C_2T = 0 \qquad (21)$$

which gives the optimum value time t_1 i.e.

$$t_1 = t_1^* = \frac{C_2T}{C_1 + C_2}$$

Substituting $\theta=0$ in the equation (18) *and* (19) respectively, we get

$$S^{*}=D_0\mu\left(\frac{C_2T}{C_1+C_2}-\frac{\mu}{2}\right) \tag{22}$$

and

$$Q^{*}=D_0\mu\left(T-\frac{\mu}{2}\right) \tag{23}$$

Similarly, putting $\theta=0$ in (20), we get

$$C\left(t_1^{*}\right)=\frac{D_0\mu C_1}{2T}\left(t_1^{*2}-\frac{\mu^2}{3}\right)+\frac{D_0\mu C_2}{2T}\left(T-t_1^{*}\right)^2 \tag{24}$$

5. NUMERICAL EXAMPLE

Let us consider the values of parameters of inventory model as follows:

C_1=\$4 per unit per year, C_2=\$15 per unit per year, C_3=\$5 per year, θ=0.002, α =2, D_0=100 units, μ=0.13year, T=1year.

Using the above values of the parameters and according to the equation (13), we find the optimum value of t_1 by Newton-Raphson method

$$t_1^{*}=0.788 \text{ year}$$

This value of $t_1$also satisfies the sufficient condition for optimality.

Taking t_1^{*} =0.788 year, we get the optimal values $Q^*$and S^*as follows:

Q^*=12.169 units and S^*=9.413 units

The minimum average total cost $C\left(t_1^{*}\right)$ =\$20.470 per year.

6. SENSITIVITY ANALYSIS

To study the effects of changes of the parameters on the optimal cost, optimal initial inventory and optimal total production quantity, a sensitive analysis is executed considering the numerical example given above. Sensitive analysis is executed by changing (increasing or decreasing) the parameters by 25% and 50% taking one parameter at a time, keeping the remaining parameters at their original values. The results are shown in the following table (table-1):

Table-1

Parameters	Change (%)	Change in		
		$C^*(t_1)$	S*	Q*
	-50	12.474	13.387	12.191
C_1	-25	16.472	12.733	12.187
	+25	24.468	11.647	12.180
	+50	28.466	11.176	12.177
	-50	18.279	10.366	12.173
	-25	19.448	11.477	12.179
C2	+25	21.566	12.628	12.186
	+50	22.661	12.955	12.188
	-50	20.422	12.170	12.183
	-25	20.446	12.170	12.183
C3	+25	20.494	12.157	12.183
	+50	20.519	12.157	12.183
	-50	20.405	9.409	12.165
α	-25	20.442	9.441	12.167
	+25	20.494	9.416	12.172
	+50	20.520	9.419	12.175
	-25	10.289	4.918	6.296
μ	-50	15.400	7.218	9.285
	+25	25.485	11.502	14.947
	+50	30.432	13.502	17.620
	-50	20.425	9.406	12.162
θ	-25	20.448	9.409	12.165
	+25	20.493	9.417	12.173
	+50	20.516	9.420	12.176
	-50	10.235	4.706	6.085

D0	-25	15.353	7.060	9.127
	+25	25.588	11.767	15.212
	+50	30.706	14.120	18.254

A careful study of above table shows that

(i). S^*, Q^* increases while C^* decreases with decrease in the value of the parameter C_1. However, C^* and S^* are highly sensitive and Q^* is slightly sensitive to changes in the value of C_1.

(ii). C^*, S^* and Q^* increases with increase in the value of parameter C_2. Q^* is moderately sensitive to changes in the value of C_2.

(iii). To increase in the value of the parameter C_3, C^*, S^* and Q^* are slightly sensitive.

(iv). C^*, S^* and Q^* are moderately sensitive to changes in the value of parameter α .

(v). C^*, S^* and Q^* decreases with decrease in the value of parameter μ . Also C^*, S^* and Q^* are highly sensitive to changes in the value of μ .

(vi). C^*, S^* and Q^* are moderately sensitive to changes in the value of parameter θ .

(vii). C^*, S^* and Q^* increases with increase in the value of parameter D_0. Also C^*, S^* and Q^* are highly sensitive to changes in the value of D_0.

7. CONCLUSION

In this work, an inventory model is derived for deteriorating items which follows deterioration rate as a Gompertz distribution function and demand rate as a ramp type function of time. Here, we have derived the average total cost, total optimal production quantity and initial inventory level. The model is solved analytically by minimizing the average total cost. The proposed model is illustrated with numerical example and verified by graphically (figure-1).

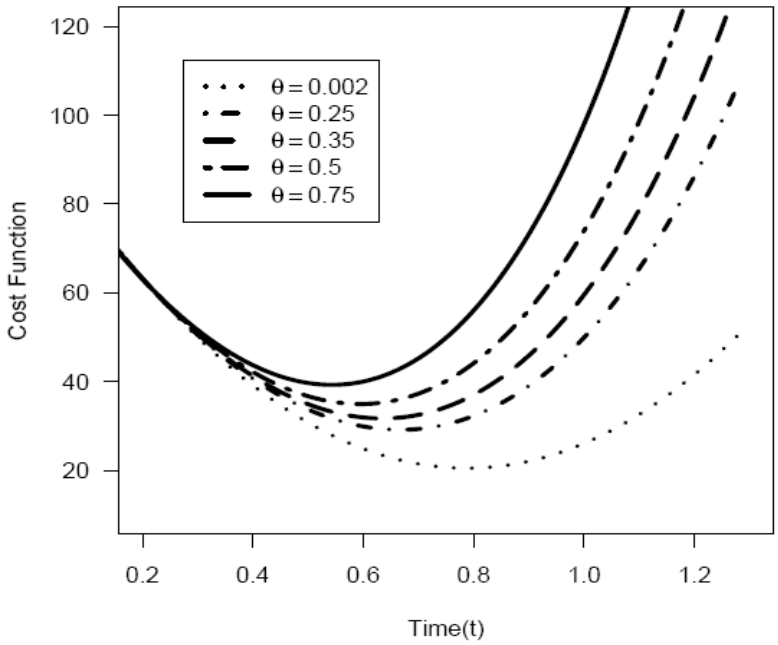

Figure-1: Variation of cost function with time

We could generalize this model to allow for partial backlogging, shortages at starting and others.

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