Queuing Analysis of Markovian Queue Having Two Heterogeneous Servers with Catastrophes using Matrix Geometric Technique

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Abstract

We consider M/M/2 queuing system with intermittently available server and catastrophes. Server1 is always available and Server 2 is intermittently available. This model is solved by Matrix Geometric technique. Numerical study has been done for analysis of various performance measures for various values of parameters.

Keywords: Intermittently available server, Catastrophes, Matrix Geometric Technique.

INTRODUCTION

Interruptions in service occur due to the breakdown of the server or due to the in availability of the server. In the former, Interruption in service occurs during the period of the service and the customer in service along with all other customers will have to wait till the service facility resume service. In the later, interruption in service occurs for a random length of time but occurs only after the service in hand is completed. This service is termed as intermittently available service. The intermittently available server either goes for rest or goes to attend some very urgent odd/different jobs when the queue length is greater than or equal to zero. In exhaustive service, also known as vacation, the server goes for rest or goes to attend some very urgent odd/different jobs when the queue length is zero. In non-exhaustive service, the server goes for rest or goes to attend some very urgent odd/different jobs
when the queue length is greater than zero. Due to the presence of intermittently available server in computer networks, communication networks and electrical grids, we analysed Markovian queues with intermittently available server. Whenever a catastrophe occurs, all the customers are forced to leave the system immediately. Queuing models with intermittently available server have been studied by many researchers in the past. The concept of intermittently available server was introduced by Aggarwal (1965). Sharda (1968) gave the solution of a queuing model in which arrivals and departures occur in batches of variable size and the server is intermittently available. Gaur (1973) considered an intermittent bulk queuing system with multiphase capacity of the service channel. Chaudhary and Lee (1972) studied a single channel constant capacity bulk service queuing process with an intermittently available server and Chaudhary (1974) gave the transient/steady state solution of a single channel with intermittently available server. Sharda and Garg (1986) obtained explicitly the probability of exact number of arrivals and departures of a continuous time queuing model with intermittently available server.

Queuing models with catastrophes have also been analysed by many researchers in the past. Many researchers have studied queuing models with vacations. Krishnamoorthy (2012) studied M/M/2 queuing system with heterogeneous servers including one with working vacation. But no study pertains to the queuing models in which intermittently available server and catastrophes have been analysed together. Therefore, we studied a queuing model having intermittently available server with catastrophes.

Matrix geometric method was conceptualized by Marcel F. Neuts in 1974. Many researchers used this technique in their respective models.

In the present study, we have considered a Markovian queuing system, with catastrophes, having two heterogeneous servers out of which one server is intermittently available. Steady state probabilities have been found using Matrix geometric technique.

The rest of the paper is organized as follows: In section 2, we describe assumptions of the model. Then the description of the model is presented. In section 3, numerical study has been done and various performance measures have been calculated.

SECTION 2
Assumptions of the Model
1) Arrivals of customers follow a poisson process with arrival rate \( \lambda \). Arriving customers form a single waiting line based on the order of their arrivals.
2) Two servers have heterogeneous service rates \( \mu_1 \) and \( \mu_2 \) (i.e. \( \mu_1 \neq \mu_2 \)) . The service time at both the servers follow exponential distribution. Server 1 is always available and server 2 is intermittently available.
3) Customers are served on first come first serve basis.
4) Availability time of server 2 follows exponential distribution with rate $v$.
5) Catastrophe occurs according to exponential distribution with rate $\xi$.

Model Description
We consider $M/M/2$ queuing model with two heterogeneous servers, server1 and server2. Customers arrive according to poisson distribution with rate $\lambda$. These arriving customers form a single waiting line based on the order of their arrivals. If the arriving customer finds a free server, he enters the service immediately. Otherwise he joins the queue. The two servers have heterogeneous service rates $\mu_1$ for server 1 and $\mu_2$ for server 2 where $\mu_1 \neq \mu_2$. Server 1 is always available and server 2 is intermittently available. Server 2 goes to perform some important different/odd jobs when the queue length is greater than or equal to zero. But before going to perform these odd jobs, the server 2 must first complete his in hand service.

Let $n(t)$ be the number of customers in the system at time $t$ and let $k(t)$ be the status of server 2 at time $t$, defined as follows:

$$k(t) = \begin{cases} 0 & \text{if server 2 is intermittently available at time } t \\ 1 & \text{if server 2 is working at time } t \end{cases}$$

We define the system state by $n(t)$ and $k(t)$. Then $\{n(t),k(t)\}$, a markovian process with the state space set arranged in lexicographical order is as follows: 

$\{(0,0) \cup \{(i,j); i \geq 1, j=0,1\}$

Infinitesimally generator matrix $Q$ is as follows:

$$Q = \begin{pmatrix}
B_{00} & B_{01} & \cdots & \cdots & \cdots & \cdots & \cdots \\
B_{10} & B_{11} & A_0 & \cdots & \cdots & \cdots & \cdots \\
A_3 & A_1 & B_1 & A_0 & \cdots & \cdots & \cdots \\
A_3 & \cdots & A_2 & B_2 & A_0 & \cdots & \cdots \\
A_3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & A_2 & B_2 & A_0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Where

$$B_{00} = ( -\lambda ) ; \; B_{01} = ( \lambda \; \; 0 ) ; \; B_{10} = \left( \begin{array}{c}
\mu_1 + \xi \\
\mu_2 + \xi
\end{array} \right) ; \; A_3 = \left( \begin{array}{c}
\xi \\
\xi
\end{array} \right) ;$$

$$B_{11} = \left( \begin{array}{cc}
-\lambda - \mu_1 - \xi - \nu & \nu \\
0 & -\lambda - \mu_2 - \xi
\end{array} \right) ; \; A_0 = \left( \begin{array}{c}
\lambda \\
0
\end{array} \right) ; \; A_1 = \left( \begin{array}{cc}
\mu_1 & 0 \\
\mu_2 & \mu_1
\end{array} \right) ;$$

$$B_1 = \left( \begin{array}{cc}
-\lambda - \mu_1 - \xi - \nu & \nu \\
0 & -\lambda - \mu_1 - \mu_2 - \xi
\end{array} \right) ; \; A_2 = \left( \begin{array}{cc}
\mu_1 & 0 \\
\mu_2 & \mu_1 + \mu_2
\end{array} \right) ;$$
\[ B_2 = \begin{pmatrix} -\lambda - \mu_1 - \xi - \nu & \nu \\ 0 & -\lambda - \mu_1 - 2\mu_2 - \xi \end{pmatrix}; \]

The stability condition of the system is \( \rho < 1 \) where \( \rho = \frac{\lambda}{\mu_1 + \mu_2} \)

Let \( n \) and \( k \) be the stationary random variables for the number of customers in the system and the status of the server 2.

We define \( \pi_{ij} = \{ n = i, k = j \} = \lim_{t \to \infty} P\{n(t) = i, k(t) = j\} \) where \( (i, j) \in \) state space set.

The stationary probability matrix \( \pi \) is given by

\[
\pi = (\pi_0, \pi_1, \pi_2, \ldots) \quad \text{where}
\]

\[
\pi_0 = \pi_{00} \quad \text{and} \\
\pi_i = (\pi_{i0}, \pi_{i1}) \quad \text{for } i \geq 1
\]

The stationary probability matrix \( \pi \) is solved by using \( \pi Q = 0 \).

\[
\pi_0 B_{00} + \pi_1 B_{10} + \pi_2 (I - R)^{-1} A_3 = 0 \quad \ldots(1)
\]

\[
\pi_0 B_{01} + \pi_1 B_{11} + \pi_2 A_1 = 0 \quad \ldots(2)
\]

\[
\pi_1 A_0 + \pi_2 (B_1 + RA_2) = 0 \quad \ldots(3)
\]

\[
\pi_i = \pi_2 R^{i-2} \quad \text{for } i \geq 3 \quad \ldots(4)
\]

Here \( R \) (called rate matrix) is minimal non negative solution of equation

\[
A_0 + R B_2 + R^2 A_2 = 0 \quad \ldots(5)
\]

The normalizing equation is given by

\[
\pi_0 e + \pi_1 e + \pi_2 (I - R)^{-1} e = 1 \quad \ldots(6)
\]

Here ‘e’ is column vector of appropriate length of 1’s.

First of all, using equation (5) matrix \( R \) is obtained. Then substituting \( R \) in equations (1), (2) and (3) sub vectors \( \pi_0, \pi_1 \) and \( \pi_2 \) are obtained. Using equation (4), \( \pi_i \) for \( i \geq 3 \) are calculated. Finally using equation (6) all sub vectors are normalized.
SECTION 3
The stationary probabilities are calculated using codes developed in Maple. The following performance measures are calculated:

Probability that the system is empty: $P_{emp} = \pi_0$

Probability that the server 1 is idle: $P_{1 \text{ idl}} = \pi_0 + \pi_{11}$

Mean number of customers in the system: $MNS = \pi_1 e + \pi_2 (I - R)^{-1} e + \pi_2 (I - R)^{-2} e$

Probability that the server 2 is intermittently available: $P_{2 \text{ int}} = \sum_{j=0}^{\infty} \pi_{j0} + \pi_0$

Probability that the server 2 is working: $P_{2 \text{ working}} = 1 - P_{2 \text{ int}}$

**Table 1:** Effect of $\lambda$ on Performance measures:
This table has been generated using $\mu_1 = 10$; $\mu_2 = 5$; $\xi = 0.1$; $\nu = 1$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$P_{emp}$</th>
<th>$P_{1 \text{ idl}}$</th>
<th>$P_{2 \text{ int}}$</th>
<th>$MNS$</th>
<th>$P_{2 \text{ working}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.7923101</td>
<td>0.8200158</td>
<td>0.9659524</td>
<td>0.2530131</td>
<td>0.0340475</td>
</tr>
<tr>
<td>3</td>
<td>0.6944084</td>
<td>0.7307392</td>
<td>0.9499030</td>
<td>0.4166662</td>
<td>0.0500969</td>
</tr>
<tr>
<td>5</td>
<td>0.5110127</td>
<td>0.5555324</td>
<td>0.9198381</td>
<td>0.8695044</td>
<td>0.0801618</td>
</tr>
<tr>
<td>8</td>
<td>0.2707275</td>
<td>0.3084431</td>
<td>0.8804471</td>
<td>2.2930237</td>
<td>0.1195528</td>
</tr>
<tr>
<td>10</td>
<td>0.1419550</td>
<td>0.1660163</td>
<td>0.85933683</td>
<td>4.9483053</td>
<td>0.1406631</td>
</tr>
<tr>
<td>13</td>
<td>0.0395744</td>
<td>0.0463760</td>
<td>0.84255257</td>
<td>19.656344</td>
<td>0.1574474</td>
</tr>
</tbody>
</table>

Since $\mu_1$ and $\mu_2$ are fixed for this table, therefore as $\lambda$ increases value of $\rho$ increases $P_{emp}$, $P_{1 \text{ idl}}$ and $P_{2 \text{ int}}$ decreases, while $MNS$ and $P_{2 \text{ working}}$ increases.

**Table 2:** Effect of $\mu_1$ on Performance Measures:
This table has been generated using $\lambda = 2$; $\mu_2 = 5$; $\xi = 0.1$; $\nu = 1$

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$P_{emp}$</th>
<th>$P_{1 \text{ idl}}$</th>
<th>$P_{2 \text{ int}}$</th>
<th>$MNS$</th>
<th>$P_{2 \text{ working}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3637174</td>
<td>0.4053813</td>
<td>0.8956913</td>
<td>1.4617087</td>
<td>0.1043086</td>
</tr>
<tr>
<td>3</td>
<td>0.4854631</td>
<td>0.5293330</td>
<td>0.9156496</td>
<td>0.9401606</td>
<td>0.0843503</td>
</tr>
<tr>
<td>5</td>
<td>0.6356333</td>
<td>0.6753869</td>
<td>0.9402677</td>
<td>0.5341769</td>
<td>0.0597322</td>
</tr>
<tr>
<td>8</td>
<td>0.7489279</td>
<td>0.7807489</td>
<td>0.9588406</td>
<td>0.3208036</td>
<td>0.0411593</td>
</tr>
<tr>
<td>10</td>
<td>0.7923101</td>
<td>0.8200158</td>
<td>0.9695924</td>
<td>0.2530131</td>
<td>0.0340475</td>
</tr>
<tr>
<td>15</td>
<td>0.8550845</td>
<td>0.8757950</td>
<td>0.9762433</td>
<td>0.1654600</td>
<td>0.0237566</td>
</tr>
<tr>
<td>20</td>
<td>0.8887521</td>
<td>0.9052105</td>
<td>0.9817626</td>
<td>0.1229134</td>
<td>0.0182373</td>
</tr>
<tr>
<td>0</td>
<td>0.0798187</td>
<td>0.0933426</td>
<td>0.8491506</td>
<td>6.9714887</td>
<td>0.1508493</td>
</tr>
</tbody>
</table>
With increase in the value of $\mu_1$ Pemp, P1idl and P2int increases while MNS and P2working decreases.

**Table 3**: Effect of $\mu_2$ on Performance Measures:
This table has been generated using $\lambda = 2$ ; $\mu_1 = 5$ ; $\xi = 0.1$ ; $\nu = 1$

<table>
<thead>
<tr>
<th>$\mu_2$</th>
<th>Pemp</th>
<th>P1idl</th>
<th>P2int</th>
<th>MNS</th>
<th>P2working</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5843482</td>
<td>0.6714381</td>
<td>0.8659187</td>
<td>0.6163709</td>
<td>0.1340812</td>
</tr>
<tr>
<td>3</td>
<td>0.6113288</td>
<td>0.6736277</td>
<td>0.9052021</td>
<td>0.5725593</td>
<td>0.0947978</td>
</tr>
<tr>
<td>5</td>
<td>0.6356333</td>
<td>0.6753869</td>
<td>0.9402677</td>
<td>0.5341769</td>
<td>0.0597322</td>
</tr>
<tr>
<td>8</td>
<td>0.6505470</td>
<td>0.6763345</td>
<td>0.9615985</td>
<td>0.5112502</td>
<td>0.0384014</td>
</tr>
<tr>
<td>10</td>
<td>0.6557382</td>
<td>0.6766355</td>
<td>0.9689854</td>
<td>0.5033993</td>
<td>0.0310145</td>
</tr>
<tr>
<td>15</td>
<td>0.6628411</td>
<td>0.6770191</td>
<td>0.9790584</td>
<td>0.4927764</td>
<td>0.0209415</td>
</tr>
<tr>
<td>20</td>
<td>0.6664726</td>
<td>0.6772016</td>
<td>0.9841930</td>
<td>0.4874015</td>
<td>0.0158069</td>
</tr>
<tr>
<td>0</td>
<td>0.1723198</td>
<td>0.6270198</td>
<td>0.2475635</td>
<td>1.3509940</td>
<td>0.7524364</td>
</tr>
</tbody>
</table>

As $\mu_2$ increases Pemp and P2int increases, while MNS and P2working decreases. P1idl remains constant.

**Table 4**: Effect of $\nu$ on Performance Measures:
This table has been generated using $\lambda = 2$ ; $\mu_1 = 10$ ; $\mu_2 = 5$ ; $\xi = 0.1$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>Pemp</th>
<th>P1idl</th>
<th>P2int</th>
<th>MNS</th>
<th>P2working</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7923101</td>
<td>0.8200158</td>
<td>0.9659524</td>
<td>0.2530131</td>
<td>0.0340475</td>
</tr>
<tr>
<td>3</td>
<td>0.7768552</td>
<td>0.8449452</td>
<td>0.9173538</td>
<td>0.2651886</td>
<td>0.0826461</td>
</tr>
<tr>
<td>5</td>
<td>0.7655940</td>
<td>0.8620046</td>
<td>0.8839574</td>
<td>0.2750813</td>
<td>0.1160425</td>
</tr>
<tr>
<td>8</td>
<td>0.7534059</td>
<td>0.8796041</td>
<td>0.8494082</td>
<td>0.2865403</td>
<td>0.1505917</td>
</tr>
<tr>
<td>10</td>
<td>0.7472983</td>
<td>0.8881275</td>
<td>0.8326479</td>
<td>0.2925283</td>
<td>0.1673520</td>
</tr>
<tr>
<td>15</td>
<td>0.7362502</td>
<td>0.9030997</td>
<td>0.8031718</td>
<td>0.3037127</td>
<td>0.1968281</td>
</tr>
<tr>
<td>20</td>
<td>0.7288260</td>
<td>0.9128671</td>
<td>0.7839251</td>
<td>0.3114516</td>
<td>0.2160748</td>
</tr>
<tr>
<td>0</td>
<td>0.8024616</td>
<td>0.8024616</td>
<td>1</td>
<td>0.2461654</td>
<td>0</td>
</tr>
</tbody>
</table>

As availability time of second server $\nu$ increases Pemp, P2int decreases, while P1idl, P2working and MNS increases. With $\nu$ taking zero value, P2int and P2working takes respectively 1 and 0 value.
Table 5: Effect of $\xi$ on Performance Measures:
This table has been generated using $\lambda = 2$ ; $\mu_1 = 10$ ; $\mu_2 = 5$ ; $\nu = 1$

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>Pemp</th>
<th>P1idl</th>
<th>P2int</th>
<th>MNS</th>
<th>P2working</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8140755</td>
<td>0.8358701</td>
<td>0.9734394</td>
<td>0.2225848</td>
<td>0.0265606</td>
</tr>
<tr>
<td>3</td>
<td>0.8476639</td>
<td>0.8617882</td>
<td>0.9830737</td>
<td>0.1771598</td>
<td>0.0169262</td>
</tr>
<tr>
<td>5</td>
<td>0.8700369</td>
<td>0.8800400</td>
<td>0.9881851</td>
<td>0.1480547</td>
<td>0.0118148</td>
</tr>
<tr>
<td>8</td>
<td>0.8928324</td>
<td>0.8994319</td>
<td>0.9923451</td>
<td>0.1194373</td>
<td>0.0076548</td>
</tr>
<tr>
<td>10</td>
<td>0.9038308</td>
<td>0.9090655</td>
<td>0.9939894</td>
<td>0.1060227</td>
<td>0.0060105</td>
</tr>
<tr>
<td>15</td>
<td>0.9231833</td>
<td>0.9264341</td>
<td>0.9963420</td>
<td>0.0830588</td>
<td>0.0036579</td>
</tr>
<tr>
<td>20</td>
<td>0.9358863</td>
<td>0.9381112</td>
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<td>0.0684347</td>
<td>0.0024659</td>
</tr>
<tr>
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<td>0.8180221</td>
<td>0.9649168</td>
<td>0.2569922</td>
<td>0.0350831</td>
</tr>
</tbody>
</table>

With increase in catastrophe rate $\xi$, Pemp and P1idl increases, P2int takes value which is close to 1 while MNS and P2working takes value which are close to 0.
If we remove the condition of catastrophe and intermittently available server from this model we get a Markovian queue having two heterogeneous servers. When we remove condition of vacation and working vacation from model discussed in Krishnamoorthy (2012), we also get an M/M/2 queue with two heterogeneous servers. This particularisation of our model and matching of particularisation with model discussed in Krishnamoorthy (2012) verifies the authenticity of our model.

CONCLUSIONS
We studied M/M/2 queuing system having intermittently available server with catastrophes. Server 1 is always available and Server 2 is intermittently available. Steady state probabilities using matrix geometric method were found. Effect of various parameters on performance measures were analysed.

REFERENCES

