Entropy of Lomax Probability Distribution and its Order Statistic

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Abstract
The term entropy was introduced by Shannon in 1948. The study of underlying distribution and associated properties is necessary to explore and understand the nature of many statistical phenomenon. As every probability distribution has certain kind of uncertainty associated with it, which is measured by the entropy. In the present communication, we have derived the entropy of Lomax probability distribution as well as their order statistics which is used in business, economics, actuarial modeling, queuing problems and biological sciences.

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1. Introduction
Every probability distribution has some kind of uncertainty associated with it and Entropy is used to measure this uncertainty. Entropy is also considered as a measure of randomness of a probabilistic system. The concept of entropy was introduced by Shannon (1948) as a measure of information, which provides a quantitative measure of the uncertainty. It is a measure of average uncertainty in a random variable. When a system is in a number of states randomly with equal probability of occurrences entropy attains its maximum value and attains minimum value i.e 0 when the system is in one particular state, which means there is no uncertainty in its description.

Differential entropy extends the idea of Shannon’s entropy. It is a measure of average uncertainty to a continuous random variable and so is also known as continuous entropy. Let X be a random variable with probability density function f(x) whose support is a set x then the differential or Shannon’s entropy of the variable X is given by

\[ H(x) = -\int_x f(x) \times ln(f(x)) \, dx \]  

(1.1)
where the existence of integral is one of our main condition. The units of entropy depend on the base of logarithm. If the base of the logarithm is $e$, then entropy is measured in nats and if the base of the logarithm is 2, then the entropy is measured in bits.

Entropy is used to understanding a wide variety of physical, chemical and biological phenomenon. We see that the movement of molecule in biological sciences is random so, to study the different properties of the molecules we have to find the entropy of probability distribution of the molecule. In the present communication, we have derived the expressions of the entropy for Lomax distribution and we have computed the entropy of the related $i^{th}$, $i^{th}$ and the $n^{th}$ order statistics of this distribution whenever they exists (in closed form).

2. Lomax Distribution

The Lomax distribution is a heavy tail distribution. Johnson et al. (1994) has used the Lomax in business, economics, actuarial modeling, queuing problems and biological sciences. It is conditionally also called Pareto Type II distribution. It is essentially a Pareto distribution that has been shifted so that its support begins at zero. Specifically, the Lomax distribution is a Pareto Type II distribution with $X_m = \lambda$ and $\mu = 0$.

The cumulative density function of Lomax distribution is given by

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-(\alpha + 1)}, \alpha > 0, \lambda > 0, x \geq 0 \quad (2.1)$$

The probability density function of Lomax distribution is given by

$$f(x) = \frac{\alpha}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha + 1)}, x \geq 0 \quad (2.2)$$

with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$.

The density function can also be written as

$$f(x) = \frac{\alpha\lambda^\alpha}{(x + \lambda)^{\alpha + 1}} \quad (2.3)$$

which shows its relation to Pareto Type I distribution.

Many applications of the Lomax distribution can be found in the literature. Balkema and Hann (1974) have used the Lomax distribution in reliability modeling and life testing. Bryson (1974) has found its application as an alternative of exponential distribution when the data are heavy tailed. Ahsanullah (1991) studied the record values of Lomax distribution. Balakrishnan and Ahsanullah (1994) introduced some recurrence relations between the moments of record values from Lomax distribution. Arnold et al. (1998) has studied Lomax distribution by Bayesian point of view by many authors. In this paper, our main focus is on finding the entropy expressions for the Lomax distribution and its related order statistics.
The entropy of the Lomax distribution can be found as follows:

\[
H(x) = -\int_x f(x) \times \log(f(x)) \, dx = E[-\ln(f(x))]
\]

\[
H(x) = -\ln \left( \frac{\alpha}{\lambda} \right) + (\alpha + 1)E \left[ \ln \left( 1 + \frac{x}{\lambda} \right) \right] \quad (2.4)
\]

To compute the entropy in the above expression we need to find

\[
E \left[ \ln \left( 1 + \frac{x}{\lambda} \right) \right] = \frac{\alpha}{\lambda} \int_0^\infty \ln \left( 1 + \frac{x}{\lambda} \right) \left( 1 + \frac{x}{\lambda} \right)^{-\alpha(n+1)} \, dx
\]

On substituting \( 1 + \frac{x}{\lambda} = t \) and solving it for \( t \), we get

\[
E \left[ \ln \left( 1 + \frac{x}{\lambda} \right) \right] = \frac{1}{\alpha} \quad (2.5)
\]

Thus,

\[
H(x) = -\ln \left( \frac{\alpha}{\lambda} \right) + \frac{(\alpha + 1)}{\alpha} \quad (2.6)
\]

This is the required expression of entropy of Lomax distribution which can be seen as a function of parameters \( \alpha \) and \( \lambda \).

3. Entropy for Order Statistic

Let \( X_1, X_2, \ldots, X_n \) be a random sample of the probability density function (2.2) and cumulative density function (2.1) and let \( X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n-1)} \leq X_{(n)} \) denotes the corresponding order statistics.

3.1. Entropy for the First Order Statistic

The probability density function of the first order statistics is given by

\[
f_1(x) = nf(x)(1 - F(x))^{(\alpha-1)}
\]

\[
f_1(x) = n\frac{\alpha}{\lambda} (1 + \frac{x}{\lambda})^{-\alpha(n+1)} \quad (3.1)
\]

So the Shannon’s entropy of the first order statistics is given by

\[
H_1(x) = E[-\ln(f(x))]
\]

\[
= -\ln \left( \frac{\alpha}{\lambda n} \right) + \alpha(n + 1)E \left[ \ln \left( 1 + \frac{x}{\lambda} \right) \right]
\]

So, we will solve the expression

\[
E \left[ \ln \left( 1 + \frac{x}{\lambda} \right) \right] = n\frac{\alpha}{\lambda} \int_0^\infty \ln \left( 1 + \frac{x}{\lambda} \right) \left( 1 + \frac{x}{\lambda} \right)^{-\alpha(n+1)} \, dx
\]
On putting \((1 + \frac{x}{\lambda}) = t\) and solving it for \(t\) we get

\[
E \left[ \ln \left(1 + \frac{x}{\lambda}\right) \right] = \frac{1}{\alpha n}
\]  
(3.2)

Thus,

\[
H_1(x) = -\ln \left(\frac{\alpha}{n}\right) + \frac{(n + 1)}{n}
\]  
(3.3)

This is the entropy expression for the first order statistics of the Lomax distribution which is the function of not only the parameters \(\alpha\) and \(\lambda\) but also of the sample size \(n\).

### 3.2. Entropy for the ith Order Statistic

The probability density function of the \(i^{th}\) order statistics is given by

\[
f_i(x) = \frac{(n - 1)!}{(i - 1)! (n - i)!} n F(x)^{(i-1)} (1 - F(x))^{(n-i)} f(x)
\]

So putting the value of cumulative density function and probability density function of Lomax distribution we will have,

\[
f_i(x) = n \frac{\alpha}{\lambda} \frac{(n - 1)!}{(i - 1)! (n - i)!} \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]^{(i-1)} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha n + \alpha i + 1 + \alpha)}, \quad x \geq 0
\]  
(3.4)

The entropy expression of \(f_i(x)\) is

\[
H_i(X) = - \ln \left[n \frac{\alpha}{\lambda} \frac{(n - 1)!}{(i - 1)! (n - i)!}\right] + (i - 1) E \left[ \ln \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]\right]
\]

\[
+ (\alpha n + \alpha i + 1 + \alpha) E \left[ \ln \left(1 + \frac{x}{\lambda}\right)\right]
\]

Next, we solve the following two expectations to find the entropy expression of the \(i^{th}\) order statistics of Lomax distribution:

\[
E \left[ \ln \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]\right]
\]

and

\[
E \left[ \ln \left(1 + \frac{x}{\lambda}\right)\right]
\]

So, firstly we will evaluate

\[
E \left[ \ln \left(1 + \frac{x}{\lambda}\right)\right]
\]

\[
= n \frac{\alpha}{\lambda} \frac{(n - 1)!}{(i - 1)! (n - i)!} \int_0^\infty \ln \left(1 + \frac{x}{\lambda}\right) \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]^{(i-1)} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha n + \alpha i + 1 + \alpha)} dx
\]
On substituting $1 + \frac{x}{\lambda} = t$ and solving for $t$, we get

$$E \left[ \ln \left( 1 + \frac{x}{\lambda} \right) \right] = \infty$$

(3.5)

And

$$E \left[ \ln \left( 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right) \right] = n\alpha \frac{(n - 1)!}{(i - 1)! (n - i)!} \int_{0}^{\infty} \ln \left[ 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right]$$

$$\times \left[ 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right]^{(i-1)} \left( 1 + \frac{x}{\lambda} \right)^{-(\alpha n + \alpha i + \alpha + 1)} dx$$

On substituting $1 + \frac{x}{\lambda} = t$ and solving for $t$, we get

$$E \left[ \ln \left[ 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right] \right] = n\alpha \frac{(n - 1)!}{(i - 1)! (n - i)!} \int_{1}^{\infty} \ln \left[ 1 - (t)^{-\alpha} \right] \left[ 1 - t^{-\alpha} \right]^{(i-1)} t^{-(\alpha n + \alpha i + \alpha + 1)} dt$$

As above integral has no closed form so, we cannot solve this by usual methods of integration which implies that the entropy for the $i^{th}$ order statistics cannot be found yet. But an attempt can be made to get the solution by using some suitable approximation method or some iterative procedure. We are working in this direction.

### 3.3. Entropy for the $n^{th}$ Order Statistic

The probability density function of the $n$th order statistics is given by

$$f_n(x) = nf(x)F(x)^{(n-1)}$$

$$= n\frac{\alpha}{\lambda} \left( 1 + \frac{x}{\lambda} \right)^{-(\alpha+1)} \left[ 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right]^{(n-1)}$$

The Shannon’s entropy for the above density is given by

$$H_n(x) = -\ln \left[ n\frac{\alpha}{\lambda} \right] + (\alpha + 1)E \left[ \ln \left( 1 + \frac{x}{\lambda} \right) \right] - (n - 1)E \left[ \ln \left[ 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right] \right]$$

Now,

$$E \left[ \ln \left( 1 + \frac{x}{\lambda} \right) \right] = n\alpha \int_{1}^{\infty} \ln(t) t^{-(\alpha+1)}(1 - t^{-\alpha})^{n-1} dt$$

$$= \infty$$

(3.6)
Again,

\[ E \left[ \ln \left( 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right) \right] = n\alpha \frac{\lambda}{\alpha} \int_0^\infty \ln \left( 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right) \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \left( 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right)^{(n-1)} dx \]

On substituting \[ \ln \left( 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right) = y \] and solving for above equation, we get

\[ E \left[ \ln \left( 1 - \left( 1 + \frac{x}{\lambda} \right)^{-\alpha} \right) \right] = \infty \quad (3.7) \]

By putting the values of all the expectations in the entropy expression we will get

\[ H_n(x) = \infty \quad (3.8) \]

This shows that the entropy of the \( n^{th} \) order statistics of the Lomax distribution is infinite.

4. Conclusions

In this paper, we have found the exact entropy expressions for the Lomax distribution. In case of order statistics, we have found the entropy expression of \( 1^{st} \) order statistics of Lomax distribution exactly and it has been proved that entropy of the \( n^{th} \) order statistics of the Lomax distribution is infinite.

References


