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Quantum Computing through Quaternions

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Abstract

Using quaternions, we study the geometry of the single and two qubit states of quantum computing. Through the Hopf fibrations, we identify geometric manifestations of the separability and entanglement of two qubit quantum systems.

Introduction

Ever since the invention of "quaternions [1-6]" in 1843 by Sir William Hamilton to model the three dimensional motion of rigid bodies, these magic numbers have fascinated mathematicians and physicists worldwide with application growing by the day. Quaternions have provided a successful and elegant means for the representation of three dimensional rotations, Lorentz transformations of special relativity, robotics, computer vision, problems of electrical engineering and so on. Quaternionic Quantum Mechanics has aso shown potential of possible unification with General Relativity. In fact, there is belief in some schools of thought that the conventional quantum mechanics in complex spacetime is an asymptotic version of the Quaternionic Quantum Mechanics.

In this paper, an attempt is made to apply these "quaternions" in quantum information processing.

What are "Quaternions [1-6]"

We summarize below the salient properties of the "quaternion algebra" to facilate completeness and continuity in this article.

The "quaternions" are generalized complex numbers of the form $q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with $w, x, y, z \in \mathbb{R}$, the set of real numbers and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being imaginary units that satisfy the quaternionic algebra $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$.

Furthermore, $\operatorname{Re} q = \frac{1}{2}(q + \bar{q}) = w, \operatorname{Im} q = \frac{1}{2}(q - \bar{q}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\bar{q} = \operatorname{Re} q - \operatorname{Im} q$ is the conjugate of $q = \operatorname{Re} q + \operatorname{Im} q$.

Quaternionic multiplication is associative and distributive but not commutative. In fact, we have, for any two quaternions $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $y = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$

$$xy = (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + (x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)\mathbf{i}$$
$$+ (x_0y_2 + x_2y_0 + x_3y_1 - x_1y_3)\mathbf{j} + (x_0y_3 + x_3y_0 + x_1y_2 - x_3y_2)\mathbf{k}$$

which can be succinctly expressed as $xy = x_0y_0 - \mathbf{x.y} + x_0\mathbf{y} + \mathbf{x}y_0 + \mathbf{x} \times \mathbf{y}$. For pure quaternions i.e. quaternions with $\operatorname{Re} q = 0$, this simplifies to $xy = -\mathbf{x.y} + \mathbf{x} \times \mathbf{y}$. Furthermore, since $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$, we also have $\frac{1}{2}(xy + yx) = x_0y_0 - \mathbf{x.y} + x_0\mathbf{y} + \mathbf{x}y_0$, $\frac{1}{2}(xy - yx) = \mathbf{x} \times \mathbf{y}$ with the corresponding values for pure quaternions being $\frac{1}{2}(xy + yx) = -\mathbf{x.y}$, $\frac{1}{2}(xy - yx) = \mathbf{x} \times \mathbf{y}$. The product of two quaternions is again a quaternion being the sum of a real number $(\mathbf{x.y})$ and a pure quaternion $(\mathbf{x} \times \mathbf{y})$. The cross product $\mathbf{x} \times \mathbf{y}$ also satisfies the Jacobi identity that makes the vector space $\mathfrak{R}^3$ with the bilinear map $\mathfrak{R}^3 \times \mathfrak{R}^3 \to \mathfrak{R}^3 : (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \times \mathbf{y}$ into a Lie algebra.

We define the norm of a quaternion as $N(q) = \|q\| = (q\bar{q})^{1/2} = w^2 + x^2 + y^2 + z^2$. The inverse of a quaternion is naturally defined by $q^{-1} = \frac{\bar{q}}{\|q\|^2}$.

Writing the quaternions as $q = \operatorname{Re} q + \operatorname{Im} q$, we can split the quaternion algebra $\mathbb{Q}$ into the direct sum of two orthogonal subspaces $\mathbb{Q} \equiv \mathbb{R} \oplus \mathbb{R}^3$ where the real part of the quaternion maps onto the straight line $\mathbb{R}$ and the imaginary part maps onto the orthogonal three dimensional real plane.

The quaternion algebra also provides a representation of the group of symplectic transformations $Sp(1)$ (defined as the group of all linear quaternion transformations φ that leave the origin unchanged and preserve the real valued scalar product defined below) [7].

For this purpose, we define, in the quaternion algebra, a real valued symmetric scalar product as $\langle x|y\rangle = \operatorname{Re} x\overline{y}$ which coincides with the conventional dot product of vectors i.e. $\langle x|y\rangle = \sum_{i=0}^{3} x_i y_i$ as is easily verified. To explicitly set out the representation of the symplectic group $Sp(1)$, we identify the quaternion algebra $\mathbb{Q}$ with the complex space $\mathbb{C}^2$ by writing an arbitrary quaternion $q \in \mathbb{Q}$ as $q = (q_0 + q_1 i) + j(q_2 - q_3 i) = q_\alpha + j\overline{q}_\beta$ with $q_\alpha = (q_0 + q_1 i), q_\beta = (q_2 + q_3 i) \in \mathbb{C}$. Under this canonical identification, the quaternion valued form $\langle x|y\rangle_{\mathbb{Q}} = x\overline{y}$, $x, y \in \mathbb{Q}$ becomes $\langle x|y\rangle_{\mathbb{Q}} = x\overline{y} = (x_\alpha \overline{y}_\alpha + x_\beta \overline{y}_\beta) + (x_\beta y_\alpha - x_\alpha y_\beta)\, j = \langle x|y\rangle_{\mathbb{C}} + (x, y)_{\mathbb{C}}$ with the former form being hermitian and the latter skew-symmetric. It can be shown that a transformation that preserves the scalar product $\langle x|y\rangle = \operatorname{Re} x\overline{y} = \operatorname{Re}\langle x|y\rangle_{\mathbb{Q}}$ also preserves the scalar product $\langle x|y\rangle_{\mathbb{Q}} = x\overline{y}$ and vice versa. This follows from the fact that a transformation preserving $\langle x|y\rangle_{\mathbb{Q}} = x\overline{y}$ would, obviously, preserve the real and imaginary components of the scalar product separately. Conversely, let a quaternionic transformation $\varphi \in Sp(1)$ preserve the real valued product so that $\langle \varphi x|\varphi y\rangle = \langle x|y\rangle = \operatorname{Re}\langle x|y\rangle_{\mathbb{Q}}$. Since this expression holds for quaternionic vectors of the form $|ix\rangle$ as well, we have $\operatorname{Re}\langle ix|y\rangle_{\mathbb{Q}} = \operatorname{Re}\langle \varphi(ix)|\mathbb{Q}y\rangle_{\mathbb{Q}}$. Now, since, for the transformation $\varphi \in Sp(1)$, we have $\varphi(ix) = i\varphi(x)$ so that $\operatorname{Re}\langle ix|y\rangle_{\mathbb{Q}} = \operatorname{Re}\langle i\varphi(x)|\varphi y\rangle_{\mathbb{Q}}$ which implies that the $\mathbf{i}$th component of the quaternionic product is preserved if the real part is preserved by the transformation $\varphi \in Sp(1)$. Similarly, the $\mathbf{j},\mathbf{k}$th components can also be shown to be preserved. It follows that if a quaternionic transformation $\varphi \in Sp(1)$ preserves the real product, then it also preserves the imaginary part and hence the complete quaternionic product.

With the identification of $\mathbb{Q}$ with $\mathbb{C}^2$, the group $Sp(1)$ is embedded as a subgroup in $U(2)$. This follows from the fact that every quaternion transformation $\varphi \in Sp(1)$

preserves the quaternionic product $\langle x|y\rangle_{\mathbb{Q}} = \langle x|y\rangle_{\mathbb{C}} + (x,y)_{\mathbb{C}} >$ therefore, such transformation must necessarily preserve the hermitian complex form $\langle x|y\rangle_{\mathbb{C}}$ and also the skew symmetric form $(x,y)_{\mathbb{C}}$. Hence, $\varphi \in Sp(1)$ is a unitary transformation in $\mathbb{C}^2$ and so it belongs to $U(2)$.

Any element $\varphi \in Sp(1)$ can, therefore, be written as a 2×2 unitary matrix, say $\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, the unitary and symplectic nature of $\varphi \in Sp(1)$ translate to the constraints $\varphi \mathbf{E} \varphi^T = \mathbf{E}$, $\varphi^\dagger = \bar{\varphi}^T = \varphi^{-1}$ or $\varphi \mathbf{E} = \mathbf{E}\left(\varphi^T\right)^{-1} = \mathbf{E}\left(\varphi^{-1}\right)^T = \mathbf{E}\varphi^*$ where $\mathbf{E} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that $b = -\bar{c}$, $d = \bar{a}$,with a,b being determined from the unitarity conditions $a\bar{a} + b\bar{b} = 1$ and $ab = ba$. In the case of an infinitesimal $\varphi \in Sp(1)$, we can write it in the neighbourhood of the identity transformation as $\varphi = \mathbf{I} + \varepsilon \begin{pmatrix} \alpha & \beta \\ \chi & \gamma \end{pmatrix}$. The constraints on the transformation $\varphi \in Sp(1)$ translate into the following constraints on $\alpha, \beta, \chi, \gamma$ viz. $\gamma = \bar{\alpha}$, $\chi = -\bar{\beta}$ and $\bar{\alpha} = -\alpha$.

The fact that the group of quaternions is isomorphic to $Sp(1)$ and also to the sphere S^3 in $\mathbb{R}^4$, then follows from the fact that elements of the group $Sp(1)$ act on the space $\mathbb{Q}$ of quaternions as $\varphi q = q\bar{a}$ for $q \in \mathbb{Q}$ and $a \in \mathbb{Q}$ being determined by the transformation $\varphi \in Sp(1)$. Since $\varphi \in Sp(1)$ preserves the quaternionic product, we have $\langle x|y\rangle_{\mathbb{Q}} = x\bar{y} = x\bar{a}a\bar{y} = \|a\|^2 x\bar{y}$ whence $\|a\| = 1$. Since the identity $\|ab\| = \|a\|\|b\|$ holds for all quaternions, it follows that the group $Sp(1)$ is isomorphic to the group of unit quaternions that form a sphere S^3 in $\mathbb{R}^4$ for $1 = \|a\|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

The Geometry of a Single Qubit

The “quantum bit” or “qubit” plays the role of a “bit” in quantum computing [8] and constitutes a unit of quantum information [8-9]. It is represented by a state vector of a two-level quantum system. The representation space is, therefore, a two dimensional Hilbert space of the complex numbers and the basis vectors are usually chosen as $|0\rangle \equiv (1 \quad 0)^T$ and $|1\rangle \equiv (0 \quad 1)^T$, being the eigenvectors of the “spin” operator $\boldsymbol{\sigma}_3$ in the direction of the z axis.

The fundamental difference between the "classical bit" and the "qubit" is that the former can have only two possible values viz. 0,1. The "qubit", on the other hand, can occur in an infinite number of states being the superposition of the "pure states" represented by the basis vectors. We can, therefore, express a qubit as a linear combination of the two basis states as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. $\alpha, \beta \in \mathbb{C}$ are the probability amplitudes whose squares provide a measure of the probability of the qubit being in state $|0\rangle$ and state $|1\rangle$ respectively. We must, therefore, have $|\alpha|^2 + |\beta|^2 = 1$

The state space of a single qubit quantum register admits a geometrical representation as a Bloch sphere [10]. This is established as follows:-

The state space of a two level quantum system is conventionally taken as the Hilbert space $\mathrm{H} \equiv \mathbb{C} \otimes \mathbb{C}$ [11]. Now, if two physical states $|\psi\rangle, |\phi\rangle$ that differ merely by a phase i.e. a complex number of unit magnitude i.e. $|\psi\rangle = e^{i\omega}|\phi\rangle$, then they represent the same physical state. It follows, therefore, that the proper space for a two level quantum system is the above Hilbert space $\mathrm{H} \equiv \mathbb{C} \otimes \mathbb{C}$ quotiented by the equivalence relation $|\psi\rangle \sim |\phi\rangle$ iff $|\psi\rangle = e^{i\omega}|\phi\rangle$. It will, thus, be the projective Hilbert space created by this equivalence relation and may be defined as $\Pi(\mathrm{H}) = \mathrm{H}/\sim$. Sets of points in H differing only in phase (i.e. the same quantum ray) will be mapped onto the same point in $\Pi(\mathrm{H})$. Thus, $\psi \mapsto \Pi(\psi) =: \dfrac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$. Now, the complex space $\mathbb{C}^2$ has already been identified with the algebra of quaternions $\mathbb{Q}$ through the symplectic decomposition of an arbitrary quaternion $q \in \mathbb{Q}$ as $q = (q_0 + q_1 i) + j(q_2 - q_3 i) = q_\alpha + j\overline{q}_\beta \quad q_\alpha = (q_0 + q_1 i)$, $q_\beta = (q_2 + q_3 i) \in \mathbb{C}$. The set of normalized quaternions i.e. quaternions with unit modulus get mapped into a sphere S^3 embedded in $\mathbb{R}^4$. It, therefore, follows that normalized state vectors in $\mathbb{C}^2$ can also be canonically identified with the sphere S^3 embedded in $\mathbb{R}^4$. Quotienting $\mathbb{C}^2$ by the equivalence relation $|\psi\rangle \sim |\phi\rangle$ iff $|\psi\rangle = e^{i\omega}|\phi\rangle$ to get the projective Hilbert space $\Pi(\mathrm{H}) = \mathrm{H}/\sim$, amounts to constructing the complex projective space $CP(1)$ i.e. $S^3/U(1)$ which yields the sphere S^2 usually referred to in the literature as the Bloch sphere. In other words, the geometry of the two level quantum system (qubits) can be conveniently represented by the Bloch sphere.

The Hopf map

The identification of S^3 in $\mathbb{R}^4$ with the Bloch sphere $\left(S^2\right)$ is done through the well studied Hopf map. As a by product of the Hopf analysis, one also recovers the association between the geometry of qubits [12-15] and quaternions. To construct the Hopf map, we recall that the sphere S^3 is the group manifold of the special unitary group of matrices $SU(2)$ i.e. matrices with unit determinant that is isomorphic to the symplectic group $Sp(1)$ of transformations that preserve the quaternionic form. Elements on S^3 can be expressed in terms of quaternions $q \equiv \left(z_\alpha, z_\beta\right)$ through the symplectic decomposition $q = z_\alpha + j\bar{z}_\beta$, $z_\alpha, z_\beta \in \mathbb{C}$ or equivalently by matrices $q_m = \begin{pmatrix} z_\alpha & z_\beta \\ -\bar{z}_\beta & \bar{z}_\alpha \end{pmatrix}$ with $z_\alpha\bar{z}_\alpha + z_\beta\bar{z}_\beta = 1$ for, writing $z_\alpha = q_0 + iq_1$, $z_\beta = q_2 + iq_3$, we obtain $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. confirming that $q \equiv \left(z_\alpha, z_\beta\right)$ lies on the sphere S^3.

To obtain explicit expressions for the Hopf map, we make use of the canonical representation of the quaternion units by the well known Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $i \equiv -i\sigma_1$, $j \equiv -i\sigma_2$, $i \equiv -i\sigma_3$. In terms of these matrices, acting as the basis, the Hopf mapping is defined by $\mathbf{x} = \pi(q) = \left(\bar{z}_\alpha \quad \bar{z}_\beta\right)\boldsymbol{\sigma}\left(z_\alpha \quad z_\beta\right)^T$ yielding $\mathbf{x} = \left(\bar{z}_\beta z_\alpha + \bar{z}_\alpha z_\beta, i\left(\bar{z}_\beta z_\alpha - z_\beta\bar{z}_\alpha\right), \left|z_\alpha\right|^2 - \left|z_\beta\right|^2\right)$

$$= \left(2\left(q_0q_2 + q_1q_3\right), 2\left(q_0q_3 - q_1q_2\right), q_0^2 + q_1^2 - q_2^2 - q_3^2\right).$$

Let us take an element of the unitary group $U(1)$, say, $\varphi = \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix} = \lambda\mathbf{I} + \mu\sigma_3$. We, then, have $\pi(q\varphi) = (q\varphi)^\dagger \boldsymbol{\sigma} q\varphi = \varphi^\dagger \mathbf{x}\varphi = \mathbf{x}$ confirming, thereby that $\pi(q) = \pi(q\varphi)$ for $\varphi \in U(1)$ and hence, establishing the projective nature of the Hopf map taking all elements of S^3 connected through a unitary transformation to a single image. The image set is confirmed to be S^2 since $\mathbf{x}^2 = 1$ as can be easily verified. Thus, the Hopf map creates a principal bundle structure for S^3 with the base manifold being S^2 and the fibres being circles S^1 (members of the unitary group $U(1)$.

To obtain the local charts and the transition functions for the Hopf map, we parameterize the sphere S^3 by the stereographic projection coordinates. Let (X, Y) be the stereographic projection coordinates of a point in the southern hemisphere U_S of S^2 from the North Pole. Consider a complex plane that contains the equator of S^2. Then, $Z = X + iY$ lies within the circle of unit radius on the plane. Further, from the standard expressions for stereographic coordinates, we have $Z = \frac{x_1 + ix_2}{1 - x_3} = \frac{q_0 - iq_1}{q_2 - iq_3} = \frac{\overline{z}_\alpha}{\overline{z}_\beta}$. The projective nature of the Hopf map again manifests itself here as the invariance of Z under the transformation $(z_\alpha, z_\beta) \to (\lambda z_\alpha, \lambda z_\beta)$ for $|\lambda| = 1$. Similarly, the stereographic coordinates of (U, V) of a point in the northern hemisphere U_N with respect to the South Pole will be given by $W = U + iV = \frac{\overline{z}_\beta}{\overline{z}_\alpha}$.

We can, now, define the fibre bundle structure of the Hopf map. The local trivializations in the northern and southern hemisphere are respectively given by:-

(i) $\phi_N^{-1} : \pi^{-1}(U_N) \to U_N \times U(1)$ by $(z_\alpha, z_\beta) \mapsto \left(\frac{\overline{z}_\beta}{\overline{z}_\alpha}, \frac{z_\alpha}{|z_\alpha|} \right)$

(ii) $\phi_S^{-1} : \pi^{-1}(U_S) \to U_S \times U(1)$ by $(z_\alpha, z_\beta) \mapsto \left(\frac{\overline{z}_\alpha}{\overline{z}_\beta}, \frac{z_\beta}{|z_\beta|} \right)$

(Both these trivializations are well defined on the respective charts for, in the northern hemisphere $z_\alpha \neq 0$ and in the southern hemisphere $z_\beta \neq 0$).

(iii)On the equator, $x_3 = 0$ so that $|z_\alpha| = |z_\beta| = 2^{-1/2}$, whence, on the equator, the local trivializations become $\phi_N^{-1} : (z_\alpha, z_\beta) \mapsto \left(\frac{\overline{z}_\beta}{\overline{z}_\alpha}, \sqrt{2} z_\alpha \right)$ and $\phi_N^{-1} : (z_\alpha, z_\beta) \mapsto \left(\frac{\overline{z}_\alpha}{\overline{z}_\beta}, \sqrt{2} z_\beta \right)$ leading to the equatorial transition function $t_{NS} = \frac{z_\alpha}{z_\beta}$.

The Geometry of Two Qubit States & Quantum Entanglement

The Hopf map described above can easily be generalized to $\pi : S^7 \to S^4$. This motivates us to examine the geometry of a two qubit quantum state using the formalism of the Hopf map. However, when addressing multiple qubit states, one needs to carefully consider the issue of quantum entanglement. The “quaternions” again come in handy in studying the two qubit state.

The Hilbert space for the compound system H will be the tensor product of the individual Hilbert spaces $\mathrm{H}_A, \mathrm{H}_B$ of the two qubits and the basis vectors will be the direct product of the bases of the two spaces. We can, therefore, write a pure state of a two qubit system as $|\Phi\rangle = \alpha|00\rangle + \beta|01\rangle + \chi|10\rangle + \delta|11\rangle$ where $|ij\rangle \equiv |i\rangle \otimes |j\rangle$, $|i\rangle \in \mathrm{H}_A, |j\rangle \in \mathrm{H}_B$, $\alpha,\beta,\chi,\delta \in \mathbb{C}$, $\alpha = \alpha_{\mathrm{Re}} + i\alpha_{\mathrm{Im}}$, $\beta = \beta_{\mathrm{Re}} + i\beta_{\mathrm{Im}}$, $\chi = \chi_{\mathrm{Re}} + i\chi_{\mathrm{Im}}$ and $\delta = \delta_{\mathrm{Re}} + i\delta_{\mathrm{Im}}$, $|\alpha|^2 + |\beta|^2 + |\chi|^2 + |\delta|^2 = 1$. This normalization condition translates to a sphere S^7 embedded in $\mathbb{R}^8$. Now, if the two qubit state is a composition is two one qubit states, then it should be possible to write the composite state as the tensor product of the two single qubit states. Writing $|\phi\rangle_A = a_1|0\rangle_A + a_2|1\rangle_A$, $|\phi\rangle_B = b_1|0\rangle_B + b_2|1\rangle_B$, we have, for separable states $|\Phi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B = a_1b_1|00\rangle + a_1b_2|01\rangle + a_2b_1|10\rangle + a_2b_2|11\rangle$ whence, the separability condition can be inferred as $\alpha\delta - \beta\chi = 0$.

To introduce the Hopf fibration $\pi : S^7 \to S^4$ through the quaternions, we write the probability amplitudes $\alpha,\beta,\chi,\delta \in \mathbb{C}$ in the form of two quaternions using the symplectic decomposition as $q_1 = \alpha_{\mathrm{Re}} + \alpha_{\mathrm{Im}}\mathbf{i} + \beta_{\mathrm{Re}}\mathbf{j} + \beta_{\mathrm{Im}}\mathbf{k}$ and $q_2 = \chi_{\mathrm{Re}} + \chi_{\mathrm{Im}}\mathbf{i} + \delta_{\mathrm{Re}}\mathbf{j} + \delta_{\mathrm{Im}}\mathbf{k}$. Obviously, the normalization condition implies that $|q_1|^2 + |q_2|^2 = 1$. Parametrizing the sphere S^4 as $\sum_{l=1}^{5} \xi_l^2 = 1$, we obtain the Hopf map $\pi : S^7 \to S^4$ by the mapping $\xi_1 = Q_0$, $\xi_2 = Q_1$, $\xi_3 = Q_2$, $\xi_4 = Q_3$ and $\xi_5 = \sqrt{\left(1 - |Q|^2\right)}$ where $\pi(q_1, q_2) = Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k} = 2\overline{(q_1\bar{q}_2)}$. Explicit computation using the values of the quaternions q_1 and q_2 yield

$$\xi_1 = 2(\alpha_{\mathrm{Re}}\chi_{\mathrm{Re}} + \beta_{\mathrm{Re}}\delta_{\mathrm{Re}} + \alpha_{\mathrm{Im}}\chi_{\mathrm{Im}} + \beta_{\mathrm{Im}}\chi_{\mathrm{Im}})$$

$$\xi_2 = 2(\alpha_{\mathrm{Re}}\chi_{\mathrm{Im}} - \alpha_{\mathrm{Im}}\chi_{\mathrm{Re}} + \beta_{\mathrm{Re}}\delta_{\mathrm{Im}} - \beta_{\mathrm{Im}}\delta_{\mathrm{Re}})$$

$$\xi_3 = 2(\alpha_{\mathrm{Re}}\delta_{\mathrm{Re}} - \alpha_{\mathrm{Im}}\delta_{\mathrm{Im}} - \beta_{\mathrm{Re}}\chi_{\mathrm{Re}} + \beta_{\mathrm{Im}}\chi_{\mathrm{Im}})$$

$$\xi_3 = 2(\alpha_{\mathrm{Re}}\delta_{\mathrm{Im}} + \alpha_{\mathrm{Im}}\delta_{\mathrm{Re}} - \beta_{\mathrm{Re}}\chi_{\mathrm{Im}} - \beta_{\mathrm{Im}}\chi_{\mathrm{Re}})$$

$$\xi_5 = 1 - 2|q_1 q_2|$$

The Hopf map $\pi : S^7 \to S^4$ is equivalent to the mapping of S^7 onto a fibre bundle with the base space being the unit sphere S^4 and the fibres being spheres S^3 (this is evidenced by the invariance of this map under the transformation $(q_1, q_2) \mapsto (\lambda q_1, \lambda q_2), |\lambda| = 1$)

A perusal of the above expressions reveals an intriguing feature of the Hopf map. If the two qubit states are separable i.e. $\alpha\delta - \beta\chi = 0$, then $\xi_3 = \xi_4 = 0$ and the base space reduces to S^2 which is the Bloch sphere discussed in the earlier section of this manuscript. This Bloch sphere (the base space) constitutes the state space of one of the qubits of the two qubit separable system. The obvious question to be posed, then is – What about the state space of the other qubit of this separable system? A possible solution is to introduce a second Hopf map that fibres out the fibrings of the first Hopf map. As mentioned earlier the fibres of the map $\pi : S^7 \to S^4$ consist of spheres S^3 attached to the base space S^4. By means of another Hopf map $\pi' : S^3 \to S^2$ we can further, fibrate the fibres of the first map into a base space (the two sphere S^2) and fibres (being the one dimensional sphere). This creates another Bloch sphere that can be considered as the state space of the second qubit in the two qubit separable composite system. It needs be emphasized here that such a construction is not permissible in an entangled system because of the non vanishing of the coordinates ξ_3, ξ_4.

Conclusion

It is shown that the "quaternions" provide an attractive and efficient machinery to study the geometry of the one qubit and two qubit systems. One is led to the conclusion, through the Hopf map $\pi : S^3 \to S^2$, that the one qubit system has a geometrical representation as the Bloch sphere S^2 which the base space of a principal bundle with fibres consisting of the one dimensional sphere S^1. In the case of the two qubit composite system, a similar over fibration $\pi : S^7 \to S^4$ implies that the system has the geometry of a fibre bundle with the base space being the four dimensional sphere S^4 fibres consisting of S^3. As a fallout of the Hopf map analysis, we also find that unentangled two qubit systems admit a geometry as adirect product of two Bloch spheres as is intuitively to be expected. However, the Bloch sphere corresponding to one of the qubits in an unentangled system must be extracted from the S^3 fibres of the $\pi : S^7 \to S^4$ by invoking a second Hopf fibration of these S^3 fibres as $\pi : S^3 \to S^2$.

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