Bianchi type-III bulk viscous cosmological models in presence of Chaplygin gas with time varying-Λ in Lyra Geometry

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Abstract

In this paper, we have studied on Bianchi type-III space-time with time varying Λ in presence of bulk viscosity and Chaplygin gas in the context of Lyra geometry. We solve the Einstein field equations by using the relation $B = C^n$ as the expansion scalar ($Θ$) is proportional to the Shear scalar ($σ^2$). We find the cosmological parameters by using EoS of Chaplygin gas and the relation $Ω_M + Ω_Λ = 1$, also we observed the behavior of the gauge function $β^2$ and Λ against time. Some Physical and Geometrical behaviour of the model are discussed.

Keywords Bianchi type-III · Lyra Geometry · Chaplygin Gas· Cosmological model

1. INTRODUCTION

As per the observation in the different field of Cosmology, the universe entered into a phase of accelerated expansion. Although a huge effort of cosmologist helps us to understand the future evolution of the universe and comprehension the past and present state of our universe, still there is no final conclusion about the universe, is obtained. The accelerated expansion of the universe can be represented in terms of two different ways (i.e. the alternative theory or modified theory of gravitation and the dark energy and dark matter). Now a days Dark energy and dark matter is a hot topic among the researchers as it plays a significant role in understanding the universe

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which dominates the universe in positive energy density and negative pressure. Lyra Geometry (Lyra, 1951) [8] is a scalar-tensor theory. In 1918 Weyl [23] provide a modified theory of Riemannian geometry to unify electromagnetism and gravitation. But later, this theory was not taken seriously due to the non-integrability of length transfer. In 1951 G.Lyra proposed a new type of alternative theory by introducing a gauge function into the structurless manifold which removes the non-integrability of length transfer. Further, based on this theory, Sen and Dunn [15,16] proposed a scalar-tensor theory of gravitation which was an analog of the Einstein field equations. Later, Halford [4] conclude that in the normal general relativistic treatment the constant displacement vector $\phi_i$ used in Lyra's geometry plays a cosmological constant $\Lambda$. As per the Einstein theory, the scalar-tensor theory based on Lyra’s geometry predicts the same effect within observational limits. Several authors have investigated on the scalar-tensor theory and cosmology within the framework of Lyra geometry. Beesham(1988) [2], Singh and Singh (1991,1992) [18,19] and Singh& Desikan (1997)[20] have studied time-dependent displacement vector field with different space-time geometry in Lyra geometry. Mahbubur et al.[9] studied LRS Bianchi type I metric in presence of perfect fluid and solve the field equations using quadratic Eos, Rajbali et al.(2010)[1] have studied Bianchi type III cosmological model with varying G and $\Lambda$, Anirudha et al. (2010)[13] investigated Bianchi type III anisotropic model with variable EoS parameter. Panigrahi et.al. (2014, 2012)[10, 11] studied on five dimensional string cosmological model in Lyra’s manifold. Martiros Khurshudyan(2015)[6] have investigated the behavior of FRW metric in presence of extended Chaplygin gas with a varying $\Lambda$ term and concluded that modified field equations, compare to field equations of GR, provide a new parametrization of the dark energy sector of the large scale universe. Recently Singh et al.(2016)[21] instigated Bianchi type I cosmological model in presence of Chaplygin gas in Lyra geometry and solved the field equations. Y. Heydarzade et al.(2016)[5] have studied Einstein static universe on the brane embedded in a non-constant bulk space in the context of extended Chaplygin gas.

The above discussion and investigation inspire us to study on Bianchi type III cosmological model with Chaplygin gas and time varying $\Lambda$ in the context of Lyra geometry.

The paper is organized as follows : the next section deals with the metric and its field equations, in section 3, we have solved the field equations with the help of some realistic physical properties and EoS of Chaplygin gas, in section 4, we focus on the model and its physical properties, the last section deals with the summary and conclusion.

2. THE METRIC AND FIELD EQUATIONS

The field equations based on Lyra geometry as proposed by Sen(1957)[16] and Sen & Dunn(1971)[17] in normal gauge function is written as
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\[ R_{ij} - \frac{1}{2} g_{ij} \nabla^2 \phi - \frac{3}{2} g_{ij} \phi_j - \frac{3}{4} g_{ij} \phi^k \phi_k = -T_{ij}, \]  

(1)

where we use \( \frac{8\pi G}{c^4} = 1 \), and the displacement vector \( \phi_i \) is given by

\[ \phi_i = (0,0,0,\frac{2}{\sqrt{3}}\beta(t)). \]  

(2)

As the constant displacement field \( \phi_k \) of this theory was considered as cosmological constant \( \Lambda \) in the relativistic treatment (Halford 1970)[3], we are interested in the modified field equations consisting cosmological constant \( \Lambda(t) \) and which can be written as (Shchigolev 2013)[15]

\[ R_{ij} - \frac{1}{2} g_{ij} \nabla^2 \phi - \frac{3}{2} g_{ij} \phi_j - \frac{3}{4} g_{ij} \phi^k \phi_k = -T_{ij}. \]  

(3)

Here we consider the four-dimensional Bianchi type III space-time metric as

\[ ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{2x} dy^2 + c^2 dz^2, \]  

(4)

where \( A, B, C \) are functions of cosmic time \( t \) only.

The energy momentum tensor corresponding to bulk viscous fluid is given by

\[ T_{ij} = (\bar{p} + \rho) u_i u_j - \bar{p} g_{ij}. \]  

(5)

Here the flow vector \( u^i \) satisfies \( u^i u_i = -1 \) and

\[ \bar{p} = p - 3\xi H, \]  

(6)

\[ p = D\rho - \frac{E}{\rho^\alpha}, \]  

(7)

where \( D, E \) and \( \alpha \) are constant parameters.

For \( \alpha=1 \), equation (7) reduces to

\[ p = D\rho - \frac{E}{\rho}. \]  

(8)

In a co-moving coordinate system, the Einstein’s field equations (3) for the metric (4) with the help of (5),(6) and (8) leads to the following system of equation :

\[ \frac{\dot{A}}{AB} + \frac{\dot{A}}{AC} + \frac{\dot{B}}{BC} - \frac{1}{A^2} + \beta^2 - \Lambda = -\rho \]  

(9)

\[ \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{C}}{BC} + \beta^2 - \Lambda = -(D\rho - \frac{E}{\rho} - 3\xi H) \]  

(10)
\[ \frac{\ddot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{C} \dot{A}}{CA} + \beta^2 - \Lambda = -(D\rho - \frac{E}{\rho} - 3\xi H) \]  
\[ \frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A} \dot{B}}{AB} - \frac{1}{A^2} + \beta^2 - \Lambda = -(D\rho - \frac{E}{\rho} - 3\xi H) \]  
and \[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} = 0 \]

where the overhead dot (.) denote the derivative w.r.t cosmic time \( t \).

Now, let us define some parameters which are useful for the model given by (4)

The average scale factor \( F(t) \) and spatial volume are given by

\[ F(t) = (ABC)^{\frac{1}{3}} \]  
\[ V = F^3(t) = ABC \]

The average Hubble parameter \( H \) is given by

\[ H = \frac{\dot{F}}{F} = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \]

The shear scalar \( \sigma^2 \) and scalar expansion \( \Theta \) are given by

\[ \Theta = 3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \]

And

\[ \sigma^2 = \frac{1}{2} \left[ \left( \frac{\dot{A}}{A} \right)^2 + \left( \frac{\dot{B}}{B} \right)^2 + \left( \frac{\dot{C}}{C} \right)^2 \right] - \frac{\Theta^2}{6} \]

The average anisotropic parameter \( \Delta \) and the deceleration parameter are defined as

\[ \Delta = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2 \]  
\[ \text{Where} \quad H_1 = \frac{\dot{A}}{A}, H_2 = \frac{\dot{B}}{B}, H_3 = \frac{\dot{C}}{C} \]

\[ q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1. \]

The jerk parameter \( j(t) \) is defined as the dimensionless third derivative of the scale factor with respect to cosmic time and expressed as

\[ j(t) = \frac{F^2 F'''}{F'^3}. \]
The study of jerk parameter is essential as it is describing the models close to \( \Lambda \)CDM model.

### 3. SOLUTIONS OF THE FIELD EQUATION

As the system of the five equations (9)-(13) contains seven unknown parameters \( A, B, C, \rho, \beta, \Lambda, \) and \( \xi, \) so to find explicit solutions for the system, two additional constraints are required. Since the expansion of the model is proportional to the Shear scalar, assume that

\[ B = C^n \]  \hspace{1cm} (22)

where \( n \neq 1 \) is an arbitrary constant.

Equation (13) leads to

\[ A = m_1 B \]  \hspace{1cm} (23)

where \( m_1 \) is integrating constant.

Subtracting (12) from (11) and using (22) and (23), we get

\[ 2\frac{\ddot{C}}{C} + 4n \frac{\dot{C}^2}{C} = \frac{2}{m_1^2(n^2 - 1)} \cdot \frac{1}{C^{2n-1}} \]  \hspace{1cm} (24)

Multiply \( C^{4n} \) in both sides and integrating, equation (24) leads to

\[ \dot{C}^2 = \frac{C^{2-2n}}{m_1^2(n^2 - 1)} + m_2 C^{-4n} \]  \hspace{1cm} (25)

For simplification, we take \( m_2 = 0 \) in equation (25), and get

\[ \dot{C}^2 = \frac{C^{2-2n}}{m_1^2(n^2 - 1)} \]

\[ \Rightarrow \frac{dC}{dt} = \frac{C^{1-n}}{m_1 \sqrt{n^2 - 1}} \]  \hspace{1cm} (26)

On integrating both sides w.r.t t we get,

\[ C = (at + b)^{\frac{1}{n}} \]  \hspace{1cm} (27)

where \( a = \frac{n}{m_1 \sqrt{n^2 - 1}}, \) \( b = nm_3 \) and \( m_3 \) is integrating constant.

Using the value of \( C \) in equation (22) and (23), we obtain

\[ A = m_1 (at + b) \]  \hspace{1cm} (28)

And

\[ B = at + b \]  \hspace{1cm} (29)
Hence, the space-time (4) reduces to the form

\[ ds^2 = -dt^2 + m_1^2(at + b)^2 dx^2 + [e^{x}(at + b)]^2 dy^2 + (at + b)^2 dz^2 \]  

(30)

4. SOME PHYSICAL AND GEOMETRICAL PROPERTIES OF THE MODEL

The expressions from the scale factor \( F \), Hubble’s parameter \( H \), shear scalar \( \sigma^2 \), the average anisotropic parameter \( \Delta \), deceleration parameter \( q \), and proper volume \( V \) for the model (30) are given by

\[
F(t) = m_1^{\frac{1}{3}}(at + b)^{\frac{2n+1}{3n}}
\]  

(31)

\[
H = \frac{a(2n + 1)}{3n(at + b)}
\]  

(32)

\[
\Theta = \frac{a(2n + 1)}{n(at + b)}
\]  

(33)

\[
\sigma^2 = \frac{a^2(n - 1)^2}{3n^2(at + b)^2}
\]  

(34)

\[
\Delta = \frac{2(n - 1)^2}{(2n + 1)^2}
\]  

(35)

\[
q = \frac{n - 1}{2n + 1}
\]  

(36)

\[
j(t) = \frac{(1 - n)(1 - 4n)}{(2n + 1)^2(at + b)^2}
\]  

(37)

\[
V = m_1(at + b)^{\frac{2n+1}{n}}
\]  

(38)

\[
\frac{\sigma}{H} = \frac{\sqrt{3}(n - 1)}{(2n + 1)}.
\]  

(39)

If \( n \leq 10.265 \), then the ratio \( \frac{\sigma}{H} \) reduces to red-shift limit.
Here we observe that the deceleration parameter and jerk parameter are vanished for n=1 and, for n=-0.5 Hubble’s parameter and scalar expansion are become zero, for t=0 Hubble’s parameter (H) and the expansion (Θ) remains constant, and when t gradually increases, the value of H and Θ are decreased to zero.

Now, by subtracting (10) from (9) and using (27)-(29) we obtain,

$$\rho = \frac{Q(t) \pm \sqrt{Q^2(t) + 4E(D-1)}}{2(D-1)}, D \neq 1 \quad (40)$$

Where

$$Q(t) = \frac{m_1^2a^2(n^2 + 2n - 1) - n^2 + nm_1^2a^2(2n + 1)(at + b)}{m_1^2n^2(at + b)^2} \quad (41)$$

Now, by equation (8) and (40), we follow the cases

Case-I:

For D=0, we have

$$p = -\frac{E}{\rho} = \frac{2E}{Q(t)\pm\sqrt{Q^2(t)-4E}} \quad (42)$$

Here we note that if $Q^2(t) \geq 4E$, equation (47) reduces to the generalized Chaplygin gas, but, the case is not acceptable for $Q^2(t) < 4E$.

Case-II:

For E=0, we have either p=0 or $p=\frac{DQ(t)}{D-1}, D\neq1 \quad (43)$

Here, we obtain EoS of perfect fluid.

For $\rho$ to be positive and real, we consider $Q^2(t) + 4E(D-1) \geq 0$.

For representative case we assume $n=m_1=1.1, m_3=0.3$. 
Fig. 1 and 2 represent the variation of the energy density $\rho$ versus time $t$ for different $\xi$ with $D=E=0.7$ but in fig. 3 and 4, we study the variation of the energy density $\rho$ versus time $t$ for different $\xi$ for $D=0$ and $E=0$, $D\neq 1$ respectively.

Now, in absence of any curvature, let’s use the relation

$$\Omega_M + \Omega_\Lambda = 1$$

(44)

where $\Omega_M = \frac{\rho}{3H^2}$, $\Omega_\Lambda = \frac{\Lambda}{3H^2}$, which help us to find the value of cosmological constant ($\Lambda$) and the displacement vector $\beta(t)$.

Now from relation (44), we get

$$\Lambda = 3H^2 - \rho$$

(45)

Using (27)-(29) and (45) in (9), we get,

$$\beta^2 = \frac{a^2(2n+1)^2}{3n^2(at+b)^2} - 2\rho - \frac{m_4}{(at+b)^2}$$

(46)

Where

$$m_4 = \frac{m_1^2a^2(n+2)-n}{m_1^2n}$$

(47)
In Fig-5, we focus on the variation of the gauge function $\beta^2$ against time $t$ for different $\xi$ for $D=E=0.7$.

In fig. 6, we study the variation of $\Lambda$ versus time $t$ for different $\xi$ with $D=E=0.7$.

In the above figure, we observed that the gauge function and time-varying term $\Lambda$ behaves alike with respect to time in the context of Chaplygin gas.
We observed from equation (38) that the volume \( V = m_1 b^{2n+1} \) in initial epoch and gradually increases as time increases. In fig.5, it can be observed that the displacement vector \( \beta^2 \) is a decreasing function of time and it approaches to a small positive value at late time (i.e. present time), which is the ultimate result of Halford [3] and the recent observations of SNe Ia (Schmidt et. al. [14]). Equation (36) shows that, the deceleration parameter \( q < 0 \) for \( 0 < n < 1 \), and \( q > 0 \) for \( n > 1 \). Thus both decelerating and accelerating phases occur in the model which is the result of latest astronomical observations. Also the jerk parameter is positive which conclude that the universe is smoothly transits from deceleration phase to acceleration phase. Also from (32) & (33), we observed that for \( t \to 0 \), the Hubble’s parameter \( H \) and expansion factor \( \Theta \) are constant as time progresses gradually, they decreases and approaches to zero as \( t \to \infty \). This means that the universe is expanding with the increase of time, but rate of expansion becomes slow. Also we observed that \( \frac{\sigma^2}{\Theta^2} = \frac{(n-1)^2}{3(2n+1)^2} \) = constant \( \neq 0, n \neq -0.5 \) for all values of \( t \). So our universe is anisotropic in nature.

5. CONCLUSION
In this paper, we have investigated a Bianchi type-III cosmological model in the context of Chaplygin gas in the framework of Lyra’s geometry in presence of bulk viscous fluid. Our work analyzes the general feature of Bianchi type-III cosmological model with time dependent \( \Lambda \). It can be observed that the density is increasing initially, and then started decreasing to approaches a small positive value as expected (see Fig-1 to Fig-4).

We observe the following:

- It is observed that the model is anisotropic.
- It is observed that the model is expanding with both accelerating and decelerating phases.
- We observed that the cosmological time varying term \( \Lambda \) and gauge function \( \beta \) behaves identically (see fig. 5 and fig.6), which is the ultimate result of Halford (1970) [3].
- The universe is smoothly transited from deceleration phase to acceleration phase.
- For late time (i.e. present time) density become a small positive value, which may be \(~1 \times 10^{-27} Kg/m^3\) as per the recent observation.

REFERENCES