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Large Deviation Principle Applied for a Solution of Stochastic Differential Equation Driven by a Sub-Fractional Brownian Motion

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Abstract

We study the behavior of the stochastic differential equation driven by a sub-fractional Brownian motion $\mathbf{S}_t^H$ with Hurst index $H \in (0;1)$.

On this we first look at the case where the drift is zero and the diffusion coefficient is equal to 1 then we generalize the study in the case where the drift is not zero. Our study is done via the large deviation principle on the set of continuous square integrable functions in the Schwartz space.

Keywords: Sub-fractional Brownian motion, Large deviation principle, Contraction principle, Stochastic differential equation.

1. INTRODUCTION

Consider the process X_t^H solution of stochastic differential equation:

$$X_t^H = x + \int_0^t \sigma(X_s^H) d\mathbf{S}_s^H + \int_0^t b(X_s^H) ds, \ s,t \in [0;T] \tag{1}$$

where

- ⋆ $x \in \mathbb{R}$ is deterministic;
- ⋆ $b, \sigma : \mathbb{R} \to \mathbb{R}$ are bounded lispchitz continuous functions.
- ⋆ $\mathbf{S}^H : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$ is sub-fractional Brownian motion (see[6]) with Hurst parameter $H \in (0;1)$ and with covariance function

$$\mathcal{C}_{S^H}(t,s) = E(\mathbf{S}_t^H \mathbf{S}_s^H) = |t|^{2H} + |s|^{2H} + \frac{1}{2}(|t+s|^{2H} - |t-s|^{2H}) = \int_0^t \int_0^s \phi(r,u) du dr,$$

$$\phi(t,s) = \frac{\partial^2 \mathcal{C}_{S^H}(t,s)}{\partial t \partial s} = H(2H-1)[|t-s|^{2H-2} - (t+s)^{2H-2}].$$

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Our goal consists in establishing the large deviation principle (**LDP**) of the process X_t^H. To be done, we give a probability space ($\mathcal{S}'(\mathbb{R})$,$\mathcal{B}(\mathcal{S}'(\mathbb{R})), \mathbb{P}_\phi^H$) where $\mathbb{P}_\phi^H$ is the probability of $\mathbf{S}_t^H$ and the space of tempered distribution $\mathcal{S}'(\mathbb{R})$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$ for rapidly decreasing infinitely differential real valued function (see [2],[10]) defined by:

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}), \forall(\alpha, \beta) \in \mathbb{R}; \|t^\alpha D^\beta f\| \leq \infty\}.$$

However, in the large deviation theory case (see [1], [4],[9]), several authors have studied the behavior of solutions of stochastic differential equations driven by standard Brownian motion $\mathcal{B}_t$. Among these authors, we may cite the works of Schilder and Wentzell [1].

This is how we have the idea to study simultaneously the behavior of the stochastic differential equation driven by the fractional Brownian motion $\mathcal{B}_t^H$ in another paper and that directed by the sub-fractional $\mathbf{S}_t^H$ here in all the interval of the Hurst parameter $H \in (0; 1)$.

Our approach here, in the first times, is to apply to a function $f \in \mathcal{S}(\mathbb{R})$ a linear product associated with a trajectory ω of $\mathbf{S}_t^H$ belonging to $\mathcal{S}'(\mathbb{R})$ to show that $\mathbf{S}_t^H$ obeys the large deviation principle(**LDP**).

It is well known by Bochner Minlos theorem (see [2] [8]) in $\mathcal{S}(\mathbb{R})$ that there exists a probability measure $\mathbb{P}_\phi^H$ on $\mathcal{S}'(\mathbb{R})$ such that

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle\omega,f\rangle} d\mathbb{P}_\phi^H(\omega) = e^{-\frac{1}{2}|f|_\phi^2}.$$

So by fractional Girsanov formulas we obtain the bounds of the large deviation principle. The second and last time, via the contraction principle and the fact that $\mathbf{S}_t^H$ satisfies the **LDP**, we also succeed in showing that the process X_t^H defined (1) satisfies the **LDP**.

This paper is organized as follows. Sections 2 and 3 contain respectively some preliminaries on the sub-fractional Brownian motion and the large deviation principle. Section 4 contain our main results.

2. SUB-FRACTIONAL BROWNIAN MOTION

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ and $\mathcal{S}'(\mathbb{R})$ be the space of tempered distribution ω on $\mathbb{R}$.

Definition 1. *The coordinate process* $\mathbf{S}^H : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$ *defined as*

$$\mathbf{S}_t^H = \omega(t), \omega \in \mathcal{S}'(\mathbb{R})$$

on the probability space ($\mathcal{S}'(\mathbb{R})$,$\mathcal{B}(\mathcal{S}'(\mathbb{R})), \mathbb{P}_\phi^H$*), is a centered Gaussian process.*

Definition 2. *Let* $\mathfrak{C}$ *be the set of step function. The* $\mathcal{S}(\mathbb{R})$ *is defined as the closure of* $\mathfrak{C}$ *with respect to the scalar product.*

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle = \mathcal{C}_{S^H}(t,s) = \int_0^t \int_0^s \phi(r,u)dudr \tag{2}$$

and for all function $f \in S(\mathbb{R})$

$$\langle f, f\rangle_\phi = |f|^2_\phi = \int_0^t \int_0^s f(r)f(u)\phi(r,u)dudr \tag{3}$$

Theorem 1. *(see [2] and [8])*
Let $\mathbb{P}^H_\phi$ *be the probability measure of* S^H_t *on* $\mathcal{S}'(\mathbb{R})$*.Then for* $f \in \mathcal{S}(\mathbb{R})$*,*

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle\omega,f\rangle} d\mathbb{P}^H_\phi(\omega) = e^{-\frac{1}{2}|f|^2_\phi} \tag{4}$$

Lemma: 1. *Let* $f \in \mathcal{S}(\mathbb{R})$ *and* $\omega \in \mathcal{S}'(\mathbb{R})$*.Then*

$$E_{\mathbb{P}^H_\phi}[\langle\omega, f\rangle] = 0 \qquad and \qquad E_{\mathbb{P}^H_\phi}[\langle\omega, f\rangle^2] = |f|^2_\phi. \tag{5}$$

Theorem 2. ***(Fractional Girsanov formula I** see [10])*
Let $\Psi \in L^p(\mathbb{P}^H_\phi)$ *for* $p > 1$ *and let* $f \in \mathcal{S}(\mathbb{R})$*. Then for all* $\gamma(t) = \int_0^t f(r)\phi(r,s)dr$*, the map* $w \mapsto \Psi(w+\gamma) \in L^q(\mathbb{P}^H_\phi)$ *for all* $q < p$ *and*

$$\int_{\mathcal{S}'(\mathbb{R})} \Psi(w+\gamma)d\mathbb{P}^H_\phi = \int_{\mathcal{S}'(\mathbb{R})} \Psi(w)\exp\{\langle w, f\rangle_\phi - \frac{1}{2}|f|^2_\phi\}d\mathbb{P}^H_\phi \tag{6}$$

Theorem 3. ***(Fractional Girsanov formula II** see [5] and [10])*
Let γ *and* g *be continuous functions with supp*$\gamma \subset [0;T]$ *and supp*$g \subset [0;T]$ *such that* $\langle g, f\rangle_\phi = \langle\gamma, f\rangle \forall f \in \mathcal{S}(\mathbb{R}), suppf \subset [0;T]$ *ie* $\gamma(t) = \int_0^t g(r)\phi(r,s)dr$*. Define a probability measure* $\tilde{\mathbb{P}}^H_\phi$ *on* $\mathcal{S}'(\mathbb{R})$ *by*

$$\frac{d\tilde{\mathbb{P}}^H_\phi}{d\mathbb{P}^H_\phi} = \exp\{-\langle w, g\rangle_\phi - \frac{1}{2}|g|_\phi\}. \tag{7}$$

Then the process defined by $\tilde{w} = w + \gamma$ *is fractional Brownian motion under* $\tilde{\mathbb{P}}^H_\phi$

3. LARGE DEVIATION PRINCIPLE

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space.

Definition 3. *The family* $(X^\varepsilon_t)_{\varepsilon>0}$ *of probability* $\mathbb{P}^\varepsilon$ *is said to satisfy a large deviation principle if there exists a rate function* I *defined on* $\mathcal{S}(\mathbb{R})$ *and a speed* ε *tending to* 0 *such that:*

- $0 \leq I(x) \leq +\infty$*, for all* $x \in \mathcal{S}(\mathbb{R})$*;*
- I *is lower semicontinuous that is, for all* $a < +\infty$*,* $\{x : I(x) \leq a\}$ *is a closed of* $\mathcal{S}(\mathbb{R})$*;*
- *for all* $a < +\infty$*,* $\{x : I(x) \leq a\}$ *is a compact of* $\mathcal{S}(\mathbb{R})$*, in which case* I *is a good rate function;*

- *for any closed set $F \subset \mathcal{S}(\mathbb{R})$,*

$$\lim_{\varepsilon \to 0} \sup \varepsilon \log \mathbb{P}^{\varepsilon}(F) \leq - \inf_{x \in F} I(x) \tag{8}$$

- *for any open set $O \subset \mathcal{S}(\mathbb{R})$,*

$$\lim_{\varepsilon \to 0} \inf \varepsilon \log \mathbb{P}^{\varepsilon}(O) \geq - \inf_{x \in O} I(x) \tag{9}$$

Theorem 4. *(**Contraction principle** see [1])*
Let E and F be two Schwartz spaces and $g : E \to F$ is a continuous function.
If the family $(X_t^{\varepsilon})_{\varepsilon>0}$ satisfies a large deviation principle of a rate function I then the family $g((X_t^{\varepsilon})_{\varepsilon>0})$ satisfies the LDP on F of a rate function J defined by:

$$J(y) = inf\{I(x) : x \in E,\ y = g(x)\},$$

for each $y \in F$.

4. MAIN RESULTS

4.1 Large deviation principle applied to sub-fractional Brownian motion

In this part, our work consists in checking if the process $\varepsilon \mathbf{S}_t^H$ obeys the large deviation principle (LDP) for any index $H \in]0; 1[$.
Let $\mathcal{S}(\mathbb{R})$ be a Schwartz space. We denote $\mathbb{P}_{\phi}^{\varepsilon,H}$ the law of process $\varepsilon \mathbf{S}_t^H$ on $\mathcal{S}'(\mathbb{R})$ and

$$\mathbb{H} = \{f \in \mathcal{S}(\mathbb{R})\, and\, s \leq t \in [0; T], |f|_{\phi}^2 = \langle f_t, f_s \rangle = \int_0^t \int_0^s f_r f_u \phi(u,r) dudr < +\infty\}$$

Theorem 5. *The family $(\varepsilon \mathbf{S}_t^H)_{(\varepsilon>0)}$ satisfies the large deviation principle of speed ε^2 with a rate function given by:*

$$I(f) = \begin{cases} \frac{1}{2}|f|_{\phi}^2 = \frac{1}{2}\int_0^t \int_0^s f(r)f(u)\phi(r,u)dudr \; if\ f \in \mathbb{H} \\ \\ +\infty \; otherwise \end{cases} \tag{10}$$

In other words :

$-For\ all\ closed\ set\ C \subset \mathcal{S}(\mathbb{R}),$

$$\lim_{\varepsilon \to 0} \sup \varepsilon^2 \log \mathbb{P}_{\phi}^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in C) \leq -\frac{1}{2}|f|_{\phi}^2$$

$-For\ any\ open\ set\ O \subset \mathcal{S}(\mathbb{R}),$

$$\lim_{\varepsilon \to 0} \inf \varepsilon^2 \log \mathbb{P}_{\phi}^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) \geq -\frac{1}{2}|f|_{\phi}^2$$

Proof:
$\omega \in \mathcal{S}'(\mathbb{R})$ is the coordinated process of $\mathbf{S}_t^H$, $\langle \omega, f\rangle = \int_0^t f_r d\mathbf{S}_r^H < +\alpha \in \mathbb{R}_+^*$ is the action between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ and $|.|_\phi$ the norm in $\mathcal{S}(\mathbb{R})$.
Lower bound: Let O be an open set of $\mathcal{S}(\mathbb{R})$ and according to the fractional Girsanov formula II of $\mathbf{S}_t^H$

$$\begin{cases} d\mathbb{P}_\phi^{H,\varepsilon} = \exp\{\frac{1}{\varepsilon}\langle \omega, f\rangle_\phi + \frac{1}{2\varepsilon^2}|f|_\phi^2\} d\tilde{\mathbb{P}}_\phi^{\varepsilon,H} \\ \\ \omega = \tilde{\omega} - \frac{1}{\varepsilon}\gamma \end{cases}$$

We have

$$\begin{aligned} d\mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) &= \exp\{\frac{1}{\varepsilon}\langle \omega, f\rangle_\phi + \frac{1}{2\varepsilon^2}|f|_\phi^2\} d\tilde{\mathbb{P}}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) \\ &= \exp\{\frac{1}{\varepsilon}\langle \tilde{\omega}, f\rangle_\phi - \frac{1}{\varepsilon^2}\langle \gamma, f\rangle_\phi + \frac{1}{2\varepsilon^2}|f|_\phi^2\} d\tilde{\mathbb{P}}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) \\ &= \exp\{-\frac{1}{2\varepsilon^2}|f|_\phi^2\} \times \exp\{\frac{1}{\varepsilon}\langle \tilde{\omega}, f\rangle_\phi\} d\tilde{\mathbb{P}}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) \\ \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) &= \exp\{-\frac{1}{2\varepsilon^2}|f|_\phi^2\} \times \tilde{\mathbb{E}}[1_O \exp\{\frac{1}{\varepsilon}\langle \tilde{\omega}, f\rangle_\phi\}] \\ \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) &\geq \exp\{-\frac{1}{2\varepsilon^2}|f|_\phi^2\} \times \tilde{\mathbb{P}}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) \exp\{\frac{1}{\varepsilon}\alpha\} \\ & \qquad\qquad (Markov's\ inequality) \\ \log \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) &\geq -\frac{1}{2\varepsilon^2}|f|_\phi^2 + \log \tilde{\mathbb{P}}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) + \frac{1}{\varepsilon}\alpha \\ \varepsilon^2 \log \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) &\geq -\frac{1}{2\varepsilon^2}|f|_\phi^2 + \varepsilon^2 \log \tilde{\mathbb{P}}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) + \varepsilon\alpha \\ \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) &\geq -\frac{1}{2}|f|_\phi^2 \quad with \quad \liminf_{\varepsilon \to 0} \varepsilon^2 \log \tilde{\mathbb{P}}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) = 0 \end{aligned}$$

Hence

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in O) \geq -\frac{1}{2}|f|_\phi^2$$

Upper bound: Let C be closed set of $\mathcal{S}(\mathbb{R})$, $\gamma = \int_0^t f_r \phi(r,s)dr$ and $\mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in C) = \mathbb{E}[1_C \exp\{\langle \omega, f\rangle_\phi\}]$ and according to the fractional Girsanov formula I,

$$\begin{aligned}
\mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in C) &= \mathbb{E}[1_C \exp\{\langle \omega, f\rangle_\phi\}] = \tilde{\mathbb{E}}[1_C \exp\{\langle \omega + \gamma, f\rangle_\phi\}] \\
&= \tilde{\mathbb{E}}[1_C \exp\{\langle \omega, f\rangle_\phi\} \exp\{\langle \omega, f\rangle_\phi - \frac{1}{2\varepsilon^2}|f|_\phi^2\}] \\
&= \tilde{\mathbb{E}}[1_C \exp\{2\langle \omega, f\rangle_\phi - \frac{1}{2\varepsilon^2}|f|_\phi^2\}] \\
&= \tilde{\mathbb{E}}[1_C \exp\{2\langle \omega, f\rangle_\phi\}] \exp\{-\frac{1}{2\varepsilon^2}|f|_\phi^2\} \\
&= \tilde{\mathbb{E}}[1_C \exp\{2\langle \tilde{\omega}, f\rangle_\phi - 2\langle \gamma_\varepsilon, f\rangle_\phi\}] \exp\{-\frac{1}{2\varepsilon^2}|f|_\phi^2\} \\
&= \tilde{\mathbb{E}}[1_C \exp\{2\langle \tilde{\omega}, f\rangle_\phi - \frac{2}{\varepsilon^2}|f|_\phi^2\}] \exp\{-\frac{1}{2\varepsilon^2}|f|_\phi^2\} \\
\mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in C) &\leq \tilde{\mathbb{E}}[\exp\{2\langle \tilde{\omega}, f\rangle_\phi - \frac{2}{\varepsilon^2}|f|_\phi^2\}] \exp\{-\frac{1}{2\varepsilon^2}|f|_\phi^2\} \\
&= 1 \times \exp\{-\frac{1}{2\varepsilon^2}|f|_\phi^2\} \\
\log \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in C) &\leq -\frac{1}{2\varepsilon^2}|f|_\phi^2
\end{aligned}$$

$\log \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in C) \leq -\frac{1}{2\varepsilon^2}|f|_\phi^2$

Hence

$$\lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{\varepsilon,H}(\varepsilon \mathbf{S}_t^H \in C) \leq -\frac{1}{2}|f|_\phi^2$$

It remains to demonstrate that the rate function I is a good rate function that is to say to show the following lemmas:

Lemma: 2. *$I(f)$ is lower semi-continuous for $f \in \mathcal{S}(\mathbb{R})$.*

Proof

We assume that f_ε simply converges to f for all $f_\varepsilon, f \in \mathcal{S}(\mathbb{R})$

We have

$$\begin{aligned}
\lim_{\varepsilon\to 0} I(f_\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{2}|f_\varepsilon|_\phi^2 &= \frac{1}{2}\lim_{\varepsilon\to 0}\int_0^t\int_0^s f_\varepsilon(r)f_\varepsilon(u)\phi(r,u)dudr \\
&\geq \frac{1}{2}\int_0^t\int_0^s \lim_{\varepsilon\to 0} f_\varepsilon(r)f_\varepsilon(u)\phi(r,u)dudr \qquad (Fatou's\ lemma) \\
&= \frac{1}{2}\int_0^t\int_0^s (\lim_{\varepsilon\to 0} f_\varepsilon(r))(\lim_{\varepsilon\to 0} f_\varepsilon(u))\phi(r,u)dudr \\
&= \int_0^t\int_0^s f(r)f(u)\phi(r,u)dudr \\
&= \frac{1}{2}|f|_\phi^2
\end{aligned}$$

$$So\, I(f) \leq \lim_{\varepsilon \to 0} I(f_\varepsilon)$$

Hence I is lower semicontinuous.

Lemma: 3. *For each $a \geq 0$, the set $\{f \in \mathcal{S}(\mathbb{R}), I(f) \leq a\}$ is a compact subset.*

Proof
$I(f) < +\infty$ for all $f \in \mathcal{S}(\mathbb{R})$, so there exists $a \in \mathbb{R}_+^*$ such that $I(f) \leq a$, we deduces that the set of level: $\{f \in \mathcal{S}(\mathbb{R}), I(f) \leq a\}$ is compact.
Hence $\{f \in \mathcal{S}(\mathbb{R}), I(f) \leq a\}$ is a compact of $\mathcal{S}(\mathbb{R})$.

We conclude that this rate function is a good rate function for the LDP of the process $\varepsilon \mathbf{S}_t^H$.

4.2 Large deviation principle applied a solution of stochastic differential equation driven by the sub-fractional Brownian motion

In this second part of our work, we are interested by the behavior of the process (1) when $b \neq 0$.
So consider on $\mathcal{S}(\mathbb{R})$, when ε tends to 0, the solution $X_t^{\varepsilon,H}$ of the following stochastic differential equation:

$$X_t^{\varepsilon,H} = x + \varepsilon \int_0^t \sigma(X_r^{\varepsilon,H}) d\mathbf{S}_r^H + \int_0^t b(X_r^{\varepsilon,H}) dr,\ r, t \in [0; T] \tag{11}$$

and we denote its law of probability by $\mu^{\varepsilon,H} = \mathbb{P}_\phi^{\varepsilon,H} o F^{-1}$ such that $F(f_t) = h(t)$, the only continuous solution of the ordinary differential equation:

$$F(f_t) = h(t) = x + \int_0^t \sigma(h_r) f_r \phi(r,s) dr + \int_0^t b(h_r) dr,\ s, t \in [0; T] \tag{12}$$

where $f \in \mathcal{S}(\mathbb{R})$ the continuous function induced by the **LDP** of the sub-fractional Brownian motion with its second partial derivative convariance function

$$\phi(t,s) = H(2H-1)[|t-s|^{2H-2} - (t+s)^{2H-2}]. \tag{13}$$

Assume that on $\mathcal{S}(\mathbb{R})$, x is determinist and b and σ are bounded lipschitz continuous functions ie there satisfy the following assumption:
(**H:**) there exists constants L and M such that for all h and $z \in \mathcal{S}(\mathbb{R})$,

- $|b(h) - b(z)| \leq L|h - z|$;
- $|\sigma(h) - \sigma(z)| \leq L|h - z|$;
- $|b(h)| \leq M$,$|\sigma(h)| \leq M$.

Lemma: 4. *Let σ be a bounded function and f and h be bounded and continuous functions.*
Then there exists $K > 0$ and $N > 0$ such that
$|f(t)\phi(t,s)| \leq K$

$|\sigma(h(t))\phi(t,s)| \leq N$ *for all $s,t \in [0,T]$ and $h \in \mathcal{S}(\mathbb{R})$.*

Proof
f is a bounded function, so there exists δ such that $|f| \leq \delta$. We have for $s,t \in [0;T]$

$$\begin{aligned}|f(t)\phi(s,t)| &= |f(t)||\phi(s,t)| \\ &= |f||H(2H-1)[|t-s|^{2H-2} - (t+s)^{2H-2}]| \\ &\leq \delta H|(2H-1)|T^{2H} = K.\end{aligned}$$

According to assumption $|\sigma(h_t)| \leq M \ \forall \ h \in \mathcal{S}(\mathbb{R})$, so we have for $s,t \in [0;T]$

$$\begin{aligned}|\sigma(h_t)\phi(s,t)| &= |\sigma(h_t)||\phi(s,t)| \\ &= |\sigma||H(2H-1)[|t-s|^{2H-2} - (t+s)^{2H-2}]| \\ &\leq MH|(2H-1)|T^{2H} = N.\end{aligned}$$

Proposition 1. *F defined in (12) is a continuous function on $\mathcal{S}(\mathbb{R})$.*

Proof
Let's first show $F(f_t) = h(t)$ is continuous.
Let $h_1 = F(f_1)$ and $h_2 = F(f_2)$ with $h(t) = x + \int_0^t \sigma(h_r)f_r\phi(r,s)dr + \int_0^t b(h_r)dr$

$$\begin{aligned}h_2(t) - h_1(t) &= \int_0^t [\sigma(h_2(r))f_2(r) - \sigma(h_1(r))f_1(r)]\phi(r,s)dr + \int_0^t [b(h_2(r)) - b(h_1(r)]dr \\ &= \int_0^t [\sigma(h_2(r)) - \sigma(h_1(r))]f_2(r)\phi(r,s)dr + \int_0^t [f_2(r) - f_1(r)]\sigma(h_1(r))\phi(r,s)dr \\ &\quad + \int_0^t [b(h_2(r)) - b(h_1(r)]dr.\end{aligned}$$

$$\begin{aligned}|h_2(t) - h_1(t)| &\leq L\int_0^t |h_2(r) - h_1(r)||f_2(r)\phi(r,s)|dr + \int_0^t |f_2(r) - f_1(r)||\sigma(h_1(r))\phi(r,s)|dr \\ &\quad + L\int_0^t |h_2(r) - h_1(r)|dr \\ &\leq LK\int_0^t |h_2(r) - h_1(r)|dr + \delta NT + L\int_0^t |h_2(r) - h_1(r)|dr \\ &= L(K+1)\int_0^t |h_2(r) - h_1(r)|dr + \delta NT\end{aligned}$$

$$|h_2(t) - h_1(t)| \leq L(K+1)\int_0^t |h_2(r) - h_1(r)|dr + \delta NT$$

$$\|h_2 - h_1\| \leq \delta NTe^{L(K+1)T}$$

So F is continuous.

The fact that $\varepsilon\mathbf{S}_t^H$ satisfies the **LDP** and F is continuous function allows us to have the following theorem:

Theorem 6. *The family $(X_t^{\varepsilon,H})_{(\varepsilon>0)}$ of the stochastic differential equation (11) satisfies the large deviation principle of the good rate function given by*

$$J(h) = \begin{cases} \frac{1}{2}|\sigma^{-1}(h)[\dot{h} - b(h)]|_{\phi^{-1}}^2 \; if \; h \in \mathcal{S}(\mathbb{R}) \\ \\ +\infty \; otherwise \end{cases} \tag{14}$$

In other words :

* *for all closed set $C \subset \mathcal{S}(\mathbb{R})$,*

$$\lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mu^{\varepsilon,H}[X_t^{\varepsilon,H} \in C] \leq -\frac{1}{2}|\sigma^{-1}(h)[\dot{h} - b(h)]|_{\phi^{-1}}^2;$$

* *for any open set $O \subset \mathcal{S}(\mathbb{R})$,*

$$\lim_{\varepsilon\to 0}\inf \varepsilon^2 \log \mu^{\varepsilon,H}[X_t^{\varepsilon,H} \in O] \geq -\frac{1}{2}|\sigma^{-1}(h)[\dot{h} - b(h)]|_{\phi^{-1}}^2;$$

* *J is lower semi-continuous;*

* *$\{h \in \mathcal{S}(\mathbb{R}) \; and \; a \in \mathbb{R}_+^*, J(h) \leq a\}$ is a compact subset.*

Proof

Now let us show the upper and the lower bound by **the contraction principle**.
We know that εS_t^H of probability $\mathbb{P}_\phi^{\varepsilon,H}$ satisfies a **LDP** with a rate function $I(f) = \frac{1}{2}|f|_\phi^2$.

Upper bound Let C be a closed of $\mathcal{S}(\mathbb{R})$, since F is continuous, we have:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mu^{\varepsilon,H}[X_t^{\varepsilon,H} \in C] &= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{\varepsilon,H} oF^{-1}[X_t^{\varepsilon,H} \in C] \\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{\varepsilon,H}[F^{-1}(X_t^{\varepsilon,H} \in C)] \\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{\varepsilon,H}[F^{-1}(X_t^{\varepsilon,H}) \in F^{-1}(C)] \\
&= \lim_{\varepsilon\to 0}\sup \varepsilon^2 \log \mathbb{P}_\phi^{\varepsilon,H}[\varepsilon S_t^H \in F^{-1}(C)] \leq -\inf_{f \in F^{-1}(C)} I(f) \\
&= -\inf_{F(f)\in C}\{\inf I(f), f \in \mathcal{S}(\mathbb{R}), F(f) = h\} = -I(h)
\end{aligned}$$

So $\limsup_{\varepsilon\to 0}\varepsilon^2\ln\mu^{\varepsilon,H}[X_t^{\varepsilon,H}\in C]\leq -I(h)=-\inf_{F(f)\in C}\{\inf I(f),f\in\mathcal{S}(\mathbb{R}),F(f)=h\}$.

lower bound Let O an open set of $S(\mathbb{R})$, F is continuous, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0}\varepsilon^2\log\mu^{\varepsilon,H}[X_t^{\varepsilon,H}\in O] &= \liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}^{\varepsilon,H}oF^{-1}[X_t^{\varepsilon,H}\in O]\\ &= \liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}_\phi^{\varepsilon,H}[F^{-1}(X_t^{\varepsilon,H}\in O)]\\ &= \liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}_\phi^{\varepsilon,H}[F^{-1}(X_t^{\varepsilon,H})\in F^{-1}(O)]\\ &= \liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}_\phi^{\varepsilon,H}[\varepsilon S_t^H\in F^{-1}(O)]\geq -\inf_{f\in F^{-1}(O)}I(f)\\ &= -\inf_{F(f)\in O}\{\inf I(f),f\in\mathcal{S}(\mathbb{R}),F(f)=h\}=-I(h)\end{aligned}$$

So $\liminf_{\varepsilon\to 0}\varepsilon^2\log\mu^{\varepsilon,H}[X_t^{\varepsilon,H}\in O]\geq -I(h)=-\inf_{F(f)\in O}\{\inf I(f),f\in\mathcal{S}(\mathbb{R}),F(f)=h\}$

lower semicontinuous of J

Let's first show that $J(h)=\frac{1}{2}|\sigma^{-1}(h)[\dot{h}-b(h)]|^2_{\phi^{-1}}$

According to (12), $h(t)=x+\int_0^t b(h_r)dr+\int_0^t\sigma(h_r)f_r\phi(r,s)dr$, donc

$$\begin{aligned}\dot{h}_t &= b(h_t)+\sigma(h_t)f_t\phi(t,s)\\ f_t &= \frac{1}{\sigma(h_t)\phi(t,s)}[\dot{h}_t-b(h_t)]=\sigma^{-1}(h_t)[\dot{h}_t-b(h_t)]\phi^{-1}(t,s)\end{aligned}$$

$$\begin{aligned}\Rightarrow J(h) &= \inf\{\frac{1}{2}|f|^2_\phi,F(f)=h\}=\frac{1}{2}|\sigma^{-1}(h_t)[\dot{h}_t-b(h_t)]\phi^{-1}(t,s)|^2_\phi\\ &= \frac{1}{2}\int_0^t\int_0^s(\sigma^{-1}(h_r)[\dot{h}_r-b(h_r)]\phi^{-1}(r,u))(\sigma^{-1}(h_u)[\dot{h}_u-b(h_u)]\phi^{-1}(r,u))\phi(r,u)dudr\\ &= \frac{1}{2}\int_0^t\int_0^s(\sigma^{-1}(h_r)[\dot{h}_r-b(h_r)])(\sigma^{-1}(h_u)[\dot{h}_u-b(h_u)])\phi^{-1}(r,u)dudr\\ &= \frac{1}{2}|\sigma^{-1}(h)[\dot{h}-b(h)]|^2_{\phi^{-1}}\end{aligned}$$

Hence

$$J(h)=\frac{1}{2}|\sigma^{-1}(h)[\dot{h}-b(h)]|^2_{\phi^{-1}}$$

Let $h_\varepsilon\in\mathcal{S}(\mathbb{R})$ such that $h_\varepsilon\longrightarrow h\in\mathcal{S}(\mathbb{R})$.

So we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} J(h_\varepsilon) &= \lim_{\varepsilon\to 0}\frac{1}{2}|\sigma^{-1}(h_\varepsilon)[\dot{h_\varepsilon} - b(h_\varepsilon)]|^2_{\phi^{-1}} \\
&= \frac{1}{2}\lim_{\varepsilon\to 0}\int_0^t\int_0^s \sigma^{-1}(h_\varepsilon(r))[\dot{h_\varepsilon}(r) - b(h_\varepsilon(r))]\sigma^{-1}(h_\varepsilon(u))[\dot{h_\varepsilon}(u) - b(h_\varepsilon(u))]\phi^{-1}(r,u)dudr \\
&\geq \frac{1}{2}\int_0^t\int_0^s \lim_{\varepsilon\to 0}[\sigma^{-1}(h_\varepsilon(r))[\dot{h_\varepsilon}(r) - b(h_\varepsilon(r))]\sigma^{-1}(h_\varepsilon(u))[\dot{h_\varepsilon}(u) - b(h_\varepsilon(u))]\phi^{-1}(r,u)dudr \\
&\qquad\qquad (Fatou\ Lemma) \\
&= \frac{1}{2}\int_0^t\int_0^s \{\lim_{\varepsilon\to 0}\sigma^{-1}(h_\varepsilon(r))[\dot{h_\varepsilon}(r) - b(h_\varepsilon(r))]\}\{\lim_{\varepsilon\to 0}\sigma^{-1}(h_\varepsilon(u))[\dot{h_\varepsilon}(u) - b(h_\varepsilon(u)]\}\phi^{-1}(r,u)dudr \\
&= \frac{1}{2}\int_0^t\int_0^s \sigma^{-1}(h(r))[\dot{h}(r) - b(h(r))]\sigma^{-1}(h(u))\dot{h}(u) - b(h(u)) - \chi_{\{h(u)=0\}}(h(u))\dot{\eta}]\phi^{-1}(r,u))dudr \\
&= \frac{1}{2}[|\sigma^{-1}(h)[\dot{h} - b(h)]|^2_{\phi^{-1}}\,.
\end{aligned}$$

So $J(h) \leq \lim_{\varepsilon\to 0} J(h_\varepsilon)$, Hence J is lower semicontinuous.

Compactness $J(h) < +\infty$ for all $h \in \mathcal{S}(\mathbb{R})$,so there exists $a > 0$ such that $J(h) \leq a$, we deduces that the set of level $\{h \in L^2_\phi(\mathbb{R}), J(h) \leq a\}$ is compact of $\mathcal{S}(\mathbb{R})$ and J is a good rate function for **LDP** of $X_t^{\varepsilon,H}$

5. CONCLUSION

In short, we were able to study the asymptotic behavior of a stochastic differential equation driven by a sub-fractional Brownian motion thanks to the large deviations principle. This study is done first the case where the drift is zero and the diffusion coefficient is equal to 1. In this first case, we have shown the upper bound of **LDP** using the fractional Girsanov formula I and the lower bound by the fractional Girsanov formula II and Markov's inequality. The Contraction principle and the study of the previous case allow us to generalize the **LDP** for a stochastic differential equation directed by a sub-fractional Brownian motion for all $H \in (0;1)$. This work leads us to consider widening the study of the behavior of the stochastic differential equations to reflected stochastic differential equations driven by the fractional to the future.

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