International Journal of Pure and Applied Mathematical Sciences.
ISSN 0972-9828 Volume 12, Number 1 (2019), pp. 1-4

http://www.ripublication.com

Existence and Uniqueness Solution of Hyper Geometric Differential Equation Using Banach Fixed Point Theorem

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Abstract

Banach Contraction mapping principle or Banach Fixed point theorem takes an important role in the new branch of mathematical Sciences, Engineering and Technology. Buy this method modeler can form a suitable model for human being. In this present Research work we will study the existence and uniqueness solution of the second order hypergeometric differential equation. The important characteristics that the coefficients of a hypergeometric series is that are the solutions of the ordinary or partial differential equations. Here we will introduce a new approach to find out the uniqueness solution of Hypergeometric differential equation by the method of well known Banach contraction mapping principal or Banach Fixed Point theorem.

Keywords:-Gauss Hypergeometric differential equation, Complete Metric spaces, Banach Contraction mapping, Banach Fixed Point Theorem.

INTRODUCTION

We begin by deriving the second order ordinary differential equation satisfied by Gauss' hypergeometric function. The solutions of hypergeometric differential equation include many of the most interesting special functions of mathematical physics. Solutions to the hypergeometric differential equation there are in different method.

The particular focus lies on the existence and uniqueness solution of Hyper Geometric Differential equation. The main technique used here is that the Banach Fixed point property.

Definition: - Let n be an positive integer. Then the Pochhemmar symbol is denoted and defined as,

(i) $(\alpha)_n = \alpha(\alpha+1)\ (\alpha+2)\ (\alpha+3)\ \ldots\ldots\ldots\ldots(\alpha+n)$

(ii) $(\alpha)_0 = 1$

(iii) $(\alpha)_n = \frac{1.2.3.4\ldots\ldots(\alpha-1)\alpha(\alpha+1)(\alpha+2)(\alpha+3)\ldots\ldots(\alpha+n)(\alpha+n-1)}{1.2.3.4\ldots\ldots(\alpha-1)}$

$= \frac{\Gamma(\alpha+n)}{\Gamma\alpha}$

The well known Gauss Hyper geometric differential equation is

$x(x-1)\delta^2F + \{ (1+a+b)x - c\}\delta F + abF = 0$……………………..(1.1)

Where δ denote the ordinary differential operator, i.e $\delta = \frac{d}{dx}$

The Gauss Hyper geometric differential equation has three singularities namely at $x = 0$, $x = 1$, $x = \infty$.

We also have Hyper geometric differential function notation as,

$\theta x\ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)n(b)n}{(c)n \llcorner n} x^n$ …………………………(1.2)

And $(\theta x + a)\ (\theta x+b)\ F = (\theta x +c)\ \delta x$ …………………………..(1.3)

Now the integral representation of (1.1) is

$x(t) = \frac{\Gamma(b)}{\Gamma(c)\,\Gamma(b-c)} \int_a^b t^{-a}\ (1-t)^{c-b-1}\ (1-st)^{-a} ds$

This can be written in the form of Freadholm second type of integral equation as,

$x(t) = \frac{\Gamma(b)}{\Gamma(c)\,\Gamma(b-c)} \int_a^b k\big(t,s,s(t)\big)u(s)\, ds$ …………………(1.4)

$x(t) = \lambda \int_a^b k\big(t,s,s(t)\big)u(s)\, ds$ …………………………(1.5)

$\lambda = \frac{\Gamma(b)}{\Gamma(c)\,\Gamma(b-c)}$

Definition: - (Banach contraction mapping principle) Let (X,d) be a complete metric spaces and f be a mapping from X into itself which satisfy,

$D(fx,fy) \le \alpha\, d(x,y)\ \forall\ x,y \in X$. This is called the Banach contraction mapping principle.

Banach Fixed Point Theorem: - Let (X,d) be a complete metric spaces and f be a mapping from X into itself which satisfy,

$D(fx,fy) \le \alpha\, d(x,y)\ \forall\ x,y \in X, \alpha \in (0,1)$ then f has an unique fixed point in X.

Definition:-The exponential function $e_{p(t,a)=\left\{\exp(\int_a^t p(s)ds)\right\}}$ for tϵT.

or $e_{p(t,a)=}\exp\left(\int_a^t \frac{Log(1+\mu(s)p(s)}{\mu(s)}ds\right)$ for tϵT, μ>0.

Where T is the non empty closed subset of R and Log is the principal logarithm function.

Definition: - Let β>0 a constant and let ||.|| denotes the Euclidean norm on R^n and define the metric,

$$d_\beta(x,y) = \operatorname*{Sup}_{t\in(a,b)T} \frac{\|X(t)-Y(t)\|}{e\beta_{(t,a)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

$$d_0(x,y) = \operatorname*{Sup}_{t\in(a,b)T} \|x(t) - y(t)\| \quad \ldots\ldots\ldots\ldots\ldots(1.5)$$

Definition: - Let $C([a,b]_T;R^n)$ couple with suitable norm,

$$\|x\|_\beta = \operatorname*{Sup}_{t\in(a,b)T} \frac{\|X(t)\|}{e\beta_{(t,a)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.6)$$

$$\|x\|_0 = \operatorname*{Sup}_{t\in(a,b)T} \|x(t)\| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.7)$$

MAIN RESULTS

Theorem: - Consider the integral equation (1.4).Where t∈ [a, b] K: [a,b] ×[a,b] ×R^n → R^n be continuous mapping and let A be a mapping from $C([a,b]_T;R^n)$ → $C([a,b]_T;R^n)$. Then (1.1) has a unique solution.

Proof:- Let us recall the non empty set $C([a,b]_T;R^n)$ couple with suitable metric defined as above and for x, y∈ $C([a,b]_T;R^n)$ we have,

$$d_\beta(x,y) = \operatorname*{Sup}_{t\in(a,b)T} \frac{\|X(t)-Y(t)\|}{e\beta_{(t,a)}} \quad \text{for every } x,y \in C([a,b]_T;R^n)$$

$$\text{Now } d_\beta(Ax,Ay) = \operatorname*{Sup}_{t\in(a,b)T} \frac{\|(Ax)(t)-(Ay)(t)\|}{e\beta_{(t,a)}}$$

$$= Sup_{t\in(a,b)} \frac{\gamma}{e_\beta(t,a)}\{\int_a^b \|[\, k\big(t,s,\ x(s)\big)x(s) - k\big(t,s,\ y(s)\big)y(s)\,]\,\big|\big|\,ds$$

$$\text{Where } L = \operatorname*{Sup}_{(t,s)\in(a,b)T} \|K(t,x,\ x(s))\|$$

$$\le L\ Sup_{t\in(a,b)} \frac{\gamma}{e_\beta(t,a)}\{\int_a^b \|[x(s) - y(s)\,]\,\big|\big|\,ds$$

$$\le L\ Sup_{t\in(a,b)} \frac{\gamma}{e_\beta(t,a)}\{\int_a^b e_\beta(s,a)\ \frac{\|x(s)-y(s)\|}{e_\beta(s,a)}\,ds$$

$\leq$ L $d_\beta(x,y)$ $Sup_{t\in(a,b)}\frac{\gamma}{e_\beta(t,a)}\{\int_a^b e_\beta(s,a)$ ds

$\leq$ L $d_\beta(x,y)$ $Sup_{t\in(a,b)}\frac{\gamma}{e_\beta(t,a)}\{\int_a^b e_\beta(s,a)$ ds

$\leq$ L $d_\beta(x,y)$ $Sup_{t\in(a,b)}\frac{\gamma}{e_\beta(t,a)}\{e_\beta(b,a)-e_\beta(a,a)\}$

$\leq \gamma$L $d_\beta(x,y)$ $Sup_{t\in(a,b)}\frac{e_\beta(b,a)-e_\beta(a,a)}{e_\beta(t,a)}$

$\leq \gamma$L $d_\beta(x,y)$ $Sup_{t\in(a,b)}\frac{e_\beta(b,a)}{e_\beta(t,a)}$

$\leq \gamma\frac{L}{\lambda}d_\beta(x,y)$, taking $1/\lambda = Sup_{t\in(a,b)}\frac{e_\beta(b,a)}{e_\beta(t,a)}$

$\Rightarrow d_\beta(Ax,Ay) \leq \gamma\frac{L}{\lambda}d_\beta(x,y)$

This shows that A is contractive and $\gamma\frac{L}{\lambda}$ is contractive constant. Therefore Banach Fixed point theorem applied and which yields that their existence of unique fixed point of A. That is the hypergeometric differential equation has an unique solution.

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