Hyers-Ulam Stability of Certain Class of Nonlinear Second Order Differential Equations

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Abstract
We investigate the Hyers-Ulam stability of certain classes of nonlinear second order differential equations using a nonlinear generalisation of Gronwall-Bellman integral inequality known as Bihari integral inequality.

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1. Introduction
In 1940, Ulam [30] gave a wide-range talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among these was the question concerning the stability of homomorphisms. Hyers [10] solved the problem of Ulam for additive functions defined on Banach spaces thus: If $X$ and $Y$ are real Banach spaces and $\epsilon > 0$. Then for every function $f : X \rightarrow Y$ satisfying $||f(x + y) - f(x) - f(y)|| \leq \epsilon$, for all $x, y, \in X$, there exists a unique additive function $A : X \rightarrow Y$ with the property $||f(x) - A(x)|| \leq \epsilon$ for all $x \in X$. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians [8, 9, 11, 16, 22].
Obloza [20, 21] investigated the Hyers-Ulam stability of linear differential equation and connections between Hyers and Lyapunov stability of the ordinary differential equation. Alsina and Ger [1] continued and investigated the Hyers-Ulam stability of the differential equation \( g' = g \). They proved that if a differentiable function \( y : I \to \mathbb{R} \) satisfies \( |y' - y| \leq \epsilon \) for all \( t \in I \), then there exists a differentiable function \( g : I \to \mathbb{R} \) satisfying \( g'(t) = g(t) \) for any \( t \in I \) such that \( |g(t) - y(t)| \leq 3\epsilon \) for all \( t \in I \). The result by Alsina and Ger has been generalised by others including: Miura [17, 18], Takhasi [29, 30] and Jung [11, 12, 13, 14, 15] who proved the Hyers-Ulam stability of linear differential equations. Rus [25, 26] investigated the Hyers-Ulam stability of differential and integral equations using the Gronwall lemma [7] and the technique of weakly Picard operators. Recently, Quusuay [24] applied the Gronwall lemma to investigate the Hyers-Ulam stability of the form \( u''(t) + f(t, u(t)) = 0 \) and Emden-Fowler nonlinear differential equation of second order \( u''(t) + h(t)u(t)\alpha = 0 \) for the case where \( \alpha \) is a positive odd integer. Quusuay did not consider the case when \( \alpha \) is even integers and the function \( f \) is of the form \( f(t, u(t), u'(t)) \). These are the problems we consider in this paper using nonlinear generalisation of Gronwall-Bellman [2, 3] called Bihari inequality [4, 5].

The result obtained in this paper generalises the works of Quusuay [23] and Qarawani [24] on nonlinear second order differential equations. In this paper, we focus on the investigation of the Hyers-Ulam stability of the nonlinear second order differential equations given below.

\[
\begin{align*}
\tag{1.1} u''(t) + f(t, u(t)) &= 0 \\
\tag{1.2} u''(t) + f(t, u(t), u'(t)) &= 0
\end{align*}
\]

2. Preliminaries

In this section, we shall state the Bihari lemma and other useful results and definitions.

**Lemma 2.1.** [4, 5] Let \( u(t), f(t) \) be positive continuous functions defined on \( a \leq t \leq b, (\leq \infty) \) and \( K > 0, M \geq 0 \), further let \( \omega(u) \) be a nonnegative nondecreasing continuous function for \( u \geq 0 \), then the inequality

\[
u(t) \leq K + M \int_a^t f(s)\omega(u(s))ds, \quad a \leq t < b
\]

implies the inequality

\[
u(t) \leq \Omega^{-1}\left(\Omega(k) + M \int_a^{t'} f(s)ds\right), \quad a \leq t \leq b' \leq b
\]

where

\[
\Omega(u) = \int_{u_0}^u \frac{dt}{\omega(t)}, \quad 0 < u_0 < u
\]
In the case \( \omega(0) > 0 \) or \( \Omega(0 +) \) is finite, one may take \( u_0 = 0 \) and \( \Omega^{-1} \) is the inverse function of \( \Omega \) and \( t \) must be in the subinterval \([a, b']\) of \([a, b]\) such that

\[
\Omega(k) + M \int_a^t f(s)ds \in \text{Dom}(\Omega^{-1})
\]

**Theorem 2.2. (Generalised First Mean Value Theorem) [19, 27]** If \( f(t) \) and \( g(t) \) are continuous in \([t_0, t] \subseteq I\) and \( f(t) \) does not change sign in the interval, then there is a point \( \xi \in [t_0, t] \) such that

\[
\int_{t_0}^t g(s)f(s)ds = g(\xi)\int_{t_0}^t f(s)ds
\]

**Definition 2.3. [6]** A function \( \omega \) is said to belong to a class \( H \) if it satisfies the following conditions

i. \( \omega(u) > 0 \) is nondecreasing and \( \omega \in C^0 \) for \( u > 0 \)

ii. \( \frac{1}{v}\omega(v) \leq \omega(u) \) for all \( u \) and \( v \geq 1 \) when \( \omega \) is a positive, nondecreasing function defined and continuous on \( I \).

**Definition 2.4.** The equation (1.1) with the initial condition \( u(t_0) = u'(t_0) = 0 \) has Hyers-Ulam stability if there exists a positive constant \( K > 0 \) with following property. For every \( \epsilon > 0 \) \( u \in C^2(I) \), if

\[
|u''(t) + f(t, u(t))| \leq \epsilon
\]

(2.1)

then, there exist a solution \( u_0(t) \in C^2(I) \) of the equation (1.1), such that

\[
|u(t) - u_0(t)| \leq K \epsilon.
\]

**Definition 2.5.** Equation (1.2) with the initial condition \( u(t_0) = u'(t_0) = 0 \) has Hyers-Ulam stability if there exists a positive constant \( K > 0 \) with following property. For every \( \epsilon > 0 \) \( u \in C^2(I) \), if

\[
|u''(t) + f(t, u(t), u'(t))| \leq \epsilon
\]

(2.2)

then, there exist a solution \( u_0(t) \in C^2(I) \) of the equation (1.2), such that

\[
|u(t) - u_0(t)| \leq K \epsilon.
\]

3. Main Result

As earlier stated, we first investigate the Hyers-Ulam stability of the second order non-linear differential equation of the form (1.1) where the function \( f(t, u(t)) \) satisfies the condition

\[
|f(t, u(t))| \leq \phi(t)\omega(|u(t)|)
\]

(3.1)
Where $\phi(t)$ is a continuous, nonnegative function for $t \geq t_0$ and $\omega(u)$ is a continuous, nondecreasing, nonnegative function for $u > 0$. Besides, the function $f(t,u(t))$ is continuous on $D := \{ t, u : t \in [t_0, \infty), u \in I \}$.

**Theorem 3.1.** Let $\int_{t_0}^{\infty} |u'(s)|ds \leq L$ for $L > 0$ be assumed and let the function $f(t,u(t))$ satisfies the following conditions:

1. $c_1$ $u(t) \leq f(t,u(t))u(t)$, where $f(t,u(t)) > 1$ for all $t \geq t_0$.
2. $c_2$ $\frac{f'(t,u(t))u(t)}{f(t,u(t))} = g(t,u(t)) $ $g$ a positive, continuous function.
3. $c_3$ $|f(t,u(t))| \leq \phi(t)\omega(|u(t)|)$, where $\omega(t)$ belongs to the class $H$
4. $c_4$ $\Omega(r) = \int_{r_0}^{r} \frac{ds}{\omega(s)}$ $r_0 \geq 0, r \geq r_0$

If $u : I \to I$ satisfying $u \in C^2(I)$ and the inequality

$$|u''(t) + f(t,u(t))| \leq \epsilon \text{ for all } t \geq t_0 \text{ and for some } \epsilon > 0,$$

then there exist a solution $u_0(t) \in C^2(I)$ of the differential equation (1.1) such that $|u(t) - u_0(t)| \leq K\epsilon$ for any $t \geq 0$, provided

$$\int_{t_0}^{\infty} \phi(s) < M < \infty \text{ and } K = L\Omega^{-1} (\Omega(1) + |g(\xi, u(\xi))|M).$$

Therefore, equation (1.1) has Hyers-Ulam stability with initial condition $u(t_0) = u'(t_0) = 0$.

**Proof.** Multiplying (2.1) by $|u'(t)|$ to get

$$-\epsilon |u'(t)| \leq u'(t)u''(t) + f(t,u(t))u'(t) \leq \epsilon |u'(t)|$$

(3.2)

for all $t \geq t_0$. Integrating each term from $t_0$ to $t$, then,

$$-\epsilon \int_{t_0}^{t} |u'(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s,u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} |u'(s)|ds$$

for any $t \geq t_0$. Integrating by part, let $\int_{t_0}^{\infty} |u'(s)|ds \leq L$ for $L > 0$ and $f_u(t,u(t)) \leq 0$.

$$-\epsilon L \leq \frac{1}{2}u'(t)^2 + f(t,u(t))u(t) - \int_{t_0}^{t} f'(s,u(s))u(s)ds \leq \epsilon L$$

for all $t \geq t_0$. Then, it follows that

$$f(t,u(t))u(t) \leq \epsilon L + \int_{t_0}^{t} f'(s,u(s))u(s)ds$$

(3.3)
Applying \( c_1 \) to (3.3),

\[
  u(t) \leq \epsilon L + \int_{t_0}^t f'(s, u(s))u(s)ds \quad \text{for} \quad t \geq t_0 \tag{3.4}
\]

We write (3.4) as,

\[
  u(t) \leq \epsilon L + \int_{t_0}^t f'(s, u(s))u(s)\frac{f(s, u(s))}{f(s, u(s))}f(s, u(s))ds \quad \text{for} \quad t \geq t_0 \tag{3.5}
\]

Applying \( c_2 \) and using generalised Mean value theorem in a closed region \( D \).

\[
  u(t) \leq \epsilon L + g(\xi, u(\xi)) \int_{t_0}^t f(s, u(s))ds \quad \text{for} \quad t \leq t_0
\]

\[
  \leq \epsilon L + |g(\xi, u(\xi))| \int_{t_0}^t |f(s, u(s))|ds \quad \text{for} \quad t \leq t_0
\]

\[
  \leq \epsilon L + |g(\xi, u(\xi))| \int_{t_0}^t |f(s, u(s))|ds \quad \text{for} \quad t \leq t_0
\]

Since \( \epsilon L > 0 \), we have

\[
  \frac{|u(t)|}{\epsilon L} \leq 1 + |g(\xi, u(\xi))| \int_{t_0}^t \phi(s)\omega \left( \frac{|u(s)|}{\epsilon L} \right) ds \quad t \leq t_0 \tag{3.6}
\]

Setting \( v(t) = R \cdot \text{H.S (3.6)} \)

since \( \omega \) is nondecreasing we have

\[
  0 < \omega \left( \frac{|u(t)|}{\epsilon L} \right) \leq \omega(v(t))
\]

\[
  v'(t) = |g(\xi, u(\xi))|\phi(t)\omega \left( \frac{|u(t)|}{\epsilon L} \right)
\]

\[
  \leq |g(\xi, u(\xi))|\phi(t)\omega(v(t))
\]

then,

\[
  \frac{v'(t)}{\omega(v(t))} \leq |g(\xi, u(\xi))|\phi(t)
\]

By the definition of \( \Omega \), this gives

\[
  \frac{d\Omega(v(t))}{dt} \leq |g(\xi, u(\xi))|\phi(t)
\]

Integrating from \( t_0 \) to \( t \) gives

\[
  \Omega(v(t)) - \Omega(v(t_0)) \leq |g(\xi, u(\xi))| \int_{t_0}^t \phi(s)ds
\]
since \( v(t_0) = 1 \) and \( \Omega^{-1}(u) \) being increasing. Also we have

\[
v(t) \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))| \int_{t_0}^{t} \phi(s)ds \right)
\]

and finally from (3.6) we obtain

\[
\frac{|u(t)|}{\epsilon L} \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))| \int_{t_0}^{t} \phi(s)ds \right), \quad t \geq t_0
\]

As \( t \to \infty \) then,

\[
\frac{|u(t)|}{\epsilon L} \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))| \int_{t_0}^{t} \phi(s)ds \right), \quad t \geq t_0
\]

provided \( \lim_{t \to \infty} \int_{t_0}^{t} \phi(s)ds \leq M < \infty \)

Hence,

\[
|u(t)| \leq \epsilon L \left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))| \int_{t_0}^{t} \phi(s)ds \right) \right) \quad \text{for all} \quad t \geq t_0.
\]

\[
|u(t) - u_0(t)| \leq K \epsilon
\]

Where \( K = \epsilon L \left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi))| M) \right) \right) \).

Hence, the equation (1.1) has Hyers-Ulam stability.

**Example 3.2.** Consider Hyers-Ulam stability of the nonlinear differential equation of the form

\[
u''(t) + t^{-4} u^2 \exp\left(u(t)\right) = 0.
\]

(3.7)

taking

\[
f(t, u(t)) = t^{-4} u^2(t) \exp\left(u(t)\right)
\]

and allow

\[
\omega(u) = u^2 \exp\left(u(t)\right), \quad \phi(t) = t^{-4}
\]

Where \( u(t_0) = u'(t_0) = 0 \) and \( u_0(t) = 0 \).

Therefore, the equation (3.7) is Hyers-Ulam stable.

In our next result we consider the Hyers-Ulam stability of the nonlinear differential equation (1.2), Where the function \( f(t, u(t), u'(t)) \) is continuous on \( D = \{t, u, u' : t \in [t_0, \infty), u, u' \in I\} \) and satisfies some conditions to be prescribed later.

**Theorem 3.3.** Let \( \int_{t_0}^{\infty} |u'(s)|ds \leq L \) for \( L > 0 \) be assumed and let function \( f(t, u(t), u'(t)) \) satisfies the following conditions:

\( H_1 \) \( u(t) \leq f(t, u(t), u'(t))u(t) \), where \( f(t, u(t), u'(t)) > 1 \) for all \( t \geq t_0 \).

\( H_2 \) \[
\frac{f(t, u(t), u'(t))u(t)}{f(t, u(t), u(t))} = g(t, u(t), u'(t)), \quad g \text{ a positive, continuous function.}
\]
We can write (3.9) as

$$H_3 \quad |f(t, u(t), u(t))| \leq h(t)\omega(|u(t)|)|u'(t)|$$

where \(\omega(t)\) belongs to the class \(H\), for \(s > 0\) the function \(\omega(s)\) is nondecreasing. Where \(h, \omega : I \rightarrow I\) are nonnegative, continuous functions.

$$H_4 \quad \Omega(r) = \int_{r_0}^{r} \frac{ds}{s\omega(s)} r_0 \geq 0, r \geq r_0$$

If \(u \in C^2(I), |u'(t)| \leq \frac{|u(t)|}{\epsilon L}\) and the inequality

$$|u''(t) + f(t, u(t), u'(t))| \leq \epsilon$$

for all \(t \geq t_0\) and for some \(\epsilon > 0\), then there exist a solution \(u_0(t) \in C^2(I)\) of the differential equation (1.2) such that

$$|u(t) - u_0(t)| \leq K\epsilon$$

for any \(t \geq 0\), provided

$$\int_{t_0}^{t} h(s)ds < M < \infty \text{ and } \Omega^{-1}(\Omega(1) + |g(\xi, u(\xi), u'((\xi)))|M) < \infty,$$

where \(K = L\Omega^{-1}(\Omega(1) + |g(\xi, u(\xi), u'((\xi)))|M)\).

Therefore, equation (1.2) has Hyers-Ulam stability with initial condition \(u(t_0) = u'(t_0) = 0\).

**Proof.** Multiplying (2.2) by \(|u'(t)|\) to get

$$-\epsilon|u'(t)| \leq u'(t)|u''(t)| + f(t, u(t), u'(t))\leq \epsilon|u'(t)|$$

for all \(t \geq t_0\). Integrating each term from \(t_0\) to \(t\), then,

$$-\epsilon \int_{t_0}^{t} |u(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s, u(s), u'(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} |u'(s)|ds$$

for any \(t \geq t_0\).

Integrating by part, let \(f_u(t, u(t), u'(t)) + f'_u(t, u(t), u'(t)) \leq 0\) and using hypothesis in the theorem, we have

$$-\epsilon L \leq \frac{1}{2}u'(t)^2 + f(t, u(t), u'(t))u(t) - \int_{t_0}^{t} f'(s, u(s), u'(s))u(s)ds \leq \epsilon L$$

for all \(t \geq t_0\). Then,

$$f(t, u(t), u'(t))u(t) \leq \epsilon L + \int_{t_0}^{t} f'(s, u(s), u'(s))u(s)ds$$

Using \(H_1\)

$$u(t) \leq \epsilon L + \int_{t_0}^{t} f'(s, u(s), u'(s))u(s)ds \text{ for } t \geq t_0$$

(3.9)

We can write (3.9) as

$$u(t) \leq \epsilon L + \int_{t_0}^{t} \frac{f'(s, u(s), u'(s))u(s)}{f(s, u(s), u'(s))} f(s, u(s), u'(s))ds \text{ for } t \geq t_0$$
Applying $H_2$

$$u(t) \leq \epsilon L + \int_{t_0}^{t} g(s, u(s), u'(s)) f(s, u(s), u'(s)) ds \quad \text{for} \quad t \geq t_0$$

By application of generalised Mean value theorem of integral in the closed region $D$,

$$u(t) \leq \epsilon L + g(\xi, u(\xi), u'(\xi)) \int_{t_0}^{t} f(s, u(s), u'(s)) ds \quad \text{for} \quad t \geq t_0$$

$$|u(t)| \leq \epsilon L + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} |f(s, u(s), u'(s))| ds \quad \text{for} \quad t \geq t_0$$

$$|u(t)| \leq \epsilon L + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} h(s) \omega(|u(s)|) |u'(s)| ds \quad \text{by} \quad H_3$$

$$\frac{|u(t)|}{\epsilon L} \leq 1 + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} h(s) \omega(\frac{|u(s)|}{\epsilon L}) |u'(s)| ds \quad (3.10)$$

Setting $z(t) = \text{R.H.S of (3.10)}$

Hence,

$$z(t) \leq 1 + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} h(s) \omega(z(s)) z(s) ds \quad (3.11)$$

Setting $v(\tau) = \text{R.H.S of (3.11)}$ since $\omega$ is nondecreasing we have

$$0 < \omega(z(t)) \leq \omega(v(t))$$

$$v'(t) = |g(\xi, u(\xi), u'(\xi))| h(t) \omega(z(t)) z(t) \leq |g(\xi, u(\xi), u'(\xi))| h(t) \omega(v(t)) v(t)$$

$$\frac{v'(t)}{\omega(v(t)) v(t)} \leq |g(\xi, u(\xi), u'(\xi))| h(t)$$

Application of $H_4$, this gives,

$$\frac{d\Omega(v(t))}{dt} \leq |g(\xi, u(\xi), u'(\xi))| h(t)$$

Integrating from $t_0$ to $t$ gives

$$\Omega(v(t)) - \Omega(v(t_0)) \leq |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} h(s) ds$$

since $v(t_0) = 1$ and $\Omega^{-1}(u)$ being increasing also we have

$$z(t) \leq v(t) \leq \Omega^{-1} \left( \Omega(1) + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} h(s) ds \right)$$
Finally from (3.11) we obtain
\[
\frac{|u(t)|}{\epsilon L} \leq \Omega^{-1} \left( \Omega(1) + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} h(s)ds \right) \quad \text{for } t \geq t_0
\]
As \( t \to \infty \), then,
\[
\frac{|u(t)|}{\epsilon L} \leq \Omega^{-1} \left( \Omega(1) + |g(\xi, u(\xi), u'(\xi))| M \right)
\]
provided \( \lim_{t_0} \int_{t_0}^{t} h(s)ds \leq M < \infty \)

Hence,
\[
|u(t)| \leq \epsilon L \left( \Omega^{-1} \left( \Omega(1) + |g(\xi, u(\xi), u'(\xi))| M \right) \right) \quad \text{for all } t \geq t_0.
\]

Where,
\[
K = L \left( \Omega^{-1} \left( \Omega(1) + |g(\xi, u(\xi), u'(\xi))| M \right) \right) \quad \text{for all } t \geq t_0.
\]

Hence, it holds that \( |u(t)| \leq K \epsilon \) for any \( t \geq t_0 \), with initial condition \( u_0(t) = u'(t) = 0 \) satisfies (1.2) and \( u_0 \in C^2(I) \) such that \( |u(t) - u_0(t)\) \( \leq K \epsilon \).

\[\square\]

**Example 3.4.** To investigate Hyers-Ulam stability of the second order nonlinear differential equation of the form
\[
u''(t) + (2t)^{-4}u^2 \exp (u'(t))u'(t) = 0 \quad (3.12).
\]

We take
\[
f(t, u(t), u'(t)) = (2t)^{-4}u^2 \exp (u'(t))u'(t)
\]
and allow \( \omega (u) = u^2 \), \( h(t) = (2t)^{-4} \),
Where \( u(t_0) = u'(t_0) = 0 \) and \( u_0(t) = 0 \). Therefore, the equation(3.12) is Hyers-Ulam stable.

**References**


