Simultaneous Dual Series Equations Involving Laguerre Polynomials with Matrix Argument

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Abstract

In this paper an exact solution is obtained for the simultaneous dual series equations involving Laguerre polynomials of matrix argument by multiplying factor technique.

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1. INTRODUCTION

The dual series equations are of the type

\[ \sum_{n=0}^{\infty} \sum_{j=1}^{s} a_{ij} C_{nj} \Gamma_m (\alpha + \beta + ni)L_{ni} (\alpha; x) = f_i (x), 0 \leq x \leq D \]  
\( (1.1) \)

and

\[ \sum_{n=0}^{\infty} \sum_{j=1}^{s} b_{ij} C_{nj} \Gamma_m (\alpha + \beta + \frac{m+1}{2})L_{ni} (\alpha, x) = g_i (x), \quad D \leq x \leq \infty \]  
\( (1.2) \)
where

\[ L_n(\alpha, x) = \frac{\prod_m (\alpha + n)}{\prod_m (\alpha)} \frac{1}{1}F_1(-n; \alpha + \frac{m+1}{2}; x) \]

is the Laguerre polynomial of matrix argument for \( R(\alpha) > -1 \),

\( R(n + \alpha) > -1 \) and \( \Pi_m(a) = \Gamma_m(a + \frac{m+1}{2}) \)

\[ \Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \left(a - \frac{i-1}{2}\right) \]

\( n=0,1,2,\ldots, j=1,2,\ldots,s \); \( f_i(x) \) and \( g_i(x) \) are known functions of non-singular symmetric matrix \( x \) of order \( n \), and \( a_{ij}, b_{ij} \) are known constants. By using a multiplying factor technique [5,7] the unknown function \( C_{nj} \) is determined.

2. SOME USEFUL RESULTS

(i) The following Integrals are required from Erdelyi et al [2] and Erdelyi et al [3]

\[ \int_0^\infty [y-x]^{\beta-(m+1)/2} L_n(\alpha; x) dx \]

\[ \frac{\Gamma_m(\beta)}{\Gamma_m(\alpha + n + \frac{m+1}{2})} \left|y\right|^\beta L_n(\alpha + \beta; y) \]

for \( \alpha > -1, \beta > \frac{m+1}{2} - 1 \), and \( \int_y^\infty \left|x-y\right|^{\beta} etr(-x) \ L_n(\alpha; x) \)

\[ = \Gamma_m(\frac{m+1}{2} - \beta) \ etr(-y)L_n(\alpha + \beta - \frac{m+1}{2}; y) \]

for \( \alpha > -1, \beta > \frac{m+1}{2} - 1 \)

(ii) The orthogonality relation for Laguerre polynomial with matrix argument given by Erdelyi et al [2]

\[ \int_{x=0}^\infty |x|^\alpha etr(-x) L_p(\alpha; x) L_q(\alpha; x) \]

\[ = \frac{\Gamma_m(\alpha + p + \frac{m+1}{2})}{\Gamma_m(\alpha + p + \frac{m+1}{2})} \delta_{pq} \]

for \( \alpha > -1 \)

where \( \delta_{pq} \) is the Kronecker delta.
(iii) The differential formula Erdelyi et al [2] in the form

\[ D_x \left[ |x|^\alpha L_n (\alpha; x) \right] = |x|^{\alpha-(m+1)/2} \cdot (\alpha - \frac{m+1}{2}, x) \]  

(2.4)

3. SOLUTION OF THE EQUATIONS

Multiplying eq. (1.1) by \( |x|^{\alpha} |y-x|^{\beta-(m+1)/2} \) and eq. (1.2) by \( |x-y|^\beta \text{etr} (-x) \) and integrating with respect to \( x \) over \((0,y), (y, \alpha)\) respectively, we find after using the results (2.1) and (2.2) that

\[
\sum_{h=0}^{\infty} \sum_{j=1}^{\infty} a_{ij} c_{nj} \frac{\Gamma_m (\alpha + n_i + \frac{m+1}{2}) \Gamma_{ni} (\alpha + \beta + ni)}{\Pi_m (\alpha + \beta + ni)} L_{ni} (\alpha + \beta; y) = \frac{|x|^{\alpha-\beta}}{\Gamma_m (\beta)} \int_{0}^{y} |x|^{\alpha} |y-x|^{\beta-(m+1)/2} f_i(x) dx 
\]

(3.1)

for \( \beta > \frac{m+1}{2}, \quad \alpha > -1, \quad 0 < y < D \)

and

\[
\sum_{h=0}^{\infty} \sum_{j=1}^{\infty} b_{ij} c_{nj} \frac{\Gamma_m (\alpha + n_i + \frac{m+1}{2}) \Gamma_m (\alpha + \beta + ni)}{\Pi_m (\alpha + \beta + ni)} L_{ni} (\alpha + \beta - \frac{m+1}{2}; y) \]

\[
= \frac{\text{etr}(y)}{(\Gamma_m (\frac{m+1}{2} - \beta))} \int_{y}^{\infty} \text{etr}(-x) |x-y|^\beta g_i(x) dx 
\]

(3.2)

for \( \beta > \frac{m+1}{2}, \quad \alpha + \beta > \frac{m+1}{2} - 1, \quad D < y < \infty \)

Now multiply the equation (3.1) by \( |y|^{\alpha+\beta} \), differentiating w.r.t. \( y \) and use the formula (2.4), we see that it becomes

\[
\sum_{h=0}^{\infty} \sum_{j=1}^{\infty} b_{ij} c_{nj} \frac{\Gamma_m (\alpha + n_i + \frac{m+1}{2}) \Gamma_m (\alpha + \beta + ni)}{\Pi_m (\alpha + \beta + ni)} L_{ni} (\alpha + \beta - \frac{m+1}{2}; y) = \sum_{j=1}^{s} C_{ij} \frac{|y|^{\alpha-\beta}}{\Gamma_m (\beta)} \cdot D_y \int_{0}^{y} |x|^{\alpha} |y-x|^{\beta-(m+1)/2} f_j(x) dx 
\]

(3.3)
for \( 0 < y < D, \quad \beta > \frac{m+1}{2} - 1, \quad \alpha > -1 \);

where \( C_{ij} \) are the elements of the matrix \( [b_{ij}][a_{ij}]^{-1} \), \( i = 1, 2, 3, \ldots, s \).

Now the left hand sides of the equations (3.2) and (3.3) are identical and an application of the orthogonality relation (2.3) yields the solution of equations (1.1) and (1.2) in the form

\[
C_{ij} = \sum_{j=1}^{s} d_{ij} \frac{\Gamma_m (ni + \frac{m+1}{2}) \Gamma_m (\alpha + \beta + ni)}{\Gamma (\alpha + ni + \frac{m+1}{2}) [\Gamma_m (\alpha + \beta + ni)]^2} \cdot B_{ij} (\alpha, \beta; D) \quad (3.4)
\]

for \( \alpha + \beta > \frac{m+1}{2} - 1, \quad 0 < \beta < 1 \)

where \( n = 0, 1, 2, \ldots; \quad j = 1, 2, \ldots, s \), and \( d_{ij} \) are the element of the matrix \( [b_{ij}]^{-1} \)

and

\[
B_{ij} (\alpha, \beta; D) = \sum_{j=1}^{s} C_{ij} \frac{1}{\Gamma_m (\beta)} \int_{D}^{\infty} \text{etr} (-y) L_m (\alpha + \beta - \frac{m+1}{2}; y) F_i (y) dy
\]

\[
+ \frac{1}{\Gamma_m (\frac{m+1}{2} - \beta)} \int_{D}^{\infty} \left| y \right|^\alpha \text{etr} (-\left| y \right|) L_m (\alpha + \beta - \frac{m+1}{2}; \left| y \right|) G_i (y) dy \quad (3.5)
\]

\[
F_i (y) = D_1 \int_{0}^{y} \left| x \right|^\alpha \left| y - x \right|^{\beta - (m+1)/2} f_i (x) dx \quad (3.6)
\]

and

\[
G_i (y) = \int_{y}^{\infty} \left| x - y \right|^\beta \text{etr} (-x) g_i (x) dx \quad (3.7)
\]

REFERENCES


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