Certain inclusion properties of subclass of starlike and convex functions of positive order involving Hohlov operator

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Abstract

In this paper we investigate some inclusion properties of subclasses of convex and starlike functions of positive order involving Hohlov operator.

Keywords and Phrases: Srivastava-Wright convolution operator, Starlike functions, Convex functions, Uniformly Starlike functions, Uniformly Convex functions, Hadamard product, Hohlov operator.

1. INTRODUCTION

Let $\mathbb{H}$ be the class of functions analytic in the unit disk.

$$\mathbb{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$$

Let $\mathcal{A}$ be the class of functions $f \in \mathbb{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

The Gaussian hypergeometric function $F(a,b;c,z)$ given by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad (z \in \mathbb{U}) \quad (1.2)$$

is the solution of the homogenous hypergeometric differential equation

$$z(1-z)w''(z) + [c-(a+b+1)z]w'(z) - abw(z) = 0$$
and has rich applications in various fields such as conformal mappings, quasi
conformal theory, continued fractions, and so on. Here, $a, b, c$ are complex numbers
such that $c \neq 0, -1, -2, -3, \ldots$, $(a)_0 = 1$ for $a \neq 0$, and for each positive integer $n$, $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$ is the Pochhammer symbol.

In the case of $c = -k$, $k = 0, 1, 2, \ldots$, $F(a, b; c; z)$ is defined if $a = -j$ or $b = -j$ where $j \leq k$. We refer to [1, 8] and references therein for some important results.

For $f \in \mathcal{A}$, we recall the operator $I_{a,b,c}(f)$ of Hohlov [4] which maps $\mathcal{A}$ into itself defined by means of Hadamard product as

$$I_{a,b,c}(f)(z) = zF(a, b; c; f(z)). \quad (1.3)$$

It is a special case of Srivastava-Wright convolution operator.

Therefore, for a function $f$ defined by (1.1) we have

$$I_{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}(c)_{n-1}}{(n-1)_{n-1}} a_n z^n. \quad (1.4)$$

Using the integral representation,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-t)^{b-1} t^{c-b-1} \frac{dt}{(1-tz)^a}, \quad \Re(c) > \Re(b) > 0,$$

We can write

$$[I_{a,b,c}(f)](z) = \left( \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{(1-t)^{b-1} t^{c-b-1} f(tz)}{t} \frac{dt}{(1-tz)^a} \right) * \frac{z}{(1-tz)^a}.$$

When $f(z)$ equals the convex function $\frac{z}{1-z}$, then the operator $I_{a,b,c}(f)$ in this case becomes $zF(a, b; c; z)$. If $a = 1, b = 1 + \delta, \quad c = 2 + \delta$ with $\Re(\delta) > -1$ then the convolution operator $I_{a,b,c}(f)$ turns into Bernardi operator

$$B_f(z) = [I_{a,b,c}(f)](z) = \frac{1 + \delta}{z^{\delta}} \int_0^1 t^{\delta-1} f(t) dt.$$

Indeed, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively.
Let us denote (see [5], [6])

\[ P_l(k) = \begin{cases} \frac{8(\arccos k)^2}{\pi^2(1-k^2)} & \text{for } 0 \leq k < 1 \\ \frac{8}{\pi^2} & \text{for } k = 1 \\ \frac{\pi^2}{4\sqrt{1+t}(k^2-1)k^2(t)} & \text{for } k > 1 \end{cases} \quad (1.5) \]

where \( t \in (0,1) \) is determined by \( k = \cosh(\pi \mathcal{K}(t)/[4\mathcal{K}(t)]) \), \( \mathcal{K} \) is the Legendre's complete Elliptic integral of the first kind

\[ \mathcal{K}(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}} \]

and \( \mathcal{K}(t) = \mathcal{K}(\sqrt{1-t^2}) \) is the complementary integral of \( \mathcal{K}(t) \). Let \( \Omega_k \) be a domain such that \( 1 \in \Omega_k \) and

\[ \partial \Omega_k = \{ w = u + iv : u^2 = k^2(u^2 + k^2v^2) \}, \quad 0 \leq k < \infty. \]

The domain \( \Omega_k \) is elliptic for \( k > 1 \), hyperbolic when \( 0 < k < 1 \), parabolic when \( k = 1 \), and a right half-plane when \( k = 0 \). If \( p \) is an analytic function with \( p(0) = 1 \) which maps the unit disc \( \mathbb{U} \) conformally onto the region \( \Omega_k \), then \( P_l(k) = p'(0) \). \( P_l(k) \) is strictly decreasing function of the variable \( k \) and it values are included in the interval \((0,2]\).

Let \( f \in \mathcal{A} \) be of the form (1.1). If \( f \in k - \mathcal{UCV} \), then the following coefficient inequalities hold true (cf.[5]):

\[ |a_n| \leq \frac{(P_l(k))_{n-1}}{n!}, \quad n \in \mathbb{N} \setminus \{1\}. \quad (1.6) \]

Similarly, if \( f \) of the form (1.1) belongs to the class \( k - \mathcal{ST} \), then (cf., [6])

\[ |a_n| \leq \frac{(P_l(1))_{n-1}}{(n-1)!}, \quad n \in \mathbb{N} \setminus \{1\}. \quad (1.7) \]

A function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{R}^\tau(A,B), \quad (\tau \in \mathbb{C} \setminus \{0\}, -1 \leq B < A \leq 1) \), if it satisfies the inequality

\[ \left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1, \quad (z \in \mathbb{U}). \quad (1.8) \]
The class $\mathcal{R}^\tau(A, B)$ was introduced earlier by Dixit and Pal [3]. Two of the many interesting subclasses of the class $\mathcal{R}^\tau(A, B)$ are worthy of mention here. First of all, by setting
\[
\tau = e^{i\eta} \cos \eta (\pi/2 < \eta < \pi/2), A = 1 - 2\beta (0 \leq \beta < 1) \text{ and } B = -1,
\]
the class $\mathcal{R}^\tau(A, B)$ reduces essentially to the class $\mathcal{R}_\eta(\beta)$ introduced and studied by Ponnusamy and Ronning [8], where
\[
R_\eta(\beta) = \{ f \in A : \Re(e^{i\eta}(f'(z) - \beta)) > 0 \ (z \in \mathbb{U}; -\pi/2 < \eta < \pi/2, 0 \leq \beta < 1) \}.
\]
Secondly, if we put \(\eta = 1, A = \beta\) and \(B = -\beta (0 < \beta \leq 1)\), we obtain the class of functions \(f \in A\) satisfying the inequality
\[
\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in \mathbb{U}; 0 < \beta \leq 1)
\]
which was studied by (among others) Padmanabhan [7] and Caplinger and Causey [2].

Motivated by the earlier work of Srivastava et al.[9], we introduce two new subclasses of $\mathcal{S}$ namely $\mathfrak{M}(\lambda, \alpha)$ and $\mathfrak{N}(\lambda, \alpha)$. We need the following results, to prove our main results.

**Definition 1.1.** For some $\lambda (0 \leq \lambda < 1)$, we let $\mathfrak{M}(\lambda, \alpha)$ and $\mathfrak{N}(\lambda, \alpha)$ be two new subclass of $\mathcal{S}$ consisting of functions of the form (1.1) with positive order $\alpha$ ($1 < \alpha \leq \frac{4}{3}$) and satisfying the analytic criteria

\[
\mathfrak{M}(\lambda, \alpha) := \left\{ f \in S : \Re \left( \frac{z(I_{a,b,c}f(z))'}{(1-\lambda)(I_{a,b,c}f(z)) + \lambda z(I_{a,b,c}f(z))} \right) < \alpha, z \in \mathbb{U} \right\} \quad (1.9)
\]

and

\[
\mathfrak{N}(\lambda, \alpha) := \left\{ f \in S : \Re \left( \frac{(I_{a,b,c}f(z))' + z(I_{a,b,c}f(z))''}{(I_{a,b,c}f(z))' + \lambda z(I_{a,b,c}f(z))''} \right) < \alpha, z \in \mathbb{U} \right\} \quad (1.10)
\]

respectively.
COEFFICIENT BOUNDS:

Lemma 1.2: A function $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathfrak{M}(\lambda, \alpha)$ if $I_{a,b,c}(f) / (z'I_{a,b,c}(f)) \in \mathbb{H}$ and if

$$
\sum_{n=2}^{\infty} \left[ n - (1 + n\lambda - \lambda) \alpha \right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n \leq \alpha - 1.
$$

(1.11)

Proof:

Let $f \in \mathfrak{M}(\lambda, \alpha)$, then by (1.9)

$$
\left| \frac{\left( z(I_{a,b,c}(f)) \right)'}{\left( (1 - \lambda)(I_{a,b,c}(f)) + \lambda (I_{a,b,c}(f)) \right)} \right|^{-1} < 1,
$$

that is

$$
\left| \frac{\left( z(I_{a,b,c}(f)(z)) \right)'}{\left( (1 - \lambda)(I_{a,b,c}(f)(z)) + \lambda (I_{a,b,c}(f)(z)) \right)} \right|^{-1} < 1,
$$

that implies

$$
\sum_{n=2}^{\infty} \left[ n - (1 + n\lambda - \lambda) \alpha \right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n \leq \alpha - 1.
$$

(1.12)

Hence the theorem is proved.
Corollary 1.1:

Let \( f \in \mathfrak{M}(\lambda, \alpha) \), then

\[
\alpha - 1 \leq \sum_{n=2}^{\infty} \left[ n - (1 + n\lambda - \lambda)\alpha \right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n
\]

Corollary 1.2:

A function \( f \in \mathcal{A} \) of the form (1.1) belongs to the class \( \mathfrak{M}(\lambda, \alpha) \)

if \( I_{a,b,c}(f) / (zI - \alpha, b, c(f)) \in \mathbb{H} \) and if

\[
\sum_{n=2}^{\infty} \left[ n(n - (1 + n\lambda - \lambda)\alpha) \right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \leq \alpha - 1.
\] (1.13)

Proof:

It is well known that \( f \in \mathfrak{M}(\lambda, \alpha) \) if and only if \( zf \in \mathfrak{M}(\lambda, \alpha) \).

Since

\[
(zI_{a,b,c}f(z)) = z + \sum_{n=2}^{\infty} n \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n,
\]

we may replace \( a_n \) with \( na_n \) in Lemma (1.2).

Theorem 1.3. (CONVEX LINEAR COMBINATION) : The class \( \mathfrak{M}(\lambda, \alpha) \) is closed under convex linear combination.

Theorem 1.4 : The class \( \mathfrak{M}(\lambda, \alpha) \) is closed under convex linear combination.

Theorem 1.5( ARITHMETIC MEAN)

Let \( f_1, f_2, \ldots, f_n \) defined by

\[
f_j(z) = \sum_{n=2}^{\infty} a_{nj} z^n, (a_{nj} \geq 0, i=1,2, \ldots, j, n \geq 2)
\] (1.14)

be in the class \( \mathfrak{M}(\lambda, \alpha) \). Then the arithmetic mean of \( f_j(z) (j=1,2,\ldots, J) \) defined by

\[
h(z) = \frac{1}{J} \sum_{n=1}^{J} f_j(z),
\] (1.15)

is also in the class \( \mathfrak{M}(\lambda, \alpha) \).

Theorem 1.6 :

Let \( f_1, f_2, \ldots, f_n \) defined by
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\[ f_j(z) = \sum_{n=2}^{\infty} a_{n,i} z^n, \quad (a_{n,i} \geq 0, i = 1, 2, \ldots, n \geq 2) \]  

(1.16)

be in the class \( \mathcal{M}(\lambda, \alpha) \). Then the arithmetic mean of \( f_j(z) (j = 1, 2, \ldots) \) defined by

\[ h(z) = \frac{1}{j} \sum_{n=1}^{j} f_j(z), \]  

(1.17)

is also in the class \( \mathcal{M}(\lambda, \alpha) \).

In the following section we obtain Inclusion results for the classes \( \mathcal{M}(\lambda, \alpha), \mathcal{M}(\lambda, \alpha) \) using Hohlov operator.

2. INCLUSION PROPERTIES

Making use of the following lemma, we will study the action of the hypergeometric function on the classes \( k - UC\mathcal{V}, k - ST \).

Lemma 2.1. (Dixit et al.[3]) : If \( f \in \mathcal{R}^\ell (A, B) \) is of form (1.8), then

\[ |a_n| \leq (A - B) \frac{11}{n}, \quad n \in \mathbb{N} \setminus \{1\}. \]  

(2.1)

The result is sharp.

Theorem 2.2:

Let \( a, b \in \mathbb{C} \setminus \{0\}, \quad |a| \neq 1, |b| \neq 1. \) Also, let \( c \) be a real number such that

\[ c > |a| + |b| + 1. \]

If \( f \in \mathcal{R}^\ell (A, B), \quad I_{a,b,c} (f) / (z I_{a,b,c} (f) \in \mathbb{H} \) and if the inequality

\[ \frac{\Gamma(\ell)\Gamma(-|a| - |b| - 1)}{\Gamma(-a)\Gamma(-b)} \left[ \left(1 - (A - B)c \right) + \frac{(\ell \alpha - \alpha)}{(|a| - 1)(|b| - 1)} \right] \leq (\alpha - 1) \frac{1}{(A - B)|r|} + 1 + (\ell \alpha - \alpha) \frac{c - 1}{(|a| - 1)(|b| - 1)}, \]  

(2.2)

is satisfied, then \( I_{a,b,c} (f) \in \mathcal{M}(\lambda, \alpha) \).

Proof:

Let \( f \) be of the form (1.1) belong to the class \( \mathcal{R}^\ell (A, B) \). By virtue of Lemma (1.2), it suffices to show that
\[
\sum_{n=2}^{\infty} \left[ n - (1+n\lambda - \lambda) \alpha \right] \left[ \frac{(b_n - 1)(c_n - 1)}{(a_n - 1)(c_n - 1)} a_n \right] \leq \alpha - 1.
\] (2.3)

Taking into account the inequality (2.1) and the relation \( |a|_{n-1} \leq (|a|)_{n-1} \), we deduce that

\[
\sum_{n=2}^{\infty} \left[ n - (1+n\lambda - \lambda) \alpha \right] \left[ \frac{(b_n - 1)(c_n - 1)}{(a_n - 1)(c_n - 1)} a_n \right] 
\leq (A-B) \left[ (1-\lambda \alpha) \sum_{n=2}^{\infty} \left\{ \frac{(|a|_{n-1})(|b|_{n-1})}{(c_n - 1)(c_n - 1)} \right\} + (A-B) \left| \frac{(\lambda \alpha - \alpha)}{(|a|_{n-1})(|b|_{n-1})} \right| \sum_{n=2}^{\infty} \left\{ \frac{(|a|_{n-1})(|b|_{n-1})}{(c_n - 1)(c_n - 1)} \right\} \right.
\]

\[
\leq (A-B) \left| (1-\lambda \alpha)(A-B) \left[ F(|a|_{n-1},|b|_{n-1},c;1) - 1 \right] \right| + (A-B) \left| (\lambda \alpha - \alpha) \left[ F(|a|_{n-1},|b|_{n-1},c;1) - 1 \right] \right|
\]

here we use the relation

\[
(a)_{n} = a(a+1)_{n-1}.
\] (2.4)

The required result now follows by an application of Gauss summation theorem and (2.2)

\[
\sum_{n=2}^{\infty} \left[ n - (1+n\lambda - \lambda) \alpha \right] \left[ \frac{(b_n - 1)(c_n - 1)}{(a_n - 1)(c_n - 1)} a_n \right] \leq \alpha - 1.
\] (2.5)

**Theorem 2.3:**

Let Also, \( a,b \in \mathbb{C} \setminus \{0\} \). Also, let \( c \) be a real number and \( P_1 = P_1(k) \) be given by (1.5). If \( f \in k - UCV \) for some \( k \) (0 \( \leq k < \infty \)) and the inequality

\[
(1 - \lambda \alpha) 3^f 2[|a|,|b|,P_1; c; 1, 1] + (\lambda \alpha - \alpha) 3^f 2[|a|,|b|,P_1; c; 2, 1] \leq 2(\alpha - \lambda) - 1.
\] (2.6)

is satisfied, then \( I_{a,b,c}(f) \in \mathbb{M}(\lambda, \alpha) \).

**Proof.** Let \( f \) be given by (1.1). By (1.11), to show it is sufficient \( I_{a,b,c}(f) \in \mathbb{M}(\lambda, \alpha) \), to prove that
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\[ \sum_{n=2}^{\infty} \left[ n - (1 + n\lambda - \lambda)\alpha \right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \leq \alpha - 1 \]  \tag{2.7}

Applying the estimates for the coefficients given by (1.7) and making use of the relations (2.4) and \( \|a\|_n \leq (|a|)_n \), we get

\[ \sum_{n=2}^{\infty} \left[ n - (1 + n\lambda - \lambda)\alpha \right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \leq \sum_{n=2}^{\infty} \left[ n(1 - \lambda\alpha) + (\lambda\alpha - \alpha) \right] \frac{(|a|)_{n-1}(|b|)_{n-1}(R)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n}} \]

\[ = (1 - \lambda\alpha) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}(R)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n}} + (\lambda\alpha - \alpha) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}(R)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n}} \]

\[ = (1 - \lambda\alpha) \left[ 3^F 2(|a|, |b|, P_i; c, 1: 1) - 1 \right] + (\lambda\alpha - \alpha) \left[ 3^F 2(|a|, |b|, P_i; c, 2: 1) - 1 \right] \]

\[ \leq \alpha - 1, \]

provided the condition (2.6) is satisfied.

**Theorem 2.4.** Let \( a, b \in \mathbb{C} \setminus \{0\} \). Also, let \( c \) be a real number such that \( c > |a| + |b| + 1 \). If \( f \in R^\varphi(A, B) \) and if the inequality

\[ \frac{\Gamma(c)\Gamma(c - |a| - |b| - 1)}{\Gamma(c - |a|)\Gamma(c - |b|)} \left[ (1 - \lambda\alpha)|ab| - (\alpha - 1)(c - |a| - |b| - 1) \right] \]

\[ \leq (\alpha - 1) \left( \frac{1}{(A - B)|\tau|} - 1 \right). \]  \tag{2.8}

is satisfied, then \( I_{a,b,c}(f) \in \mathfrak{M}(\lambda, \alpha) \).

**Proof.** Let \( f \) be of the form (1.1) belong to the class \( R^\varphi(A, B) \). By virtue of Corollary(1.2),

it suffices to show that

\[ \sum_{n=2}^{\infty} n \left[ n - (1 + n\lambda - \lambda)\alpha \right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \leq 1 - \alpha. \]  \tag{2.9}

Taking into account the inequality (2.1) and the relation \( (a)_{n-1} \leq (|a|)_{n-1} \), we deduce that

\[ \sum_{n=2}^{\infty} n \left[ n - (1 + n\lambda - \lambda)\alpha \right] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \]
\[ (A - B)\|r\|(1 - \lambda \alpha) \sum_{n=2}^{\infty} \frac{|(a)n-1(b)n-1|}{(c)n-1(1)n-1} + (A - B)\|r\| \frac{(|a|)n-1(|b|)n-1}{(c)n-1(1)n} \]
\[
\leq (A - B)\|r\| \left\{ (1 - \lambda \alpha) \sum_{n=2}^{\infty} \frac{|(a)n-1(b)n-1|}{(c)n-1(1)n-1} - (\alpha - 1) \sum_{n=2}^{\infty} \frac{|(a)n-1(b)n-1|}{(c)n-1(1)n-1} \right\}
\]
\[
= (A - B)\|r\| \left\{ (1 - \lambda \alpha) \frac{|ab|}{c} F \left( 1 + |a|, 1 + |b|, 1 + c; 1 \right) - (\alpha - 1) \left( F\left(|a|, |b|, c; 1\right) - 1 \right) \right\}
\]
\[
= (A - B)\|r\| \left\{ (1 - \lambda \alpha) \frac{|ab|}{c} \frac{\Gamma(c-a-b-1)\Gamma(c+1)}{\Gamma(c-a)\Gamma(c-a)} - (A - B)\|r\| (\alpha - 1) \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \right\}
\]
\[
= (A - B)\|r\| \left\{ (1 - \lambda \alpha) \frac{|ab|}{c} \frac{\Gamma(c-a-b-1)\Gamma(c+1)}{\Gamma(c-a)\Gamma(c-a)} - (A - B)\|r\| (\alpha - 1) \right\}
\]
\[
\leq \alpha - 1,
\]
provided the condition (2.8) is satisfied.

**Theorem 2.5.** Let \(a, b \in \mathbb{C} \setminus \{0\}\). Also, let \(c\) be a real number and \(P_1 = P_1(k)\) be given by (1.5). If, for some \(k \in [0, \infty)\), \(f \in k - UCV\) and the inequality
\[
(1 - \lambda \alpha) \frac{|ab|}{c} 3^F_2 \left( 1 + |a|, 1 + |b|, 1 + P_1; 1 + c, 2; 1 \right)
\]
\[
+ (\lambda \alpha - \alpha) 3^F_2 \left( |a|, |b|, P_1; c, 1; 1 \right) \leq \alpha(2 - \lambda) - 1.
\]
is satisfied, then \(I_{a,b,c}(f) \in \mathcal{M}(\lambda, \alpha)\).

**Proof.** Let \(f\) be given by (1.1). By (1.11) to show that \(I_{a,b,c}(f) \in \mathcal{M}(\lambda, \alpha)\), it is sufficient to prove that
\[
\sum_{n=2}^{\infty} n\left(n - 1 + n(1 - \lambda) \lambda \right) \frac{|(a)n-1(b)n-1|}{(c)n-1(1)n-1} a_n \leq 1 - \alpha.
\]
(2.11)

Applying the estimates for the coefficients given by (1.6) and making use of the relations (2.4) and \(|a|_n \leq (|a|)_n\), we get
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\[ \sum_{n=2}^{\infty} n \left[ n - \left( 1 + n\lambda - \lambda \right) \alpha \right] \left[ \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \right] \]

\[ \leq \sum_{n=2}^{\infty} n \left[ n(1 - \lambda \alpha) + (\lambda \alpha - \alpha) \right] \left[ \frac{(\|a\|)_{n-1}(\|b\|)_{n-1}(P)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right] \]

\[ = (1 - \lambda\alpha) \sum_{n=2}^{\infty} \frac{|a|b||P}{c} \left[ \frac{1 + |a|_{n-2}(1 + |b|_{n-2}(1 + P_{1})_{n-2}}{(1 + c)_{n-2}(1)_{n-2}(2)_{n-2}} \right] \]

\[ - (\alpha - \lambda\alpha) \sum_{n=2}^{\infty} \frac{(\|a\|)_{n-1}(\|b\|)_{n-1}(P)_{n-1}}{(c)_{n-1}(1)_{n-1}} \]

\[ = (1 - \lambda\alpha) \frac{|a|b||P}{c} \left( 3 \frac{2(1 + |a| + |b| + P_{1}; 1 + c, 2; 1)}{1} \right) \]

\[ - (\alpha - \lambda\alpha) \left( 3 \frac{2(\|a\| + \|b\| + P_{1}; c, 1; 1)}{1} \right) \]

\[ \leq \alpha - 1. \]

provided the condition (2.10) is satisfied

**Theorem 2.6:**

Let \( a, b, c \in \mathbb{C} \setminus \{0\} \). Also, let \( c \) be a real number and \( P = P_{k}(k) \) be given by If \( f \in k - ST \), for some \( k(0 \leq k < \infty) \) and the inequality

\[ (1 - \lambda\alpha) \frac{|a|b||P}{c} \left( 3 F_{2}(1 + |a| + |b| + P_{1}; 1 + c, 2; 1) \right) + (2 - \lambda\alpha - \alpha) \frac{|a|b||P}{c} \left( 3 F_{2}(1 + |a| + |b| + P_{1}; c, 1; 1) \right) \]

\[ \leq 2(\alpha - 1), \]

is satisfied, then \( I_{a,b,c}(f) \in \mathfrak{N}(\lambda, \alpha) \).

**Proof:**

Let \( f \) be given by (1.1). We will repeat the method of proving used in the proof of Theorem (2.4). Applying the estimates for the coefficients given by (1.7), and making use of the relations (2.4) and \( |(\alpha)_{n}| \leq |(\|a\|)_{n}| \), we get

\[ \sum_{n=2}^{\infty} n \left[ n -(1 + n\lambda - \lambda \alpha) \alpha \right] \left[ \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \right] \leq \sum_{n=2}^{\infty} n \left[ n(1 - \lambda \alpha) + (\lambda \alpha - \alpha) \right] \]
\[
\frac{\Gamma(a)\Gamma(b)}{(a-1)(b-1)} = \sum_{n=2}^{\infty} \frac{(n-1)\Gamma(a-1)\Gamma(b)\Gamma(c-1)\Gamma(n)\Gamma(n+1)}{(n-1)(n-2)(n-2)(n-2)} + (1 - \lambda \alpha) + (1 - \alpha) \left( \frac{\Gamma(a)\Gamma(b)}{(a-1)(b-1)} \right)
\]

\[
\sum_{n=2}^{\infty} \frac{(n-1)(1-\lambda \alpha) + (1 - \alpha) \left( \frac{\Gamma(a)\Gamma(b)}{(a-1)(b-1)} \right)}{(c-n+1)(n-1)\Gamma(n)}
\]

\[
= (1 - \lambda \alpha) \sum_{n=2}^{\infty} \frac{\Gamma(a)\Gamma(b)\Gamma(c-n+1)}{(n-1)(n-2)(n-2)} - (\alpha - 1) \sum_{n=2}^{\infty} \frac{\Gamma(a)\Gamma(b)\Gamma(c-n+1)}{(n-1)(n-1)(n-1)}
\]

\[
+ (1 - \lambda \alpha) \sum_{n=2}^{\infty} \frac{\Gamma(a)\Gamma(b)\Gamma(c-n+1)}{(n-1)(n-2)(n-2)} - (\alpha - 1) \sum_{n=2}^{\infty} \frac{\Gamma(a)\Gamma(b)\Gamma(c-n+1)}{(n-1)(n-1)(n-1)}
\]

\[
= (1 - \lambda \alpha) \sum_{n=2}^{\infty} \frac{\Gamma(a)\Gamma(b)\Gamma(c-n+1)}{(n-1)(n-2)(n-2)} + (2 - \lambda \alpha - \alpha) \sum_{n=2}^{\infty} \frac{\Gamma(a)\Gamma(b)\Gamma(c-n+1)}{(n-1)(n-1)(n-1)}
\]

\[
= (1 - \lambda \alpha) \left\{ \frac{\Gamma(a)\Gamma(b)\Gamma(c-n+1)}{(n-1)(n-2)(n-2)} \right\} + (2 - \lambda \alpha - \alpha) \left\{ \frac{\Gamma(a)\Gamma(b)\Gamma(c-n+1)}{(n-1)(n-1)(n-1)} \right\}
\]

provided the condition (2.12) is satisfied.

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REFERENCES


