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Dominator Coloring of Sun Let, Gear and Helm Graph Families

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Abstract

A dominator coloring is a coloring of the vertices of a graph such that every vertex is either alone in its color class or adjacent to all vertices of atleast one other class. In this paper , we obtain the dominator chromatic number for the Sun let graph S_n, Middle graph of Sun let graph $M(S_n)$, Line graph of Sun let graph $L(S_n)$, Total graph of Sun let graph $T(S_n)$, Middle graph of Helm graph $M(H_n)$, Line graph of Helm graph $L(H_n)$, Middle Graph of Gear graph M(Gn), Line Graph of Gear Graph $L(G_n)$, Total Graph of Gear Graph $T(G_n)$.

Keywords: Coloring, Domination, Dominator Coloring, Middle, total, line Graph.

Mathematics subject Classification: *05C15, 05C69*

1. INTRODUCTION

All graphs considered here are finite, undirected, simple graphs. Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. The neighbors of a vertex $v \in V(G)$ are all the vertices u such that $uv \in E(G)$. Any vertex of G is said to dominate itself and all its neighbors. A set $D \subseteq V(G)$ is a dominating set if every vertex of $V(G) \setminus D$ has a neighbor in D.

A proper coloring of a graph G is a function from the set of vertices of a graph to a set of colors such that any two adjacent vertices have different colors. A subset of vertices colored with the same color is called a color class. The chromatic number is the minimum number of colors needed in a proper coloring of a graph and is denoted by χ (G).

A dominator coloring of a graph G is a proper coloring of graph such that every vertex of V dominates all vertices of at least one color class (possibly its own class). i.e., it is coloring of the vertices of a graph such that every vertex is either alone in its color class or adjacent to all vertices of at least one other class and this concept was introduced by Ralucca Michelle Gera in 2006 [4]. The dominator chromatic number χ_d (G) is the minimum number of color classes in a dominator coloring of G. The relation between dominator chromatic number, chromatic number and domination number of some classes of graphs were studied in [7], [5]. The dominator coloring of bipartite graph, star and double Star graphs, Central and Middle graphs, Fan, Double Fan, Helm graphs etc. were also studied in various papers [6], [1], [2], [3].

2. PRELIMINARIES

The Middle graph M(G) of a graph G is defined as follows. The vertex set of M(G) is V(G) $\cup$E(G). Two vertices x, y in the vertex set of M(G) are adjacent in M(G) if either (i) x, y are in E(G) and x, y are adjacent in G or (ii) x is in V(G), y is in E(G) and x, y are incident in G. In other words, M(G) is obtained by subdividing each edge of G exactly once and joining all these newly added middle vertices of adjacent edges of G.

The Total graph T(G) of a graph G is defined as a graph with vertex set V(G)$\cup$ E(G) and two vertices x, y of T(G) are adjacent in T(G) if either (i) x, y are in V(G) and x is adjacent to y in G or (ii) x, y are in E(G) and x, y are adjacent in G or (iii) x is in V(G), y is in E(G) and x, y are incident in G.

A dominator coloring of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. The convention is that if {v} is a color class, then v dominates the color class {v}. The dominator chromatic number χ_d(G) is the minimum number of colors required for a dominator coloring of G.

3. DOMINATOR CHROMATIC NUMBER OF SUN LET HELM AND GEAR GRAPH FAMILIES

THEOREM 3.1. For any $n \geq 4$, $\chi_d (M(S_n)) = n+3$.

PROOF.

Let $V(S_n) = \{v_1,v_2,v_3,\ldots,v_n\} \cup \{u_1,u_2,u_3,\ldots,u_n\}$ and $E(S_n) = \{e'_i: 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$ where e_i is the edge v_iv_{i+1} $(1 \leq i\leq n-1)$, e_n is the edge v_nv_1 and e_i' is the edge $v_i u_i$ $(1 \leq i \leq n)$. by the definition of middle graph $V(M(S_n)) = V(S_n) \cup E(S_n) = \{v_i : 1 \leq i$

$\leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u'_i : 1 \leq i \leq n\}$, where v'_i and u'_i represent the edge e_i and e'_i $(1 \leq i \leq n)$ respectively.

Let u'_i be colored by color i, $1 \leq i \leq n$. since u_i and u'_i are adjacent a new color n+1 is assigned to u_i. assign n+1 color to v_i, $2 \leq i \leq n-1$, v_1 is colored by n+3.$\langle v'_i \rangle$ contains a clique and also adjacent to u'_i and v_i so that we assign a new color to v'_i for proper dominator coloring. Let the vertex v'_i i=1,3,5,..n-1 colored by color n+2, v'_i, i=2,4,6,…n-1 is colored by color n+3. Assign n+1 color to v'_n. If v_n is odd assign n+2 colors to v_n, then n+3 to v_n.

In $(M(S_n))$, u_i, v_i, v'_i are adjacent to u'_i, the vertex u_i, v_i, v'_i dominate the color class of u'_i, $1 \leq i \leq n$, u'_i dominate themselves. Therefore, $\chi_d (M(S_n)) \leq n+3$.

To prove $\chi_d (M(S_n)) \geq n+3$. Let us assume that $\chi_d (M(S_n))$ is less than n+3 i.e.,

$\chi_d (M(S_n)) = n+2$. We must assign n+2 colors for $\{v'_i, v_i, u'_i: 1 \leq i \leq n\}$for dominator coloring, if we assign n+2 colors then an easy check shows that one of those vertices are bi colored. dominator coloring with n+2 color is not possible. Thus, $\chi_d (M(S_n)) \geq n+3$.

Hence, $\chi_d (M(S_n)) = n+3$.

THEOREM 3.2: For any $n \geq 5$, $\chi_d (L(S_n)) = \begin{cases} \lceil n/2 \rceil +2 \text{ if n is odd} \\ [n/2] +2 \text{ if n is even} \end{cases}$

PROOF.

Let $V(S_n) = \{v_1, v_2, v_3, \ldots, v_n\} \cup \{u_1, u_2, u_3, \ldots, u_n\}$ and $E(S_n) = \{e'_i: 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n-1\} \cup \{e_n\}$ where e_i is the edge $v_i v_{i+1}$ $(1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i$ $(1 \leq i \leq n)$. by the definition of line graph $V(L(S_n)) = E(S_n) = \{ u'_i : 1 \leq i \leq n) \cup \{v'_i : 1 \leq i \leq n-1\} U \{v'_n\}$ where v'_i and u'_i represents the edge e_i and e'_i $(1 \leq i \leq n)$ respectively.

Assign the following coloring for $L(S_n)$ as dominator chromatic:

i. For $1 \leq i \leq n$, assign the color i to odd vertices of v'_i.
ii. If n is odd
    Assign $\lceil n/2 \rceil$ +1 color to v'_i,i =2,4,6,…, and $\lceil n/2 \rceil$ +2 color to u'_i, $1 \leq i \leq n$.
iii. If n is even Assign [n/2] +1 color v'_i ,i = 2,4,6,…, and [n/2] +2 color to u'_i, $1 \leq i \leq n$.

v'_i,i=1,3,5,7,… dominate themselves .If n is even, even vertices of v'_i, $2 \leq i \leq n-4$ dominate the color class i+1,& v_{n-1} dominate the color class 1. If n is odd, even vertex of v'_i dominate the color class i+1, i = 2,4,6,…, u_i and u_{i+1}, i = 2, 4, 6, …, dominate the color class i+1 and u_1 dominate the color class 1.

$$\text{Hence } \chi_d(L(S_n)) = \begin{cases} \lceil n/2 \rceil + 2 \text{ if n is} \\ [n/2] + 2 \text{ if n is even} \end{cases}$$

THEOREM 3.3: For any $n \geq 5$, $\chi_d(T(S_n)) = n+3$

Proof.

By the definition of total graph $V(T(S_n)) = V(S_n) \cup E(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u'_i : 1 \leq i \leq n\}$, where v'_i and u'_i represent the edge e_i and $e'_i (1 \leq i \leq n)$ respectively.

The following procedure gives a dominator coloring of $(T(S_n))$. $(T(S_n))$, contains independent set u'_i, u_i,v_i are adjacent to u'_i. Let the Vertex v_i, is colored by color i, $1 \leq i \leq n$.

(i) if n is even the following procedure holds.

Assign n +1 color to u_i and n+2 color to u'_i .v'_i , i=1,3,5,.. are colored by color n+ 1 and v'_i, i =2,4,6,… colored by color n +3.

(ii) If n is odd the following procedure holds.

Assign n + 1 color to u_i, $1 \leq i \leq n-1$, n+2 and n+ 3 colors to u_n & u'_{n-1} . Assign n +1 color to u'_n. u'_i, $1 \leq i \leq n-2$ colored by color n+2. Assign n+1 color to v'_i, i=1,3,5,…n-2, and v'_i, i=2,4,6,…n-3 colored by color n+3. v'_n & v'_{n-1} to n +3 & n +2 colors.

u_i, u'_i are adjacent to v_i, u_i, u'_i dominate the color class of v_i, $1 \leq i \leq n$. v_i, $1 \leq i \leq n-1$ dominate the color class of v_{i+1} , v_n dominate the color class v_1. V'_i, $1 \leq i \leq n-1$ dominate the color class i+1, v'^{1}_{n} dominate the color class v_1. Therefore, $\chi_d(T(S_n)) \leq n+3$.

To prove $\chi_d(T(S_n)) \geq n+3$. Let us assume that $\chi_d(T(S_n))$ is less than n+3 i.e.,

$\chi_d(T(S_n)) = n+2$. We must assign n+2 colors for $\{v'_i, v_i, u'_i: 1 \leq i \leq n\}$for dominator coloring, if we assign n+2 colors to another then an easy check shows that one of those vertices are bi colored. dominator coloring with n+2 color is not possible. Thus, $\chi_d(T(S_n)) \geq n+3$.

Hence, $\chi_d(T(S_n)) = n+3$.

Theorem 3.4. For any **$n \geq 4$**, $\chi_d(M(H_n)) = 2n+1$

Proof.

Let $V(H_n) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$ where e_i is the edge vv_i $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \leq i \leq n)$. By the definition of middle graph $V(M(H_n)) = \{v\} \cup V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

Let s_i be colored by color i. As $M(H_n)$ contains complete graph $\langle e_1,e_2,e_3,e_4,..\rangle$,assign i+n color to e_i. the vertex u_i,v_i,e_i,e'_i are adjacent to s_i, a new color 2n+1 is assigned to u_i,v,e'_1. v_i is adjacent to e_i,s_i a new color 2n is assigned to v_i, $1 \leq i \leq n-1$, v_n is colored by color 2n-1.,

(i) if n is odd

Assign n+1,n+2 color consecutively to e'_i, $2 \leq i \leq n$.

(ii) if n is even

Assign n+1, n+2 color to e'_i, $2 \leq i \leq n-1$, and n+3 color to e'_n

The vertices u_i, e_i,v_i and e'_i are adjacent to s_i. u_i, e_i,v_i,e'_{is} Dominate the color class of s_i,$1 \leq i \leq n$. And the vertex v dominates at least one of the color classes of e_i. Each vertex of s_i dominate themselves.

Hence $\chi_d(M(H_n)) \leq 2n+1$

To prove $\chi_d(M(H_n)) \geq 2n+1$. Let us assume that $\chi_d(M(H_n))$ is less than 2n+1 i.e.,

$\chi_d(M(H_n)) = 2n$. We must assign 2n colors for $\{e'_i, v_i, s_i: 1 \leq i \leq n\}$for dominator coloring, an easy check shows that one of these vertices are bi colored., dominator coloring with 2n color is not possible. Thus, $\chi_d(M(H_n)) \geq 2n+1$.

Hence, $\chi_d(M(H_n)) = 2n+1$.

Theorem 3.5: For any $n \geq 5$, $\chi_d(L(H_n)) = n+3$

Proof:

Let $V(H_n) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i: 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$ where e_i is the edge vv_i $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \leq i \leq n)$. By the definition of line graph $V(L(H_n)) = E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

Let e_i is colored by color i, $1 \leq i \leq n$. As $L(H_n)$ contain complete graphs $\langle e_1,e_2,\ldots,e_n\rangle$. s_i,e_i^1 is adjacent to e_i. Since s_i,e_i^1 are independent, a new color n+3 is assigned to s_i, $2 \leq i \leq n$. the vertex s_1 is colored by n+2. if n is even assign n+1 & n+2 color consecutively to e_i^1. If n is odd the vertex e_i^1, $1 \leq i \leq n-1$ with color n+1 &n+2 alternatively. Assign n+3 color to e_n^1. The vertices s_i and e_i^1 are adjacent , e_i^1, s_i dominate the color classes of e_i , $1 \leq i \leq n$. e_i dominate themselves . Hence $\chi_d(L(H_n)) =$ n +3.

To prove $\chi_d(L(H_n)) \geq n+3$. Let us assume that $\chi_d(L(H_n))$ is less than n+3 i.e.,

$\chi_d(L(H_n)) = n+2$. We must assign n+2 colors for $\{e_i^1, e_i, s_i: 1 \leq i \leq n\}$for dominator coloring, since (e_1^1, e_1, s_1) forms a clique of order n+2, if we assign n+2 colors to another clique (e_2^1, e_2, s_2) then an easy check shows that one of those cliques is bi colored., dominator coloring with n+2 color is not possible. Thus, $\chi_d(L(H_n)) \geq n+2$.

Hence, $\chi_d(L(H_n)) = n+3$.

Theorem 3.6: For any $n \geq 3$ $\chi_d(M(G_n)) = 2n+1$

Proof:

Let $V(G_n) = \{v\} \cup \{v_1, v_2, \ldots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n - 1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq 2n - 1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of middle graph $V(M(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$

A procedure to obtain a dominator coloring of $M(G_n)$ is as follows. Notice that $M(G_n)$ contains a clique $<e_i>$. assign c_i color to e_i $(1 \leq i \leq n)$. v is adjacent to e_i since a new color c_{2n+1} is assigned to v. assign c_i (i=1,2,3,…,) color to e'_i since v_i $(1 \leq i \leq n)$. is colored by color c_{2n+1}. $M(G_n)$ contains a clique $<e_i>$. v_i and e_i are adjacent to e'_i assign c_{i+n} color to e_i $(1 \leq i \leq n)$. assign c_{n+1} color to e'_i (i= 2,4,6,…,n-1).

V_i v_{i+1} dominate the color class of e'_i (i= 1,3,5,…,) . v and e_i dominate the color class of e_n. Each vertex of e'_i (i= 1,3,5,…,) dominate itself hence $\chi_d(M(G_n)) = 2n+1$

Theorem 3.7: For any n $\chi_d(L(G_n)) = n+2$

Proof.

Let $V(G_n) = \{v\} \cup \{v_1, v_2, \ldots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n - 1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq 2n - 1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of line graph $V(L(G_n)) = E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$.

For $1 \leq i \leq n$ assign c_i color to e'_i. and assign c_{n+1} and c_{n+2} colors alternatively to e''_i.

e''_i. e''_{i+1}, where (i =1,3,5,…) dominate the color class of e'_i (i=1,2,3,…). Every vertex of e'_i dominate at least one color class of e'_i. hence $\chi_d(L(G_n)) = n+2$.

Theorem 3.8: For any n χ_d (T(G_n)) = 2n+1

Let V (G_n) = {v} ∪ {v_1, v_2, . . . , v_{2n}} and E(G_n) = {e_i : $1 \le i \le n$} ∪ {e'_i : $1 \le i \le 2n - 1$} ∪ {e'_n} where e_i is the edge vv_{2i-1} ($1 \le i \le n$), e'_i is the edge v_iv_{i+1} ($1 \le i \le 2n - 1$), and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of middle graph V (M(G_n)) = V (G_n) ∪ E(G_n) = {v} ∪ {v_i : $1 \le i \le 2n$} ∪ {e_i : $1 \le i \le n$} ∪ {e'_i : $1 \le i \le 2n$}

for $1 \le i \le n$ assign c_i color to all odd vertices of v_i (i=1,3,5,…) and all even vertices of v_i are colored by c_{n+1}. Assign C_{i+n} color to e_i. Assign c_{2n+1} color to v. all even vertices of e'_i colored with c_{2n+1} and odd vertices of e'_i are colored with c_{2n}.and the n^{th} vertex of e'_n is colored by c_{n+2}.

All vertices of ei, e'_i and v dominate the color class of v_i (vi, i= 1,3,5,…). And even vertices of v_i dominate the color class of (v_i, i= 1,3,5,…). all odd vertices of v_i dominate itself. Hence χ_d (T(G_n)) = 2n+1

CONCLUSION

(i) χ_d (M(S_n)) = n+3, $n \ge 4$,

(ii) χ_d (L(S_n)) = $\begin{cases} \lceil n/2 \rceil +2 \text{ if n is odd} \\ [n/2] +2 \text{ if n is even} \end{cases}$

(iii) χ_d (T(S_n)) = n+3, $n \ge 5$

(iv) χ_d (M(H_n)) = 2n+1, $n \ge 4$

(v) χ_d (L(H_n)) = n+3

(vi) χ_d (T(G_n)) = 2n+1

(vii) χ_d (L(G_n)) = n+2

(viii) χ_d (M(G_n)) = 2n+1

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