Exponential Series Method for the Solution of MHD Boundary Layer Flow over Permeable Shrinking Surface

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Abstract

The paper presents the semi-numerical solution of steady forced convection boundary layer flow past a permeable shrinking surface in a viscous and electrically conducting fluid. The governing partial differential equation of momentum equations are reduced into an ordinary differential equation (ODE) by using a classical similarity transformation along with appropriate boundary conditions. Both nonlinearity and infinite interval demand novel mathematical tools for their analysis. We use fast converging Exponential series (Dirichlet series) for the solution of these nonlinear differential equations. These methods have the advantages over pure numerical methods for obtaining the derived quantities accurately for various values of the parameters involved at a stretch and also they are valid in much larger parameter domain as compared with the classical numerical schemes.

Keywords: Magnetohydrodynamic (MHD); Boundary layer flow; shrinking sheet; Dirichlet series; Powell’s method.

1. INTRODUCTION

The study of MHD boundary layer flow over a permeable shrinking surface in a viscous and electrically conducting fluid has many important applications in engineering and science. The boundary layer problem due to a stretching sheet has relevance to extrusion problem and has received considerable interest. Such flows
have many important applications in industrial manufacturing process, the problem of the flow due to stretching/shrinking sheet has attracted the attention of many researchers, the subject of interest in the literature (Crane [1], Banks [2], Grubka and Bobba [3], Magyari and Keller [4], Lio and Pop [5] etc.). However, works on the flow problem due to shrinking sheet are scarce. The unsteady viscous flow induced by shrinking sheet was first studied by Wang [6]. Mikalavcic and Wang [7] discussed the viscous hydrodynamic flow due to shrinking surface for specific value of the suction parameter and concluded that the solution for shrinking sheet may not be unique at certain suction rates for both two-dimensional and axi-symmetric flows. Sajid and Hayat [8] studied the MHD viscous flow due to a shrinking sheet for the case of two-dimensional and axi-symmetric shrinking. Sajid et al. [9] discussed the MHD rotating flow of a viscous fluid over a shrinking sheet and showed that the stable and convergent solutions are possible for MHD flows. Some applications of shrinking sheet problems are hot rolling, paper production, metal spinning, drawing plastic films, glass blowing, continuous casting of metals and spinning of fibres etc. During the manufacture of these sheets, the melt issues from a slit and is subsequently stretched to achieve the desired thickness. The mechanical properties of the final product strictly depend on the stretching and cooling rates in the process. The phenomena of velocities on the boundary towards a fixed point are known as shrinking phenomena, which often occur in the situations such as rising shrinking balloon. Only limited attention has been focused on the study of shrinking phenomena [1-9]. In certain situations, the shrinking sheet solutions do not exist, since the velocity cannot be confined in a boundary layer. These solutions may exist if either the magnetic field or the stagnation flow is taken into account. Recently, Wang [10] studied the stagnation flow towards a shrinking sheet and found that non-alignment of the stagnation flow and the shrinking of the sheet destroys symmetry and complicates the flow. The interest has been extended to the problem of flow and heat transfer over a shrinking surface.

The present investigation is to analyse the MHD viscous flow caused by a shrinking sheet. The solution of the resulting third order nonlinear boundary value problem with an infinite interval is obtained using Dirichlet series method. The Dirichlet series solution; necessary conditions for the existence and uniqueness of these solutions may also be found in [11, 12]. For a specific type of boundary conditions i.e. \( f'(\infty) = 0 \), the Dirichlet series solution is particularly useful for obtaining solution and the derived quantities exactly. A general discussion of the convergence of the Dirichlet series may also be found in Riesz [13]. The accuracy as well as uniqueness of the solution can be confirmed using other powerful semi-numerical schemes. Sachdev et al. [14] have analysed various problems from fluid dynamics of stretching sheet using this approach and found more accurate solution compared with earlier numerical findings. Recently, Vishwanath et al [15, 16] and Ramesh et al [17] have analysed the problems from MHD boundary layer flow with nonlinear stretching sheet using these methods and found more accurate results compared with the classical numerical methods. Vishwanath and Bujurke [18] discussed the approximate analytical solutions of MHD flow of a viscous fluid on a nonlinear porous shrinking sheet. In this article, we present Dirichlet series solution and an approximate analytical method-method of
stretching of variables. This method is quite easy to use especially for nonlinear ordinary differential equations and requires less computer time compared with pure numerical methods and easy to solve compared with other approximate methods (for example, Homotopy perturbation method (HPM) Padé’ technique, Adomain decomposition methods (ADM) etc.).

The present paper is structured as follows. In section 2 the mathematical formulation of the proposed problem with relevant boundary conditions is given. Section 3 is devoted to the solution of the problem using Dirichlet series. In Section 4 detailed results obtained are compared with the corresponding numerical schemes and Section 5 is about the conclusion.

2. MATHEMATICAL FORMULATION OF THE PROBLEM
Consider the two-dimensional viscous flow and electrically conducting fluid over a permeable shrinking surface coinciding with the plane \( y = 0 \), the flow being confined to \( y > 0 \), \( y \) is the coordinate measured in the normal direction to the surface of the sheet. Let us assume that the velocity distribution of the shrinking surface is \( u_w(x) = \lambda U_w(x) \) where \( x \) is the coordinate measured along the shrinking surface and \( \lambda < 0 \) is the parameter related to the shrinking surface speed. Also it is assumed that the mass flux velocity is \( v_w(x) \) is negative for suction and positive injection or withdrawal of the fluid respectively. Further it is assumed that an external variable magnetic field \( B(x) \) is applied normal to the plate. The above conditions along with Boussinesq approximation the governing equation of the problem becomes (see Pop and Ingham[19]).

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{2.1}
\]

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2(x)}{\rho} u \tag{2.2}
\]

The relevant boundary conditions for the present flow are

\[
v = v_w(x), \quad u = u_w(x) = \lambda U_w(x) = \lambda a(x+b)^\alpha, \quad \text{at } y = 0 \tag{2.3}
\]

\[
u \to 0 \quad \text{as } y \to \infty \tag{2.4}
\]

where \( u \) and \( v \) are the velocity components along \( x \) and \( y \) axes, \( \nu = \mu/\rho \) is the kinematic viscosity, (i.e. \( \mu \) is the dynamic viscosity \( \rho \) is the fluid density) and \( \sigma \) is the electrical conductivity of the fluid and \( a,b \) and \( \alpha \) are constants with \( a > 0 \). We use following similarity variables
\[ \psi = \sqrt{\frac{v}{a(1+\alpha)}} (x+b)^{\frac{(\alpha+1)}{2}} f(\eta), \quad \eta = \sqrt{\frac{a(1+\alpha)}{v}} (x+b)^{\frac{(\alpha-1)}{2}} y \] (2.5)

where \( a \neq 0, a(1+\alpha) > 0 \) and \( \psi \) is the stream function, which is defined as \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \). Thus, we have

\[ u = a(x+b)^{\alpha} f'(\eta), \quad v = -\frac{1}{2} \sqrt{a(1+\alpha)v(x+b)^{\frac{(\alpha-1)}{2}}} \left[ f(\eta) + \frac{\alpha-1}{\alpha+1} \eta f'(\eta) \right] \] (2.6)

Thus, the Eqs. (2.1) and (2.2) admits a similarity solution, we take

\[ v_w(x) = -\frac{1}{2} \sqrt{a(1+\alpha)v(x+b)^{\frac{(\alpha-1)}{2}}} f_w, \quad B(x) = B_0(x+b)^{\frac{(\alpha-1)}{2}} \] (2.7)

where \( B_0 \) is the constant applied magnetic field and \( f_w \) is the suction /injection parameter according as \( f_w > 0 \) and \( f_w < 0 \), respectively. Substituting Eqs. (2.5-2.7) into Eqs.(2.1-2.4) reduce to the following third order nonlinear ordinary differential equation (Rosca [20]):

\[ f'''' + \frac{1}{2} f''f' - \beta f'^{2} - Mf' = 0 \quad \frac{d}{d\eta}, \] (2.8)

and the boundary conditions are

\[ f(0) = f_w, \quad f'(0) = \lambda, \quad f'(\infty) = 0 \] (2.9)

where \( \beta = \frac{\alpha}{(1+\alpha)} \) is the dimensionless parameter and \( M = \frac{\sigma B_0^2}{\rho a(1+\alpha)} \) is the magnetic field parameter. The physical quantity is the skin friction coefficients \( C_f \), which is defined as

\[ C_f = \frac{\tau_w}{\rho U_w(x)} \quad \text{and} \quad \tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} \] (2.11)

where \( \rho \) is the density of the fluid, \( \tau_w \) is the skin friction or shear stress along the shrinking surface and \( \mu \) is the dynamic viscosity of the fluid. Using (2.6) and (2.11), we get

\[ Re^{1/2} C_f = \sqrt{T-a} f''(0) \] (2.12)

where \( Re = U_w(x)(x+b)/\nu \) is the local Reynolds number.
3. DIRICHLET SERIES SOLUTION

We use Dirichlet series which is an elegant semi-numerical scheme, to solve the problem exactly. We seek Dirichlet series solution of Eq. (2.8) satisfying last boundary condition $f'(\infty)=0$ automatically in the form (Kravchenko & Yablonskii [11, 12])

$$f = \gamma_1 + \frac{6\gamma}{(1/2)} \sum_{i=1}^{\infty} b_i a^i e^{-i\eta}$$  \hspace{1cm} (3.1)

where $\gamma$ and $a$ are parameters which are to be determined. Substituting Eq. (3.1) into Eq. (2.8), we get

$$\sum_{i=1}^{\infty} \left\{ -\gamma^2 i^3 + \frac{1}{2} \gamma \eta i^2 + Mi \right\} b_i a^i e^{-i\eta} + \frac{6\gamma^2}{(1/2)} \sum_{k=1}^{\infty} \left\{ \frac{1}{2} k^2 - \beta k (i-k) \right\} b_{i+k} a^i e^{-i\eta} = 0$$  \hspace{1cm} (3.2)

For $i=1$, we have $\gamma_1 = \frac{\gamma^2 - M}{(1/2)}$.  \hspace{1cm} (3.3)

Substituting Eq.(3.3) into Eq. (3.2) the recurrence relation for obtaining coefficients is given by

$$b_i = \frac{6\gamma^2}{(1/2)i(i-1)} \left\{ \gamma^2 i + M \right\} \sum_{k=1}^{i-1} \left\{ \frac{1}{2} k^2 - \beta k (i-k) \right\} b_k b_{i-k}$$  \hspace{1cm} (3.4)

For $i=2, 3, 4, \ldots$. If the Eq. (3.1) converges absolutely when $\gamma > 0$ for some $\eta_0$, this series converges absolutely and uniformly in the half plane $\text{Re} \eta \geq \text{Re} \eta_0$ and represents an analytic $\frac{2\pi}{\gamma}$ periodic function $f = f(\eta)$ such that $f'(\infty) = 0$ ([11]).

The Eq. (3.1) contains two free parameters namely $a$ and $\gamma$. These unknown parameters are determined from the remaining boundary conditions of Eq. (1.2) at $\eta = 0$

$$f(0) = \frac{\gamma^2 - M}{(1/2)\gamma} + \frac{6\gamma}{(1/2)} \sum_{i=1}^{\infty} b_i a^i = \alpha_i$$  \hspace{1cm} (3.5)

and

$$f'(0) = \frac{6\gamma^2}{(1/2)} \sum_{i=1}^{\infty} (-i) b_i a^i = \beta_i$$  \hspace{1cm} (3.6)
The solution of the above transcendental Eq. (3.5) and (3.6) yield constants $a$ and $\gamma$. The solution of the above transcendental equations is equivalent to the unconstrained minimization of the functional

$$
\left[ \gamma^2 - M \left(1/2\right)^\gamma + \frac{6\gamma}{\left(1/2\right)^{\gamma}} \sum_{i=1}^{\infty} b_i a_i' - \alpha_i \right]^2 + \left[ \frac{6\gamma^2}{\left(1/2\right)^{\gamma}} \sum_{i=1}^{\infty} (-i)b_i a_i' - \beta_i \right]^2
$$

(3.7)

We use Powell’s method of conjugate directions (Press et al [21]) which is one of the most efficient techniques for solving unconstrained optimization problems. This helps in finding the unknown parameters $a$ and $\gamma$ uniquely for different values of the parameters $\beta$, $M$ and $f_w$. Alternatively, Newton’s method is also used to determine the unknown parameters $a$ and $\gamma$ accurately. The physical quantity of interest is the skin friction or shear stress at the surface and velocity profiles. The shear stress at the surface of the problem is given by

$$
f^\prime(0) = \frac{6\gamma}{\left(1/2\right)^{\gamma}} \sum_{i=1}^{\infty} b_i a_i' (i\gamma)^2.
$$

(3.8)

The velocity profiles of the problem is given by

$$
f'(\eta) = \frac{6\gamma^2}{\left(1/2\right)^{\gamma}} \sum_{i=1}^{\infty} (-i)b_i a_i e^{-i\eta}.
$$

(3.9)

4. RESULTS AND DISCUSSION

The third order nonlinear boundary value problems with infinite intervals are solved semi-numerically using one of most powerful technique i.e. Dirichlet series method. In this method it is important that the edge boundary layer $\eta \to \infty$ automatically satisfied. Numerical computations are performed for various values of the physical parameters involved in the equation viz. the dimensionless parameter $\beta$, magnetic field parameter $M$, the mass suction parameter $f_w$ and the stretching/shrinking parameter $\lambda$. The present solution is also validated by comparing it with the previously published work of Rosca [20] and Sakiadis [22] as shown in Tables 1.
Table 1: Comparison of the values of for several values of the similarity variable $\eta$ with the results of Sakiadis [21], Rosca [19], when $\lambda = 1$, $\beta = 0$, $M = 0$ and $f_w = 0$.

<table>
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<tr>
<th>$\eta$</th>
<th>Dirichlet Series</th>
<th>Rosca [19]</th>
<th>Sakiadis [21]</th>
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<tr>
<td></td>
<td>$a$</td>
<td>$\gamma$</td>
<td>$-f^*(\eta)$</td>
</tr>
<tr>
<td>0</td>
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<td>0.80806</td>
<td>0.4437437</td>
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The figures 1 to 3 for the function $f'(\eta)$ i.e. dimensionless velocity profiles which corresponds to velocity component $u$ and $v$ are drawn against $\eta$ for several values of the parameters $\beta$, $M$, $f_w$ and $\lambda$ which match very well with that of earlier findings depicted in their figures. It can be seen that these figures which shows that the far field boundary condition $f'(\eta)$ as $\eta \to \infty$ is satisfied asymptotically. Fig. 1, demonstrate the effect of suction parameter $f_w$ and magnetic parameter $M$ is to decrease the velocity profiles for increasing the parameter $\beta$. Fig. 2, shows the effect of the suction parameter $f_w$ and parameter $\beta$ is to increase the velocity profiles for increasing the magnetic parameter $M$. In Fig. 3, which demonstrate the effects magnetic parameter $M$ and parameter $\beta$ is to increase the velocity profiles for increasing the suction parameter $f_w$. The above computation works very well by using Dirichlet series. It is also susceptible to the computer’s memory limitations and it takes very less computer memory. In this work we use FORTRAN compiler running on a personal computer with Pentium processor.
Figure 1: Dimensionless velocity profiles $f'(\eta)$ for several Values of $\beta$.

Figure 2: Dimensionless velocity profiles $f'(\eta)$ for several Values of $M$. 
CONCLUSIONS
In this article, we describe the effect of the magnetic field on the boundary value problem for third order nonlinear ordinary differential equation over an infinite interval arising in the steady boundary-layer flow fast a permeable shrinking surface. The semi-numerical schemes described here offer advantages over solutions obtained by HAM, HPM and numerical methods etc. The convergence of the Dirichlet series method is given. The results are presented in Tables and graphically, the effects of the emerging parameters are discussed.

REFERENCES