Third Hankel Determinant for Starlike Function with respect to Symmetric Points

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Abstract

The objective of this paper is to obtain best possible upper bound to the \( H_3(1) \) Hankel determinant for starlike function with respect to symmetric points using Toeplitz determinant.

AMS subject classification:

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1. Introduction, Definition and Motivation

Let \( A \) denotes the class of function \( f \) of the form,

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]  (1.1)
in the open unit disk $E = \{ Z : |z| < 1 \}$. Let $S$ be the subclass of $A$ consisting of univalent functions. For any two analytic function $g$ and $h$ respectively with their expansions as,

$$
g(z) = \sum_{k=0}^{\infty} a_k z^k \\
h(z) = \sum_{k=0}^{\infty} b_k z^k$$

(1.2)

the Hadamard product or convolution of $g(z)$ and $h(z)$ is defined as the power series,

$$(g * h)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$$

(1.3)

The Hankel determinant of $f$ for $q \leq 1$ and $n \leq 1$ was defined by Pommerenke [1] as,

$$H_q(n) = \left| \begin{array}{cccc}
a_n & a_{n+1} & \cdots & a_{n+q+1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \\
\end{array} \right| (a_1 = 1)$$

(1.4)

One can easily observe that the Fekete-Szego functional is $H_2(1)$. Fekete-Szego then further generalized the estimate $|a_3 - \mu a_2^2|$ with $\mu$ real and $f \in S$. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for Fekete-Szego functional $|\gamma - \tau \gamma_2^2|$, where $\tau$ is real, for the inverse function of $f$ defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$.

When $f \in ST(\alpha)$, the class of strongly starlike functions of order $\alpha(0 < \alpha \leq 1)$. Further, sharp bounds for the functional

$$H_2(2) \left| \begin{array}{ccc}
a_2 & a_3 & a_4 \\
a_3 & a_4 & a_5 \\
\end{array} \right| = |a_2 a_4 - a_3^2|$$

(1.5)

when $q = 2$ and $n = 2$, known as the second Hankel determinant, were obtained from various subclasses of univalent and multivalent analytic functions. For our discussion, in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \left| \begin{array}{ccc}
a_1 & a_2 & a_3 \\
a_2 & a_3 & a_4 \\
a_3 & a_4 & a_5 \\
\end{array} \right|$$

(1.6)

For $f \in A$, $a_1 = 1$, so that we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

(1.7)
and by applying triangle inequality, we obtain

$$H_3(1) \leq |a_3||a_2a_4 - a_2^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_5^2|$$  \hspace{1cm} (1.8)

Babalola [3] obtained sharp upper bounds to the functional $|a_2a_3 - a_4|$ and $H_3(1)$ for familiar subclass namely starlike and convex functions respectively denoted by ST and CV of S. The sharp bounds to the second Hankel determinant $|a_2a_4 - a_2^2|$ for the classes ST and CV were obtained by Janteng et al. [7].

Motivated by the result obtained by D. Vamshee Krishna et al. [1], Babalola [3] and Raja & Malik [11] in finding the sharp upper bound to the Hankel determinant $H_3(1)$ for certain subclass of S.

We obtain an upper bound to the functional $|a_2a_3 - a_4|$ and hence $|H_3(1)|$ for the function $f$ given in (1.1) belonging to the class namely starlike with respect to symmetric points denoted by $S^*_c(\delta)$ and defined as follows,

**Definition 1.1.** A function $f(z) \in A$ is said to be in the class $S^*_c(\delta)$ if it satisfies the condition,

$$\Re \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > \delta, \quad \forall z \in E$$  \hspace{1cm} (1.9)

The class $S^*_c(\delta)$ was introduced and studied by Sakaguchi [15].

2. Preliminary Results

**Lemma 2.1.** If the function $\rho \in p$ is given by the series,

$$p(z) = 1 + c_1z + c_2z^2 + \cdots$$

then the following sharp estimate holds,

$$|p_k| \leq 2 \quad k = 1, 2, \ldots$$

**Lemma 2.2.** If the function $\rho \in p$ is given by the series then,

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some $x, z, |x| \leq 1, |z| \leq 1$.

**Lemma 2.3.** The power series of $p$ given in (2.1) converges in $\delta$ in to function $p$ if and only if the Toeplitz determinant

$$D_n = \begin{vmatrix} 2 & C_1 & C_2 & \cdots & C_n \\ C - 1 & 2 & C_1 & \cdots & C_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C - n & C - n + 1 & \cdots & \cdots & 2 \end{vmatrix}$$
Where, \( n = 1, 2, 3, \ldots \) & \( C_k = \bar{C}_k \) ∀ non-negative. They are strictly positive except for

\[
p(z) = \sum_{k=1}^{m} p_k p_0(e^{i\kappa z})
\]

\( p_k > 0, t_k \) real and \( t_k \neq t_j \) for \( k \neq j \) in this case \( D_n > 0 \) for \( n < m - 1 \) & \( D_n = 0 \) for \( n \geq m \).

3. Main Results

**Theorem 3.1.** If \( f(z) \) in \( ST_s \) then,

\[
|a_2a_3 - a_4| \leq \frac{1}{2}
\]

**Proof.** For the function \( f(z) = Z + \sum_{n=2}^{\infty} a_n z^n \) \( \in ST_s \) by definition (1.1), there exist an analytic function \( p \in P \) in the unit disc \( E \) with \( p(0) = 1 \) and \( \Re p(z) > 0 \) such that,

\[
\frac{2z + f'(z)}{f(z) - f(-z)} = [1 - \delta]p(z) + \delta
\]

\[
2z \left[ 1 + \sum_{n=2}^{\infty} na_n z^{n-1} \right] = \left\{ \left[ [1 - \delta]p(z) + \delta \right] \left[ \left( z + \sum_{n=2}^{\infty} a_n z^n \right) \right] - \left( -z + \sum_{n=2}^{\infty} a_n (-z)^n \right) \right\}
\]

upon simplification we obtain,

\[
a_2 = \frac{(1 - \delta)c_1}{2}
\]

\[
a_3 = \frac{(1 - \delta)c_2}{2 - \delta}
\]

\[
a_4 = \frac{(1 - \delta)(2 - \delta)c_3 + (1 - \delta)^2 c_1 c_2}{4(2 - \delta)}
\]

\[
a_5 = \frac{(1 - \delta)(2 - \delta)c_4 + (1 - \delta)^2 c_2^2}{(2 - \delta)(4 + \delta)}
\]
Substituting the values of $a_2$, $a_3$ and $a_4$ from equation (3.13), (3.14) and (3.15) in the functional $|a_2a_3 - a_4|$ for the function $f \in S^c_\delta$ we obtain,

$$
|a_2a_3 - a_4| = \left| \frac{(1 - \delta)^2 c_1 c_2}{2(2 - \delta)} - \frac{(1 - \delta)(2 - \delta)c_3 + (1 - \delta)^2 c_1 c_2}{4(2 - \delta)} \right|
$$

Applying lemma (2.2), (2.3)

$$
= \left| \frac{(1 - \delta)^2 c_1 \left[ \frac{c_1^2 + x(4 - c_1^2)}{2} \right]}{2(2 - \delta)} - \frac{(1 - \delta)(2 - \delta)}{4(2 - \delta)} \left[ c_3^2 + 2c_1(4 - c_1^2)x - c_1x(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2z) \right] \right|
$$

Now we assume $c_1 = c$ and $c \in [0, 2]$ then by using triangular inequality and $|z| \leq 1$ we have,

$$
\leq \left\{ \frac{(1 - \delta)^2 \left[ c^3 + cx(4 - c^2) \right]}{4(2 - \delta)} \right\} + \frac{(1 - \delta)(2 - \delta)}{16(2 - \delta)} \left[ c^3 + 2c(4 - c^2)x - x^2c(4 - c^2) + 2(4 - c^2)(1 - |x|^2z) \right]
$$

where $|x| = \rho \leq 1$

$$
\leq \left\{ \frac{(1 - \delta)^2 \left[ c^3 + \rho c(4 - c^2) \right]}{4(2 - \delta)} \right\} + \frac{(1 - \delta)(2 - \delta)}{16(2 - \delta)} \left[ c^3 + 2c(4 - c^2)\rho - c\rho^2(4 - c^2) + 2(4 - c^2)(1 - \rho^2) \right]
$$

Now we maximize the function $F(\rho)$ on the closed region $[0, 2] \times [0, 1]$ we get,

$$
F'(\rho) = \frac{(1 - \delta)^2}{4(2 - \delta)} \left[ c(4 - c^2) \right] + \frac{(1 - \delta)(2 - \delta)}{16(2 - \delta)} \left[ 2\rho c(4 - c^2) - 2\rho(c - 2)(4 - c^2) \right]
$$
Since \( F'(\rho) > 0 \) the \( F(\rho) \) is an increasing function. Hence, \( \text{Max } F(\rho) = F(1) \).

\[
F(1) = \left\{ \frac{(1-\delta)^2(c^3 + c(4-c^2))}{4(2-\delta)} + \frac{(1-\delta)(2-\delta)}{16(2-\delta)} \right. \\
\left. \left[ c^3 + 2c(4-c^2) - (4-c^2(c-2) + 2(4-c^2) \right) \right\} \tag{3.20}
\]

\( G(c) \) attains maximum value at \( c = 0 \).

\[
\therefore |a_2a_4 - a_3^2| \leq \frac{8(1-\delta)}{16} \leq \frac{1}{2}(1-\delta) \tag{3.21}
\]

**Theorem 3.2.** If \( f(z) \in S^\alpha_\epsilon(\delta) \) then,

\[
|a_3 - a_2^2| \leq 1 \tag{3.22}
\]

**Proof.** Substituting the values of \( a_2 \) and \( a_3 \) from equation (3.13) and (3.14) into the functional \( |a_3 - a_2^2| \), we obtain

\[
|a_3 - a_2^2| = \left| \frac{(1-\delta)c_2}{2-\delta} - \frac{(1-\delta)^2c_1^2}{4} \right| \\
= \left| \frac{(1-\delta)[c_1^2 + x(4-c_1^2)]}{2(2-\delta)} - \frac{(1-\delta)^2c_1^2}{4} \right| \\
c = c \in [0, 2] \\
\leq \frac{(1-\delta)[c_1^2 + x(4-c_1^2)]}{2(2-\delta)} + \frac{(1-\delta)^2c_1^2}{4} \tag{3.23}
\]

\[
|x| = \rho \\
\leq \frac{(1-\delta)[c_1^2 + \rho(4-c_1^2)]}{2(2-\delta)} + \frac{(1-\delta)^2c_1^2}{4} \\
= F(\rho) \\
\]

\[
F'(\rho) = \frac{(1-\delta)(4-c_1^2)}{2(2-\delta)} \tag{3.24}
\]

\( F'(\rho) > 0, \therefore F(\rho) \) is an increasing function, hence it cannot have maximum value at
any point in the interior of the closed region \([0,2] \times [0,1]\). \(c \in [0,2]\) we have,

\[
\max_{0 \leq \rho \leq 1} F(c, \rho) = F(c, 1)
\]

\[
= \frac{(1 - \delta)[c^2 + 4 - c^2]}{2(2 - \delta)} + \frac{(1 - \delta)^2 c^2}{4}
\]

\[
= \frac{4(1 - \delta)}{2(2 - \delta)} + \frac{(1 - \delta)^2 c^2}{4}
\]

\[
= G(c)
\]

\[
G'(c) = \frac{2c[1 - \delta]^2}{4} = \frac{2c[1 - \delta]^2}{4} = \frac{c[1 - \delta]^2}{2}
\]

We observe that \(G'(c) \leq 0\) for every \(c \in [0,2]\), \(\therefore\) \(G(c)\) becomes decreasing function of \(c\), whose maximum value occurs at \(c = 0\) and it is given by,

\[
|a_3 - a_2^2| \leq \frac{4(1 - \delta)}{2(2 - \delta)}
\]

\(\blacksquare\)

**Theorem 3.3.** Let \(f(z) \in S^*_c(\delta)\) then,

\[
|a_2a_4 - a_3^2| = \left\{ \frac{(1 - \delta)^2 c_1(2 - \delta)c_3 + (1 - \delta)^2 c_1c_2}{2 \times 4(2 - \delta)} - \frac{(1 - \delta)^2 c_2^2}{(2 - \delta)^2} \right\}
\]

\[
= \left\{ \frac{(1 - \delta)^2 c_1 c_3 + (1 - \delta)^2 c_1 c_2}{8} - \frac{(1 - \delta)^2 c_2^2}{(2 - \delta)^2} \right\}
\]

\[
\leq \frac{(1 - \delta)^2 c_1}{8}
\]

\[
\left[ \frac{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2 z)}{4} \right] - \frac{(1 - \delta)^2 c_2^2}{(2 - \delta)^2} \left[ \frac{c_1^2 + x(4 - c_1^2)}{2} \right]^2
\]

\[
(1 - \delta)^2 c_1 \left[ \frac{c_1^2 + x(4 - c_1^2)}{2} \right] + \frac{(1 - \delta)^2 c_2^2}{(2 - \delta)^2} \left[ \frac{c_1^2 + x(4 - c_1^2)}{2} \right]^2
\]

\[
(4 - c_1^2)(1 - \rho^2) + \frac{(1 - \delta)^2}{16} \left[ c_1^3 + c_\rho(4 - c_1^2) \right] + \frac{(1 - \delta)^2}{4(2 - \delta)^2}
\]

\[
\left[ c_1^4 + 2c_2^2 \rho(4 - c_1^2) + \rho^2(4 - c_1^2)^2 \right]
\]

\[
= F(\rho)
\]
\[ F'(\rho) > 0, \therefore F(\rho) \text{ is an increasing function. If } \rho = 1 \text{ and } c = 0, \]
\[ \frac{16(1-\delta)^2}{4(2-\delta)^2} \leq \frac{16(1-\delta)^2}{4(2-\delta)^2} \frac{4(1-\delta)^2}{(2-\delta)^2} \tag{3.29} \]

**Corollary 3.4.** If \( f(z) \in S^*_c(\delta) \) then \( |a_k| \leq 1 \) for \( k \in \{2, 3, 4, \ldots\} \)
\[ H_3(1) \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_5^2| \]
\[ \leq \frac{4(1-\delta)^2}{(2-\delta)^2} + \frac{1}{2}(1-\delta) + \frac{4(1-\delta)^2}{(2-\delta)^2} \]
\[ I f \quad \delta = 0 \quad t h e n \]
\[ \leq \frac{5}{2} \tag{3.30} \]

Result is coincide with the result of D. Vamshee et al. [1].

**References**


