Dirichlet Series Solution Approach to Coupled Equations Arising in Boundary Value Problems

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**Abstract**

The pair of differential equations

\[ f''' + A f'' + B (f')^2 + C g^2 = 0 \]
\[ g'' + D f' g' + E f' g = 0 \]

Where A, B, C, D and E are arbitrary constants subjected boundary conditions is considered. This class of equations frequently arising in problems of fluid dynamics. The proposed Dirichlet series method, conjugation with an unconstrained optimization method is found useful in analyzing the problems. The series so generated is analyzed using Euler transformation and other techniques.

**Keywords:** Boundary value problem, Dirichlet series, Euler transformation, Unconstrained optimization, Pade approximants.
INTRODUCTION
The fluid dynamics due to stretching boundary is important in the extrusion process, like the manufacturing of polymer and metal sheets, cooling of an infinite metallic plate, boundary layer along a liquid film in condensation process, etc.

The flow due to disk rotating in a viscous fluid was originally solved by Von Karman [1]. It is important in the study of flows on rotating bodies, centrifugal pumps, Viscometers etc. Von karman considered the infinite disk rotating with constant angular velocity in a fluid. Further study in the same is done by several other authors viz. Eochran [2], Mcleod [3] and Zandbergen and Dijkstra [4].

A study on steady flow produced by a rotating disc with either surface suction or injection is done by Ackroyd [5]. Further analysis of solution of free convection boundary layer on a vertical plate is also taken by Merkin et. al [6] and Ingham [7].

In all these cases, the final boundary value problem may be stated as

\[ f''' + Af'' + B(f')^2 + Cg^2 = 0 \]

\[ g'' + Dfg' + Ef'g = 0 \]

with boundary conditions of the form

\[ f(0) = \alpha_1, \quad f'(0) = \alpha_2, \quad f'(\infty) = 0, \quad g(0) = \beta_1, \quad g(\infty) = 0 \]

Where A, B, C, D, E, \( \alpha_1 \), \( \alpha_2 \) and \( \beta_1 \) are constants. The numerical studies of such boundary value problems involve more than one integration process. The use of Dirichlet series presents an attractive alternative approach. Convergence of such series is well established. In numerical methods a separate scheme is needed for calculating derived quantities. If the computation of derivatives is required, the numerical scheme to be used will be very sensitive to grid or step size. However, such problems do not arise in series solution presented here.

The proposed method is flexible and is more efficient in its implementation on a computer than a purely numerical method. For finding the values of the unknowns, the procedure employed here hardly takes any computer time and storage, where as usual numerical methods employed, take much more computer time. The proposed scheme
in conjugation with an unconstrained optimization procedure due to Press et. al [8] has found very useful in analyzing the present class of problems.

**PROBLEM FORMULATION:**

a) We give below a few examples of boundary value problems which arise from a problem of free convection boundary layer on a vertical plate [6] from special cases of system (1) and (2)

\[
\begin{align*}
    f''' + \lambda g + ff'' &= 0 \\
    g'' + fg' + 2f'g &= 0 \\
    f(0) &= 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \quad g(0) = 1, \quad g(\infty) = 0
\end{align*}
\]

b) \[
\begin{align*}
    f''' + \lambda g + ff'' &= 0 \\
    \frac{1}{pr} g'' + fg' + 2f'g &= 0
\end{align*}
\]

\[
\begin{align*}
    f(0) &= 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \quad g(0) = 1, \quad g(\infty) = 0
\end{align*}
\]

This occurs in the description of convection near a continuously moving vertical plate [7].

c) Von Karman considered the infinite disc rotating with constant angular velocity in a fluid of kinematic viscosity. Let (u, v, w) be the velocity components in the cylindrical co-ordinates \((r, \theta, z)\) respectively. Using the similarity transformations [9].

\[
\begin{align*}
    u &= \Omega rf'(\zeta), \quad v = \Omega r g(\zeta), \quad w = -2\sqrt{\gamma} \Omega f(\zeta) \text{ where } \zeta = z \sqrt{\frac{n}{\nu}}
\end{align*}
\]

and Navier-Stokes equations reduce to

\[
\begin{align*}
    f''' + 2ff'' - f''^2 + g^2 &= 0 \\
    g'' + 2fg' - 2f'g &= 0
\end{align*}
\]

\[
\begin{align*}
    f(0) &= 0, \quad f'(0) = 0, \quad f'(\infty) = 0, \quad g(0) = 1, \quad g(\infty) = 0
\end{align*}
\]
METHOD OF SOLUTION:

We seek a Dirichlet Series solution of (1) in the form

\[ f = \frac{\gamma}{A} \sum_{i=1}^{\infty} a_i e^{-i\gamma \eta} \]  

\[ g = \gamma^2 \sum_{i=1}^{\infty} i b_i e^{-i\gamma \eta} \]  

Substituting (3) into (1), we get

\[-\gamma^4 \sum_{i=1}^{\infty} i^3 a_i e^{-i\gamma \eta} + \gamma^4 \sum_{i=1}^{\infty} i^2 a_i e^{-i\gamma \eta} \]

\[+ A \gamma^4 \sum_{i=2}^{\infty} \sum_{k=1}^{i-1} k^2 a_k a_{i-k} e^{-i\gamma \eta} \]  

\[+ B \gamma^4 \sum_{i=2}^{\infty} \sum_{k=1}^{i-1} k (i-k) a_k a_{i-k} e^{-i\gamma \eta} \]

\[+ C \gamma^4 \sum_{i=2}^{\infty} \sum_{k=1}^{i-1} k (i-k) b_k b_{i-k} e^{-i\gamma \eta} = 0 \]

and

\[ \gamma^4 \sum_{i=1}^{\infty} i^3 b_i e^{-i\gamma \eta} - \gamma^4 \sum_{i=1}^{\infty} i^2 b_i e^{-i\gamma \eta} + D \gamma^4 \sum_{i=2}^{\infty} \sum_{k=1}^{i-1} k^2 b_k a_{i-k} e^{-i\gamma \eta} \]

\[+ E \gamma^4 \sum_{i=2}^{\infty} \sum_{k=1}^{i-1} k (i-k) a_k b_{i-k} e^{-i\gamma \eta} = 0 \]

We write (4) in the form

\[ i \geq 2 \]

\[ \sum_{i=2}^{\infty} \left[ -i^3 a_i + i^2 a_i + A \sum_{k=1}^{i-1} k (i-k) b_{i-k} b_k \right] e^{-i\gamma \eta} = 0 \]

Where,

\[ a_i = \frac{1}{i^2(i-1)} \sum_{k=2}^{i-1} (Ak^2 + Bk(i-k)) a_k a_{i-k} + (Ck(i-k)b_{i-k}b_k) \]  

(5)
Similarly
\[ b_i = \frac{1}{i^2(i-1)} \sum_{k=2}^{i-1} \left( Dk^2b_k a_{i-k} + E(k - k)\right) a_k b_{i-k} \] (6)
\[ i = 2, 3, 4, \ldots \]

The series (3) converges absolutely for any \( \eta > 0 \) and \( \eta = -\varepsilon \), where \( \varepsilon = -\frac{\ln |a_1|}{\gamma} + \delta \) \( (\delta > 0) \) is a sufficiently small number depending on unknowns. The series absolutely convergent and uniformly convergent on the half axis \( \eta > -\varepsilon \). A general discussion of the convergence, etc., of the Dirichlet series (3) may be found in [10]. The unknowns are determined from remaining boundary conditions

\[ f(0) = \frac{\gamma}{A} + \gamma \sum_{i=1}^{\infty} a_i = \alpha_1 \] (7)
\[ f'(0) = \gamma^2 \sum_{i=1}^{\infty} (-i)\gamma a_i = \alpha_2 \] (8)
\[ g(0) = \gamma^2 \sum_{i=1}^{\infty} ib_i = \beta_1 \] (9)

The solution of transcendental equations (7), (8) and (9) yield the constants. We employ a method of conjugate direction [8] which is one of the most efficient techniques for solving unconstrained optimization problems to determine the unknowns. To confirm the numerical values obtained for unknowns also verified by solving a system of transcendental equations (7) to (9) by modified Newton's formula [11].

The shear stress at the surface of the wall is

\[ f''(0) = \gamma \sum_{i=1}^{\infty} a_i (-i \gamma)^2 \] (10)

The radius convergence of the series is obtained by Domb-Sykes plot. The computed radius of convergence defines the presence of the singularity. However, as we shall
show, this singularity is located on the negative \( \eta \) axis. Applying the Euler's transformation \( w = \frac{R}{R+R_0} \)

We find that \( R = \infty \) corresponds to \( w = 1 \) series (10) may be written as

\[
f''(0) = \sum_{i=1}^{\infty} a_i^* R_i
\]  

(11)

Substituting \( R = \frac{wR_0}{1-w} \) in to this, we get

\[
f''(w) = \sum_{i=1}^{\infty} D_i w^{n-1}
\]

(12)

Where

\[
D_1 = 1  \\
D_2 = a_2^* R_0  \\
D_3 = a_2^* R_0 + a_3^* R_0^2
\]

(13)

\[
D_n = (-1)^{n-1} \Delta^{n-2} e_2
\]

With

\[
\Delta e_j = e_{j+1} - e_j  \\
e_j = (-R_0)^{j-1} a_j^*
\]

Relations (12) enable finding \( D_i \)'s in terms of \( a_i^* \) of (11)

**RESULTS AND CONCLUSIONS**

In this paper, we have given an exact semi analytic solution of the class of boundary value problems in the form of Dirichlet series. These are the cases which frequently occur in applications. Equation (1) is a quite general and includes several important cases. Dirichlet series is known to be convergent in the range \( \eta > \epsilon \) (is the abscissa of convergence) so include the domain \( 0 \leq \eta \leq \infty \) of interest. The computation of this series is a simple matter and is implementation very easier.
Table [1] gives the analysis of the first problem. This system describes free convection of heat on a vertical plate. Case (a), in this case, the coefficients of the series (10) decrease in magnitude using Pade approximants results are obtained table [1] and agree with the numerical solution Merkin [6] up to four decimal places.

The second problem (case b) describes convection near continuously moving vertical plate Ingham [7]. Here, the coefficients of the series (10) decrease in magnitude and alternate in sign, the singularity is found to be at \( R = 1.989 \). The Euler transformation helps in recasting the series into new series. It is found that the new series converges on the entire real line. It is seen that the results table[1] obtained almost equal to that calculated by Ingham [7]. Finally, the problem considered in case(c), the infinite disk rotating with constant angular velocity. The solution of the problem is obtained over results for \( f''(0) \), agree closely with those obtained by direct numerical scheme Table[1] Wang [9]. In the computation of the series, one may occasionally have to use Pade approximation to sum up the series.

Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>Constants</th>
<th>Boundary conditions</th>
<th>( f''(0) ) Dirichlets series</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
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<td>c</td>
<td>( A = 2, B = -1, C = 1 )</td>
<td>( D = 2, E = -2 )</td>
<td>( \alpha_1 = 0, \alpha_2 = 0, \beta_1 = 0 )</td>
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<td></td>
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<td></td>
<td></td>
<td>(Numerical Wang [9])</td>
</tr>
<tr>
<td>b</td>
<td>( A = 2, B = 0, C = 0 )</td>
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<td>( \alpha_1 = 0, \alpha_2 = 1, \beta_1 = 0 )</td>
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<td></td>
<td></td>
<td>(Numerical Ingham [7])</td>
</tr>
<tr>
<td>c</td>
<td>( A = \frac{4}{3}, B = -\frac{3}{5}, C = 0 )</td>
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<td>(Numerical Merkin [6])</td>
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REFERENCES