Unicity theorems on difference polynomials of meromorphic functions sharing one value

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Abstract

In this paper, we investigate the uniqueness of meromorphic functions $f$ and $g$ concerning polynomials with shift operator sharing one value with counting multiplicity. We extend and improved the results of K.Liu, X.L.Liu, T.B.Cao and many others.

Keywords: Meromorphic functions, Difference polynomials, Sharing value, Uniqueness, etc.

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1. Introduction and Main Results

Let $f(z)$ be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of Nevanlinna theory, for instance $T(r,f), m(r,f), N(r,f), \mathcal{N}(r,f)$ see [11], [5] and [9]. We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to +\infty$, possibly outside of a set of linear measure.

Definition 1.1. Let $f(z)$ and $g(z)$ be meromorphic functions. If $f(z) - a$ and $g(z) - a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value '$a$' CM, where '$a$' is any constant.

Definition 1.2. Let $k$ be a positive integer. We denote by $N_{(k)}(r, a, f)$ the counting function for zeros of $f - a$ with multiplicities at least $k$, and by $\mathcal{N}_{(k)}(r, a, f)$ the corresponding one for which multiplicity is not counted. Similarly, we denote by $N_{(k)}(r, a, f)$ the counting function for zeros of $f - a$ with multiplicities at most $k$, and by $\mathcal{N}_{(k)}(r, a, f)$ the corresponding one for which multiplicity is not counted. Then
Recently the topic of difference polynomial in the complex plane has attracted by many mathematicians. A number of papers have focused on value distribution and uniqueness of difference polynomials, which are analogues results of Nevanlinna theory. For a meromorphic function $f(z)$ and a constant $'c'$, $f(z + c)$ is called the shift of $f$, where $f(z)$ is not periodic function with period $c$.

In 2008, X.Y.Zhang, J.F.Chen and W.C.Lin \[12\] proved the results on uniqueness theorem of two polynomials sharing a common value.

**Theorem A.** Let $f$ and $g$ be two non-constant meromorphic functions, let $n$ and $m$ be two positive integers with $n > \max\{m + 10, 3m + 3\}$, and let $P(z) = a_mz^n + a_{m-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0$, where $a_0(\neq 0), a_1, a_2, a_3, \ldots a_{m-1}, a_m(\neq 0)$ are complex constants. If $f^nP(f)f'$ and $g^nP(g)g'$ share $1$ CM, then either $f = g$ or $g = \alpha f$, where $\alpha = \frac{m-1}{m}$.

In 2011, K.Liu, X.L.Liu and T.B.Cao \[6\] proved the following unicity theorem corresponding to difference polynomials.

**Theorem B.** Let $f$ and $g$ be two transcendental meromorphic functions with finite order. Suppose that $c$ is a non-zero constant and $n \in N$. If $n \geq 14$, $f^n(z)f(z + c)$ and $g^n(z)g(z + c)$ share $1$ CM, then $f \equiv tg$ or $fg = t$, where $t^{n+1} = 1$.

In 2015, we \[1\] have proved the following theorem on value distribution of meromorphic function $f$ concerning polynomials with shift operator.

**Theorem C.** Let $f$ be a transcendental meromorphic function with finite order, $\rho_2(f) < 1$ and $c$ be a non-zero complex constant. Let $P(z) = a_mz^m + a_{m-1}z^{m-1} + \ldots + a_1z + a_0$, where $a_0(\neq 0), a_1, a_2, a_3, \ldots a_{m-1}, a_m(\neq 0)$ are complex constants and $\alpha(z)$ be a small function of $f$. If $n \geq m + 6 (m, n \in N)$, then $f^n(z)P(f)f(z + c) - \alpha(z)$ has infinitely many zeros.

In this paper, we have proved the unicity theorem of \[1\], a result to prove the uniqueness for the meromorphic functions sharing the value $1$ with counting multiplicity. Further it extends the Theorem A by replacing $f'$ by $f(z + c)$.
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**Theorem 1.1.** Let $f$ and $g$ be two non-constant meromorphic functions of finite order. Let $n$ and $m$ be two positive integers with $n > m + 11$. Let $c$ be a non-zero complex constant and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_0(\neq 0), a_1, a_2, \ldots, a_{m-1}$, $a_m(\neq 0)$ are complex constants. If $f^n(z)P(f)(z+c)$ and $g^n(z)P(g)(z+c)$ share 1 CM, then $f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = \text{GCD}\{n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1\}$ for $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m$, or $f$ and $g$ satisfy the algebraic equation $R(\omega_1, \omega_2) \equiv 0$ where $R(\omega_1, \omega_2) = \omega_1^n(z)P(\omega_1)\omega_1(z + c) - \omega_2^n(z)P(\omega_2)\omega_2(z + c)$.

**Remark 1.1.** If $P(f) = 1$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem B.

2. Some Lemmas

We need following Lemmas to prove our results.

**Lemma 2.1.** ([2]) Let $f(z)$ be a transcendental meromorphic function of finite order and $c$ is a non-zero complex constant, then

$$T(r, f(z + c)) = T(r, f) + S(r, f)$$

**Lemma 2.2.** ([3]) Let $f$ be a transcendental meromorphic function of finite order and $c$ is a non-zero complex constant. Then

$$m\left(r, \frac{f(z + c)}{f(z)}\right) = S(r, f)$$

**Lemma 2.3.** ([2],[4]) Let $f(z)$ be a meromorphic function of finite order and $c$ is a non-zero complex constant. Then

$$m\left(r, \frac{f(z + c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z + c)}\right) = S(r, f)$$

**Lemma 2.4.** (Lemma 3 in [10]) Let $F$ and $G$ be non-constant meromorphic functions. If $F$ and $G$ share 1 CM, then one of the following three cases holds

1. $\max\{T(r, F), T(r, G)\} \leq N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G) + S(r, F) + S(r, G)$
2. $F \equiv G$,
3. $FG \equiv 1$.

**Lemma 2.5.** ([11]) Let $f(z)$ be a non-constant meromorphic function, and $a_n(\neq 0), a_{n-1}, \ldots, a_0$ be small functions with respect to $f$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f)$$

**Lemma 2.6.** ([8]) Let $f(z)$ be a transcendental meromorphic function of finite order. Then

$$N(r, f(z + c)) = N(r, f) + S(r, f)$$
Lemma 2.7. Let $f(z)$ be a transcendental meromorphic function of finite order and let $F^* = f(z)^n P(f) f(z + c)$. Then

$$(n + m - 1)T(r, f) + S(r, f) \leq T(r, F^*) \leq (n + m + 1)T(r, f) + S(r, f)$$

Proof: Since $f$ is a transcendental meromorphic function and also from Lemma 2.3, Lemma 2.5 and Lemma 2.6, we obtain

$$N\left(r, \frac{1}{f(z + c)}\right) = N\left(r, \frac{1}{f}\right) + S(r, f)$$

$$\left(\frac{1}{f(z + c)}\right)$$

$$\frac{1}{f(z + c)}$$

Now we consider

$$N_2\left(r, \frac{1}{F}\right) = N_2\left(r, \frac{1}{f^n(z)P(f) f(z + c)}\right) \leq 2N\left(r, \frac{1}{P}\right) + N\left(r, \frac{1}{f}\right) + S(r, f)$$

Then by Lemma 2.1, we obtain
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\[ N_2\left(r, \frac{1}{F}\right) = \frac{2}{n} \left[ nN\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{P(f)}\right) + \frac{1}{f(z+c)} N\left(r, \frac{1}{f}\right) \right] \\
+ \left(1 - \frac{2}{n}\right) \left[ N\left(r, \frac{1}{P(f)}\right) + \frac{1}{f(z+c)} N\left(r, \frac{1}{f}\right) \right] + S(r, f) \\
\leq \frac{2}{n} N\left(r, \frac{1}{f^n(z)P(f)^n(z+c)}\right) + \left(1 - \frac{2}{n}\right) \left[ N\left(r, \frac{1}{P(f)}\right) + \frac{1}{f(z+c)} N\left(r, \frac{1}{f}\right) \right] \\
+ \left(1 - \frac{2}{n}\right) \left( \frac{n}{n+1} \right) N\left(r, \frac{1}{P(f)}\right) + S(r, f) \\
\leq \left(\frac{3}{n + 1}\right) N\left(r, \frac{1}{f}\right) + \left(\frac{n - 2}{n + 1}\right) m[2m + N(r, f)] + S(r, f) \tag{3.3} \]

Now consider
\[ N\left(r, \frac{1}{P(f)}\right) \leq T\left(r, \frac{1}{P(f)}\right) \leq mT\left(r, \frac{1}{f}\right) \leq mT(r, f) + S(r, f) \]
\[ = m[m(r, f) + N(r, f)] + S(r, f) \]

From (3.3), we have
\[ N_2\left(r, \frac{1}{F}\right) \leq \left(\frac{3}{n + 1}\right) N\left(r, \frac{1}{f}\right) + \left(\frac{n - 2}{n + 1}\right) m[m(r, f) + N(r, f)] + S(r, f) \tag{3.4} \]

Now consider
\[ N_2(r, F) = N_1(r, f^n(z)P(f) f(z+c)) + 2N_2(r, f^n(z)P(f) f(z+c)) \]
\[ = N_1(r, f(z+c)) + 2N_2(r, f(z+c)) + S(r, f) \]
\[ \leq N(r, f(z+c)) + 2N(r, f) + S(r, f) \leq 3N(r, f) + S(r, f) \tag{3.5} \]

From (3.4) and (3.5), we have
\[ N_2(r, F) + N_2\left(r, \frac{1}{F}\right) \leq \left(\frac{3}{n + 1}\right) N\left(r, \frac{1}{f}\right) + \left(\frac{n - 2}{n + 1}\right) m[m(r, f) \]
\[ + N(r, f)] + 3N(r, f) + S(r, f) \]
\[ \leq \left(\frac{3}{n + 1}\right) N\left(r, \frac{1}{f}\right) + \left(\frac{mn - 2m + 3n + 3}{n + 1}\right) m(r, f) \]
\[ + \left(\frac{mn - 2m + 3n + 3}{n + 1}\right) N(r, f) - 3m(r, f) + S(r, f) \tag{3.6} \]
\[
\begin{align*}
&\leq \left( \frac{3}{n+1} \right)N \left( r, \frac{1}{F} \right) + \left( \frac{mn - 2m + 3n + 3}{n+1} \right)m(r,f) \\
&\quad + \left( \frac{mn - 2m + 3n + 3}{n+1} \right)N(r,f) + S(r,f) \\
&\leq \left( \frac{3}{n+1} \right)T \left( r, \frac{1}{F} \right) + \left( \frac{mn - 2m + 3n + 3}{n+1} \right)T(r,f) + S(r,f) \\
\end{align*}
\] (3.7)

W.k.t
\[
T(r, F) = T(r, f^n(z) P(f)f(z + c)) + S(r, f) \\
\leq (n + m + 1)T(r, f) + \log r + S(r, f)
\] (3.8)

From (3.7) and (3.8), we get
\[
N_2(r, F) + N_2 \left( r, \frac{1}{F} \right) \leq \left( \frac{3}{n+1} \right)T \left( r, \frac{1}{F} \right) + \left( \frac{mn - 2m + 3n + 3}{(n+1)(n+m+1)} \right)T(r, F) + \log r + S(r, f)
\]
\[
\leq \left( \frac{3(n + m + 1) + mn - 2m + 3n + 3}{(n+1)(n+m+1)} \right)T(r, F) + \log r + S(r, f)
\]
\[
\leq \left( \frac{m+6}{n+m+1} \right)T(r, F) + \log r + S(r, f)
\] (3.9)

Similarly, we have
\[
N_2(r, G) + N_2 \left( r, \frac{1}{G} \right) \leq \left( \frac{m+6}{n+m+1} \right)T(r, F) + \log r + S(r, g)
\] (3.10)

From (3.1), (3.9) and (3.10) we have
\[
T(r, F) + T(r, G) \leq 2 \left\{ N_2(r, F) + N_2 \left( r, \frac{1}{F} \right) + N_2(r, G) + N_2 \left( r, \frac{1}{G} \right) \right\} + S(r, f) + S(r, g)
\]
\[
\leq 2 \left\{ \left( \frac{m+6}{n+m+1} \right)T(r, F) + \left( \frac{m+6}{n+m+1} \right)T(r, G) \right\}
\]
\[
\quad + 4 \log r + S(r, f) + S(r, g)
\]
\[
\leq 2 \left( \frac{m+6}{n+m+1} \right) \left[ T(r, F) + T(r, G) \right] + 4 \log r
\]
\[
+ S(r, f) + S(r, g)
\]
\[
(n + m - 11)\{ T(r, F) + T(r, G) \} \leq 4(n + m + 1) \log r + S(r, f) + S(r, g)
\]
\[
\therefore T(r, F) + T(r, G) \leq (n + m + 1) \log r + S(r, f) + S(r, g)
\]

By the assumption that \( F \) and \( G \) share \( 1 \) CM and statement of the theorem 1.1, w.k.t either \( f \) and \( g \) are transcendental meromorphic functions or \( f \) and \( g \) are rational functions.
If both $f$ and $g$ are transcendental meromorphic functions, then by (3.11) we get a contradiction. If both $f$ and $g$ are rational functions, then $S(r, f) = O(1)$ and $S(r, g) = O(1)$.

Let $f(z) = \frac{p_2(z)}{p_1(z)}$ and $g(z) = \frac{q_2(z)}{q_1(z)}$, where both $p_1(z), p_2(z)$ and $q_1(z), q_2(z)$ are co-prime polynomials.

(i) If $\max \{ \deg p_1, \deg p_2 \} \geq 3$, then by $f^n(z)P(f)f(z + c)$ and $g^n(z)P(g)g(z + c)$ share 1 CM, we have $\max \{ \deg q_1, \deg q_2 \} \geq 2$.

Thus by simple computing, we get

$$T(r, F) + T(r, G) = T(r, f^n(z)P(f)f(z + c)) + T(r, g^n(z)P(g)g(z + c)) \geq 5(n + m + 1) \log r + O(1)$$

By (3.11) and (3.12), we deduce a contradiction.

If $\max \{ \deg q_1, \deg q_2 \} \geq 3$ then

$$T(r, F) + T(r, G) = T(r, f^n(z)P(f)f(z + c)) + T(r, g^n(z)P(g)g(z + c)) \geq 6(n + m + 1) \log r + O(1)$$

By (3.11) and (3.13), we deduce a contradiction.

Next we consider the case when $\max \{ \deg p_1, \deg p_2 \} \leq 2$ and $\max \{ \deg q_1, \deg q_2 \} \leq 2$.

By simple calculation, we have

$$f(z) = a \frac{(z - b_1)(z - b_2)}{(z - a_1)(z - a_2)}$$

(3.14)

where $a_1, a_2, b_1, b_2$ are four distinct non-zero constants. By the condition $n > m + 11$ and (3.14), we have

$$T(r, F) \geq 2(n + m + 1) \log r + O(1) \geq 30 \log r + O(1)$$

(3.15)

We know that $N_2(r, F) = N_2(r, f^n(z)P(f)f(z + c))$

$$= N_2(r, f^n) + N_2(r, P(f)) + N_2(r, f(z + c))$$

$$= 2N_2(r, f^n) + 2N_2(r, P(f)) + 2N_2(r, f(z + c))$$

$$+ \bar{N}(r, f(z + c))$$

$$\leq 7N(r, f) \leq 7T(r, f) \leq 7 \log r + S(r, f)$$

Now consider $N_2 \left( r, \frac{1}{F} \right) \leq N_2 \left( r, \frac{1}{f^n(z)P(f)f(z + c)} \right)$

$$\leq N_2 \left( r, \frac{1}{f^n} \right) + N_2 \left( r, \frac{1}{P(f)} \right) + N_2 \left( r, \frac{1}{f(z + c)} \right) + S(r, f)$$

$$\leq 2N_2 \left( r, \frac{1}{f^n} \right) + mN \left( r, \frac{1}{f} \right) + 2N_2 \left( r, \frac{1}{f(z + c)} \right)$$

$$+ \bar{N}(r, f(z + c)) + S(r, f)$$

$$\leq 2N \left( r, \frac{1}{f} \right) + mN \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f(z + c)} \right) + S(r, f)$$

$$\leq (m + 3) \log r + S(r, f)$$
From (3.15), we have
\[ 2 \left\{ N_2(r, F) + N_2 \left( \frac{1}{F} \right) \right\} = 2(m + 10) \log r + S(r, f) \leq 2(n + m + 1) \log r + S(r, f) \leq T(r, F) \]
\[ 2 \left\{ N_2(r, F) + N_2 \left( \frac{1}{F} \right) \right\} \leq T(r, F) \]  
(3.16)

Moreover, in the same manner as above, we have the similar results for the zeros of \( g^n(z)P(g)g(z + c) \).

By (3.16) and (3.17), we obtain
\[ 2 \left\{ N_2(r, F) + N_2 \left( \frac{1}{F} \right) + N_2(r, G) + N_2 \left( \frac{1}{G} \right) \right\} \leq T(r, F) + T(r, G) \]
Which is contradiction to case (1) of Lemma 2.4. Suppose that \( FG \equiv 1 \).

i.e \( f^n(z)P(f)g^n(z)P(g)g(z + c) \equiv 1 \)

(3.18)

Now we rewrite \( P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0 \) as \( P(z) = a_m (z - d_1)_1 (z - d_2)_2 \ldots (z - d_s)_s \) where \( l_1 + l_2 + \ldots + l_i + \ldots + l_s = m, 1 \leq s \leq m; d_i \neq d_j; i \neq j, 1 \leq i, j \leq s; d_1, d_2, \ldots, d_s \) are non-zero constants and \( l_1, l_2, \ldots, l_s \) are positive integers.

Let \( z_0 \) be a zero of order \( p_0 \), then from (3.18), \( z_0 \) is pole of \( g \) of order \( q_0 \). Again by (3.18), we obtain \( np_0 = nq_0 + mq_0 + q_0 \), that is, \( n(p_0 - q_0) = mq_0 + q_0 \), which implies that \( p_0 \geq q_0 + 1 \) and \( (m + 1)q_0 \geq n \). Hence \( p_0 \geq \frac{n + m + 1}{m} \).

Let \( z_1 \) be a zero of \( P(f) \) of order \( p_1 \) and be a zero of \( f - d_i \) of order \( m_i \), for \( i = 1, 2, \ldots, s \). Then \( p_1 = m_i \) for \( i = 1, 2, \ldots, s \). Suppose that \( z_1 \) is a pole of \( g \) of order \( q_1 \). Again by (3.18) we can obtain \( p_1 \leq m_i = \frac{n + m + 1}{m} \) for \( i = 1, 2, \ldots, s \).

Let \( z_2 \) be a zero of \( f \) \( (z + c) \) of order \( p_2 \) and then \( z_2 \) is pole of \( g(z) \) of order \( q_2 \), we get \( p_2 \geq n + m + 1 \).

Moreover, in the same manner as above, we have the similar results for the zeros of \( g^n(z)P(g)g(z + c) \). On the other hand, suppose that a pole of \( f \). Then from (3.18), we get that \( z_3 \) is the zero of \( g^n(z)P(g)g(z + c) \). So by using Lemma 2.1, we have
\[
N(r, f) \leq N \left( \frac{1}{f} \right) + N \left( \frac{1}{g - d_1} \right) + N \left( \frac{1}{g - d_2} \right) + \ldots + N \left( \frac{1}{g(z + c)} \right) + S(r, g)
\]
\[
\leq \left( \frac{m + 1}{n + m + 1} + \frac{m}{n + m + 1} + \frac{1}{n + m + 1} \right) N \left( \frac{1}{g} \right) + S(r, g)
\]
\[
\leq \left( \frac{2m + 2}{n + m + 1} \right) T(r, g) + S(r, g)
\]  
(3.19)

By second fundamental theorem and (3.19) we have
\[
sT(r, f) \leq N \left( \frac{1}{f} \right) + N \left( \frac{1}{f - d_1} \right) + N \left( \frac{1}{f - d_2} \right) + \ldots + N \left( \frac{1}{f - d_s} \right)
\]
If $1 \neq \eta$, then $\eta^m (r, f(z + c)) + N_0 (r, f') + s(r, f) \leq \left( \frac{m}{n + m + 1} + \frac{1}{n + m + 1} \right) \left( \frac{1}{r} \right) + \left( \frac{2m + 2}{n + m + 1} \right) T(r, g) + S(r, f) + S(r, g)$

Similarly, we have

$sT(r, g) \leq \left( \frac{2m + 2}{n + m + 1} \right) [T(r, f) + T(r, g)] + S(r, f) + S(r, g)$

By (3.20) and (3.21), we have

$s[T(r, f) + T(r, g)] \leq \left( \frac{2m + 2}{n + m + 1} \right) [T(r, f) + T(r, g)] + S(r, f) + S(r, g)$

Which is contradiction to $n > m + 1$.

Hence $F \equiv G$.

i.e. $f^{n}(z) (a_m f^m + a_{m-1} f^{m-1} + \cdots + a_0) f(z + c) \equiv g^n(z) (a_m g^m + a_{m-1} g^{m-1} + \cdots + a_0) g(z + c)$

(3.23)

Let $h = \frac{f}{g}$, and then substituting $f = gh$ and $f(z + c) = g(z + c) h(z + c)$ in (3.23) we deduce

$\Rightarrow (gh)^n [a_m (gh)^m + a_{m-1} (gh)^{m-1} + \cdots + a_0] g(z + c) h(z + c) = g^n (a_m g^m + \cdots + a_0) g(z + c)$

$g^m [a_m (h^{m+n} (z) h(z + c) - 1)] + g^{m-1} [a_{m-1} (h^{m+n-1} (z) h(z + c) - 1)] \ldots + [a_0 (h^n (z) h(z + c) - 1)] = 0$

If $h$ is a constant, which implies $h^d = 1$ where $d = \text{GCD}(n + m + 1 \ldots n + m + 1 - i \ldots n + 1)$ for some $i = 0, 1, 2, \ldots, m$.

If $h$ is not constant, then $f$ and $g$ satisfy the algebraic equation $R(\omega_1, \omega_2) = 0$ where $R(\omega_1, \omega_2) = \omega_1^n (z) P(\omega_1) \omega_1 (z + c) - \omega_2^n (z) P(\omega_2) \omega_2 (z + c)$.

Hence proof of theorem 1.1.

References


