International Journal of Mathematical Education.
ISSN 0973-6948 Volume 7, Number 1 (2017), pp. 1-9
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Some Mathematical Exercises on Dimensions of the Pear-shaped Earth Hemispheres

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Abstract

The orbital analysis of artificial Earth satellites have revealed the existence of a strong third harmonic in the gravitational potential of the Earth which gives its 'pear-shape' in the averaged meridional section with the 'stem' of the pear at the north pole. On account of the anti-symmetric nature of the third harmonic about the equator, the Southern Hemisphere is expected to be larger than the Northern Hemisphere in all respects. The average elevation, total surface area and total volume of the two hemispheres are calculated in this paper. The average Northern Hemisphere of a pear-shaped Earth is -2.546 m below the reference ellipsoid whereas the average Southern Hemisphere is at the same height above the reference ellipsoid. The average Southern Hemisphere is consequently 5.092 m higher than the average Northern Hemisphere. The total surface area of the Southern Hemisphere is calculated to be 408.534 km^2 larger than that of the Northern Hemisphere. Finally, the volume of the Southern Hemisphere is calculated to be 1.3042×10^6 km^3 greater than that of the Northern Hemisphere. The calculated values of the surface area and volume of the reference ellipsoid match closely to those found in the literature.

1. INTRODUCTION

The shape of the Earth to an observer depends upon the distance of the observer from the Earth. At great distances (mathematically referred to as the zeroth order approximation), the Earth can be treated as a point particle. At closer distances, in the first order of approximation, the Earth is spherical. In the second order approximation, the Earth is an oblate spheroid or ellipsoid of revolution. The orbital analysis of the Vanguard 1 satellite revealed that, in the third order approximation, the Earth was ‘***pear-shaped***’ in the averaged meridional section, with the ‘stem’ of the pear at the North Pole [1]. The spherical shape of the Earth is due to gravity alone, whereas the spheroidal shape of the Earth is due to gravitation and centrifugal force of rotation. The surface of the spheroid, symmetrical about the equator, is called the ***reference ellipsoid***. The actual surface is called the ‘***geoid***’, whose heights are measured from the reference ellipsoid. The geoid is a gravitationally equipotential surface. It coincides with the surface of the oceans, and extends under the continents to the level to which ocean water would settle if connected to oceans by open channels. A recent study advanced the proposition that the pear-shape of the Earth may be due to land-water distribution on the Earth’s surface [2].

According to the Potential Theory, the pear-shape of the Earth is described by the third harmonic of Legendre’s polynomial in the solution of Laplace’s equation [3]:

$$P_3(cos\theta) = \frac{1}{2}(5cos^3\theta - 3cos\theta) \tag{1}$$

where θ is the zenith angle or co-latitude from the North Pole. Since $P_3(cos\theta)$ is anti-symmetric about the equator, the two hemispheres of the Earth are expected to be unequal. In this paper, we calculate the comparative dimensions of the Earth’s hemispheres, viz., their average elevations, surface areas and volumes.

2. PEAR-SHAPED MODEL OF THE EARTH

Fig. 1 (from [3]) is the meridional figure of the Earth averaged over longitudes, which displays the pear-shape of the Earth. Fig. 2 shows the co-latitudinal variations of the observed geoid height h (from Fig. 1) and $P_3(cos\theta)$. Note that h represents the departure of the actual equipotential surface from the reference ellipsoid. The multiplicative factors ensure that the amplitudes of the two variables are similar and render them compatible for comparison. The closeness of the two curves indicates that the departure of the Earth’s figure from the reference ellipsoid is mainly shaped by the third harmonic of Legendre’s polynomial. Henceforth in this paper the geoid height is represented by $P_3(cos\theta)$:

$$h = AP_3(cos\theta) = \frac{A}{2}(5cos^3\theta - 3cos\theta) \tag{2}$$

where the amplitude A is determined from observed data by least-squares error analysis. Multiplying h by $cos\theta$ and summing over the data points gives [4]:

$$A = \frac{2\sum hcos\theta}{\sum(5cos^4\theta - 3cos^2\theta)} = 20.365\ m \tag{3}$$

The equation of Earth's ellipsoid in the geo-centric coordinate system is given by [5]:

$$r = \frac{a}{\sqrt{1+bcos^2\theta}} \tag{4}$$

with
$$b = \frac{e^2}{1-e^2} \tag{5}$$

and
$$e = \sqrt{1 - \frac{c^2}{a^2}} \tag{6}$$

Here, *a* is the equatorial radius of the Earth; *c* the polar radius of the Earth; and *e* the ellipticity of the Earth's figure [5]. The radial distance of the Earth' surface from the centre is therefore:

$$r = \frac{a}{\sqrt{1+bcos^2\theta}} + \frac{A}{2}(5cos^3\theta - 3cos\theta) \tag{7}$$

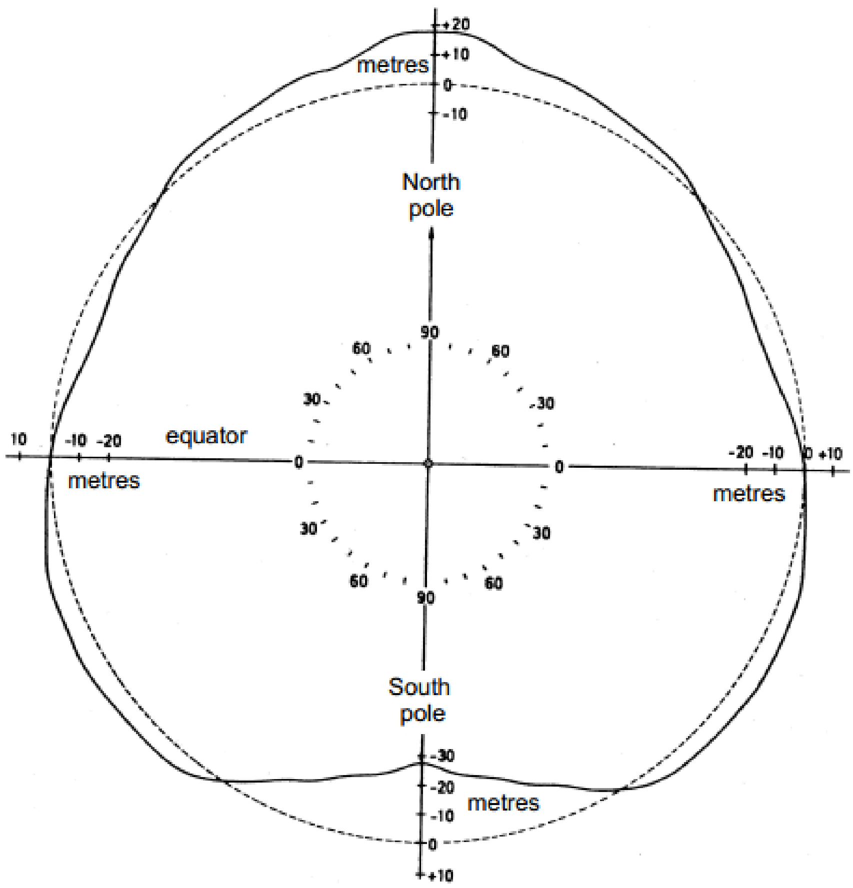

Figure 1. Average meridional section of Earth's geoid showing its pear-shape.

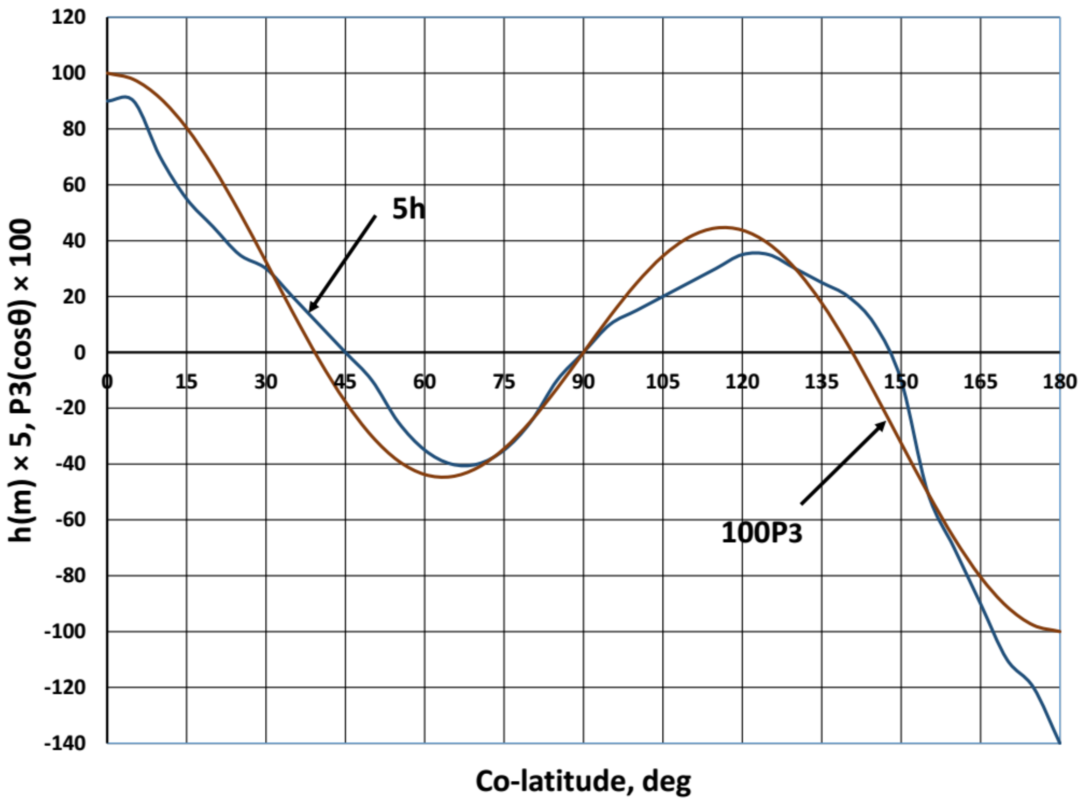

Figure 2. Co-latitudinal variations of geoid height and third harmonic of Legendre function.

3. AVERAGE ELEVATIONS OF THE HEMISPHERES

We first calculate the average elevations of the hemispheres of the Earth. In geo-centric spherical coordinate system (r, θ, ϕ), the area of a section of co-latitudinal width $d\theta$ and longitudinal width $d\phi$ is $r^2 sin\theta d\theta d\phi$. The elevation of each section is multiplied by the factor of $sin\theta$ in order to yield the weighted average. The weighted average elevations for the Northern and Southern Hemispheres, and that of the entire Earth are then as follows [4]:

$$< h >_{NH} = \frac{\int_0^{\pi/2} h sin\theta d\theta}{\int_0^{\pi/2} sin\theta d\theta} \tag{8}$$

$$< h >_{SH} = \frac{\int_{\pi/2}^{\pi} h sin\theta d\theta}{\int_{\pi/2}^{\pi} sin\theta d\theta} \tag{9}$$

and

$$< h > = \frac{\int_0^{\pi} h sin\theta d\theta}{\int_0^{\pi} sin\theta d\theta} \tag{10}$$

Substitutions from Eqs. (2) and (3) yield:

$$< h >_{NH}= \frac{A}{2}\int_0^{\pi/2}(5cos^3\theta - 3cos\theta)sin\theta d\theta = -\frac{A}{8} = -2.546\ m \tag{11}$$

$$< h >_{SH}= \frac{A}{2}\int_{\pi/2}^{\pi}(5cos^3\theta - 3cos\theta)sin\theta d\theta = \frac{A}{8} = 2.546\ m \tag{12}$$

and

$$< h >= 0 \tag{13}$$

Eqs. (11) and (12) state that the average Northern Hemisphere of a pear-shaped Earth is -2.546 m below the reference ellipsoid whereas the average Southern Hemisphere of the pear-shaped Earth is at the same height above the reference ellipsoid. This is to be expected, since $P_3(cos\theta)$ is an odd function about the equator. The average Southern Hemisphere is therefore 5.092 m higher than the average Northern Hemisphere. Eq. (13) likewise says that the pear-shaped Earth on the whole is at the same elevation as the reference ellipsoid.

These results can be better understood as follows. $P_3(cos\theta)$ is a cubic polynomial whose three roots are obtained by setting it to zero:

$$cos\theta(5cos^3\theta - 3cos\theta) = 0 \tag{14}$$

The roots of Eq. (14) are: (1) $\theta_1 = cos^{-1}\sqrt{3/5} \approx 39.23^o$; (2) $\theta_2 = cos^{-1}0 = 90^o$; and (3) $\theta_3 = \pi - cos^{-1}\sqrt{3/5} \approx 140.77^o$. The Northern Hemisphere is divided into two parts: (1) The polar section north of $\theta = \theta_1$, where geoid elevations are positive; and (2) The tropical section south of $\theta = \theta_1$, where geoid elevations are negative. The surface area of the polar section is calculated as follows:

$$A_{polar} \approx r^2\int_0^{\theta_1} sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi r^2(cos\theta_1 - 1) \approx .225(2\pi r^2) \tag{15}$$

Thus the polar section has only 22.5% of the area of the Northern Hemisphere. The remaining 77.5% of the Northern Hemisphere belongs to the tropical section. This accounts for the overall negative elevation of the entire Northern Hemisphere. A similar calculation explains the overall positive elevation of the Southern Hemisphere.

4. SURFACE AREAS OF THE HEMISPHERES

The surface area of a hemisphere of the pear-shaped Earth is evaluated as

$$S = \int_{\theta_1}^{\theta_2} r^2(\theta)sin\theta d\theta \int_0^{2\pi} d\phi \tag{16}$$

where the limits of integration are $\theta_1 = 0$ and $\theta_2 = \pi/2$ for the Northern Hemisphere; and $\theta_1 = \pi/2$ and $\theta_2 = \pi$ for the Southern Hemisphere. Expanding r^2 from Eq. (7), we can write:

$$S = S_1 + S_2 + S_3 = 2\pi a^2 I_1 + 4\pi ac I_2 + 2\pi c^2 I_3 \tag{17}$$

where
$$I_1 = \int \frac{sin\theta d\theta}{1+bcos^2\theta} \tag{18}$$

$$I_2 = \int \frac{(5cos^3\theta - 3cos\theta)sin\theta d\theta}{\sqrt{1+bcos^2\theta}} \tag{19}$$

and
$$I_3 = \int (5cos^3\theta - 3cos\theta)^2 sin\theta d\theta \tag{20}$$

The integrals (18) – (20) are conveniently evaluated by online integral calculators available on the Web (e.g., [6, 7]), giving:

$$I_1 = -\frac{tan^{-1}(\sqrt{b}cos\theta)}{\sqrt{b}} + C \tag{21}$$

$$I_2 = -\frac{\sqrt{bcos^2\theta+1}[b(5cos^2\theta-9)-10]}{3b^2} + C \tag{22}$$

and
$$I_3 = \left(-\frac{25cos^7\theta}{7} + 6cos^5\theta - 3cos^3\theta\right) + C \tag{23}$$

The surface areas of the Northern and Southern Hemispheres are now calculated term by terms using Eqs. (17) – (23) by putting the proper limits in the integrals. The values of the constants *a*, *b* and *c* are taken from the Earth data found on the Web [8]: equatorial radius *a* = 6378.137 km; polar radius *c* = 6356.752 km; from which ellipticity *e* = .081819791; and *b* = .006739596. The results are entered in Table I for ready reference. The surface area terms of the entire Earth are obtained by adding the figures from both hemispheres. As expected, the areas given by the successive terms rapidly diminished in value. The first terms S_1 represent the areas of the reference ellipsoid and are therefore equal for both hemispheres. The second terms S_2 represent the area corrections for the pear-shaped Earth from the reference ellipsoid. They are equal in magnitudes but opposite in signs for the two hemispheres amounting to -204.267 km^2 for the Northern Hemisphere and 204.267 km^2 for the Southern Hemisphere. The Southern Hemisphere of the pear-shaped Earth is thus 408.534 km^2 larger in area than the Northern Hemisphere. The third terms S_3 amount to miniscule corrections and are disregarded. Also entered in Table I is the total surface area of the Earth taken from the current data available on the Web [9]. The calculated value of *S* (510,064,070 km^2) agrees remarkably well with the reference data (510,064,472 km^2) with an accuracy of 99.99992%.

Table I. Hemispheric and Total Earth Surfaces

	S_1, km^2	S_2, km^2	S_3, km^2	S, km^2
Northern Hemisphere	255,032,035	-204.267	3.723×10^{-4}	255,031,831
Southern Hemisphere	255,032,035	204,267	3.723×10^{-4}	255,032,239
Total Earth	510,064,070	0	7.445×10^{-4}	510,064,070
Reference [9]	510,064,472	-	-	510,064,472

5. VOLUMES OF THE HEMISPHERES

The volume of a hemisphere of the pear-shaped Earth is evaluated as

$$V= \int_{\theta_1}^{\theta_2} \int_0^{r(\theta)} r^2 dr\, sin\theta d\theta \int_0^{2\pi} d\phi = \frac{2\pi}{3} \int_{\theta_1}^{\theta_2} r^3(\theta) sin\theta d\theta \tag{24}$$

where the limits of integration are $\theta_1 = 0$ and $\theta_2 = \pi/2$ for the Northern Hemisphere; and $\theta_1 = \pi/2$ and $\theta_2 = \pi$ for the Southern Hemisphere. Expanding r^3 from Eq. (7), we can write:

$$V = V_1 + V_2 + V_3 + V_4 = \frac{2\pi a^3}{3} I_1 + 2\pi a^2 c I_2 + 2\pi a c^2 I_3 + \frac{2\pi c^3}{3} I_4 \tag{25}$$

where
$$I_1 = \int \frac{sin\theta d\theta}{(1+bcos^2\theta)^{3/2}} \tag{26}$$

$$I_2 = \int \frac{(5cos^3\theta - 3cos\theta) sin\theta d\theta}{(1+bcos^2\theta)} \tag{27}$$

$$I_3 = \int \frac{(5cos^3\theta - 3cos\theta)^2 sin\theta d\theta}{\sqrt{1+bcos^2\theta}} \tag{28}$$

and
$$I_4 = \int (5cos^3\theta - 3cos\theta)^3\, sin\theta d\theta \tag{29}$$

Once again, the integrals (26) – (29) are evaluated by online integral calculators available on the Web [6, 7] to give:

$$I_1 = -\frac{cos\theta}{\sqrt{1+bcos^2\theta}} + C \tag{30}$$

$$I_2 = \frac{(3b+5) ln(bcos^2\theta+1) - 5bcos^2\theta}{2b^2} + C \tag{31}$$

$$I_3 = \frac{1}{48b^{7/2}} \left[3(72b^2 + 180b + 125) sinh^{-1}(\sqrt{b} cos\theta) - f(\theta)\right] + C \tag{32}$$

with
$$f(\theta) = \sqrt{b} cos\theta \sqrt{bcos^2\theta + 1}\, g(\theta) \tag{33}$$

$$g(\theta) = 25b^2 cos4\theta + 111b^2 - 5b(16b + 25) cos2\theta + 415b + 375 \tag{34}$$

and $$I_4 = -\frac{\left(5cos^2\theta-3\right)^4\left(20cos^2\theta+3\right)}{1000} + C \quad (35)$$

The volumes of the hemispheres of the pear-shaped Earth are now calculated term by terms using Eqs. (25) – (35) by putting the proper limits in the integrals. The results are entered in Table II.

Table II. Hemispheric and Total Earth Volumes

	V_1, km^3	V_2, km^3	V_3, km^3	V_4, km^3	V, km^3
N.H.	541,603,633,000	-652,102	2.371	2.764×10^{-7}	541,603,632,350
S.H.	541,603,633,000	652,102	2.371	-2.764×10^{-7}	541,603,633,650
Total	1,083,207,266,000	0	4.741	0	1,083,207,266,000
Ref. [9]	1,083,206,916,846	-	-	-	1,083,206,916,846

The volume terms of the entire Earth are obtained by adding the figures from both hemispheres. As in the earlier case, the volumes represented by the successive terms rapidly diminished in value. The first terms V_1 represent the volumes of the reference ellipsoid and are therefore equal for both hemispheres. The second terms V_2 represent the volume corrections for the pear-shaped Earth from the reference ellipsoid. They are equal in magnitudes but opposite in signs for the two hemispheres amounting to -652,102 km^3 for the Northern Hemisphere and 652,102 km^3 for the Southern Hemisphere. The Southern Hemisphere of the pear-shaped Earth is thus 1,304,204 km^3 larger in volume than the Northern Hemisphere. The third terms V_3 amount to corrections of merely 2.371 km^3 for both hemispheres whereas the fourth terms V_4 are virtually insignificant. Also entered in Table I is the total surface area of the Earth taken from the current data available on the Web [9]. The calculated value of V (1,083,207,266,000 km^3) agrees remarkably with the reference data (1,083,206,916,846 km^3) with an accuracy of 99.99997%”.

6. DISCUSSION

The pear-shape of the Earth was first revealed by the orbital analysis of the Vanguard 1 satellite in 1958 [1]. However, it was 58 years later and only recently that attention was brought out that the average elevation of the Southern Hemisphere pear-shaped Earth was higher than that of the Northern Hemisphere [4]. This implied that the Southern Hemisphere should also be larger in area and volume than the Northern Hemisphere. The hemispheric area and volume calculations presented in this paper confirm this idea. They also constitute illustrative examples in mathematical exercises.

The nearly complete agreement of the results with the reference values is especially satisfying.

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