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SYSTEMATIC DECOMPOSITION ALGORITHM FOR PENTAGONAL FUZZY FRACTIONAL SOLID TRANSPORTATION

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Abstract:

A novel ranking function for pentagonal fuzzy numbers is developed based on geometric centroid analysis to facilitate crisp comparison in fuzzy optimization environments. The proposed methodology introduces a Pentagonal Fuzzy Fractional Solid Transportation Decomposition (PFFSTD) algorithm, in which fuzzy parameters are systematically defuzzified and the fractional objective is decomposed into separate numerator and denominator cost matrices. Each component is optimized independently through iterative tableau reduction procedures and stepping-stone improvement strategies before integrating the results to obtain the overall three-dimensional optimal solution. The framework ensures balanced allocations among sources, destinations, and routing layers while effectively addressing multi-objective uncertainty within a structured computational setting.

Keywords: Pentagonal fuzzy numbers; geometric centroid ranking; fuzzy defuzzification; fractional solid transportation problem; SDR tableau method.

1. Introduction:

The transportation problem was first formally introduced by Hitchcock (1941), who formulated the classical distribution model for allocating commodities from multiple sources to several destinations in an optimal manner [1]. The foundational developments in linear programming and industrial optimization by Charnes and Cooper (1961) further strengthened mathematical modeling approaches applicable to transportation systems [2]. Shortly thereafter, Haley (1962) extended the framework to the solid transportation problem, incorporating an additional dimension to address source–destination–commodity interactions [7], while Shell (1963) discussed multi-property distribution structures [17]. Swarup (1965) contributed to the development of fractional programming theory, which later became essential for ratio-based optimization models [18].

The evolution of efficiency and performance measurement models was marked by Charnes, Cooper, and Rhodes (1978), introducing ratio-based evaluation techniques [4]. In the same year, Zimmermann (1978) pioneered fuzzy programming concepts for handling multi-objective uncertainty [20]. The theoretical foundations of fuzzy sets were further elaborated by

Dubois and Prade (1980) [6], followed by Chen (1985), who proposed ranking methods for fuzzy numbers [5]. Luhandjula (1987) provided an appraisal of fuzzy optimization techniques [15], and Kaufmann and Gupta (1988) presented comprehensive fuzzy mathematical models for engineering applications [11].

During the 1990s, research expanded toward multi-objective and fractional fuzzy models. Bit, Biswal, and Alam (1992) integrated fuzzy programming with multi-objective fractional programming [2], while Lai and Hwang (1992) advanced fuzzy multi-objective decision-making methods [13]. Jiménez (1996) introduced expected interval–based ranking of fuzzy numbers [9]. Entering the 2000s, Liu (2007) formalized uncertainty theory [14], and Toksarı (2007) proposed heuristic techniques for fuzzy fractional programming [19].

More recent studies focused on transportation models under fuzzy environments. Ojha and Das (2010) addressed multi-objective solid transportation problems with fuzziness [16], and Kaur and Kumar (2011) proposed improved methods for fuzzy transportation problems [10]. Abbasbandy and Hajjari (2009) introduced new ranking approaches for trapezoidal fuzzy numbers [1], while Kumar and Singh (2019) specifically discussed pentagonal fuzzy numbers and their applications in transportation problems [12].

Despite substantial progress in fuzzy transportation, fractional programming, and solid transportation models, notable research gaps persist. Most existing fuzzy transportation studies rely on triangular or trapezoidal fuzzy numbers, which may be insufficient to represent complex uncertainty and decision-maker hesitation in real-world logistics systems. Although fuzzy fractional transportation models address efficiency-based objectives, they often employ conventional defuzzification or weighted aggregation methods that do not explicitly preserve the separate effects of fractional numerators and denominators. Furthermore, fuzzy fractional solid transportation problem (FFSTP) formulations largely depend on heuristic or goal-programming techniques, with limited emphasis on structured decomposition algorithms that exploit the inherent three-dimensional source–destination–conveyance structure.

Moreover, the application of pentagonal fuzzy numbers in fractional solid transportation remains sparse, particularly with respect to robust and geometrically interpretable ranking methods. Existing ranking approaches for higher-order fuzzy numbers frequently lack computational transparency and consistency when integrated into large-scale optimization models. In addition, few studies have proposed algorithmic frameworks that independently optimize fuzzy fractional components before their systematic integration to obtain balanced and Pareto-efficient solutions. These limitations highlight the need for a unified methodology that integrates advanced fuzzy representation, reliable geometric ranking, and decomposition-based optimization for pentagonal fuzzy fractional solid transportation problems, which forms the central focus of the present study.

The remainder of this paper is organized as follows. Section 2 introduces the basic concepts of fuzzy sets, fuzzy numbers, and pentagonal fuzzy numbers. In Section 3, a new geometric centroid-based ranking method for pentagonal fuzzy numbers is proposed. Section 4 presents the Pentagonal Fuzzy Fractional Solid Transportation Decomposition (PFFSTD) algorithm in detail. A numerical example illustrating the effectiveness of the proposed approach is provided in Section 5. Finally, Section 6 concludes the paper and outlines directions for future research.

2. Preliminaries

2.1 Fuzzy Set: A fuzzy set is characterized by a membership function mapping the elements of a domain, space or universe of discourse X to the unit interval [0,1]. A fuzzy set $\tilde{A}$ in a universe of discourse X is defined as the following set of pairs: $\tilde{A}$={(x,$\mu_A(x)$;x∈X}.

Here $\mu_A(x)$:X→[0,1] is a mapping called the degree of membership function of the fuzzy set A and $\mu_A(x)$ is called the membership value of x∈X in the fuzzy set $\tilde{A}$. These membership grades are often represented by real numbers lying in [0,1].

2.2 Fuzzy Number: A fuzzy number $\tilde{A}$ is a normal and convex fuzzy subset of real line □ such that it's membership function $\mu_{\tilde{A}}$: □→[0,1] is piece-wise continuous in its domain.

2.3 Pentagonal Fuzzy Number (PFN): A fuzzy number $\widetilde{A^p} = (Q_1, Q_2, Q_3, Q_4, Q_5; \omega_1, \omega_2)$ is said to be a PFN if it satisfies the following properties:

(i) $\mu_{\widetilde{A^p}}(x)$ is a function which is continuous in [0,1].

(ii) $\mu_{\widetilde{A^p}}(x)$ is continuous and strictly increasing function in intervals $[Q_1, Q_2]$ and $[Q_2, Q_3]$.

(iii) $\mu_{\widetilde{A^p}}(x)$ is continuous and strictly decreasing function in intervals $[Q_3, Q_4]$ and $[Q_4, Q_5]$.

Here Q_1, Q_2, Q_3, Q_4 and Q_5 are real numbers such that $Q_1 \le Q_2 \le Q_3 \le Q_4 \le Q_5$. ω_1 and ω_2 are the grades of points Q_2 and Q_4 respectively, $\mu_{\widetilde{A^p}}(x)$ is the membership function of PFN and is defined as:

$$\mu_{\widetilde{A^p}}(x; \omega_1, \omega_2) = \begin{cases} \omega_1 \left(\dfrac{x - Q_1}{Q_2 - Q_1}\right), & \text{if} Q_1 \le x \le Q_2 \\ 1 - (1 - \omega_1)\left(\dfrac{x - Q_3}{Q_2 - Q_3}\right), & \text{if } Q_2 \le x \le Q_3 \\ 1, & \text{if } x = Q_3 \\ 1 - (1 - \omega_2)\left(\dfrac{x - Q_3}{Q_4 - Q_3}\right), & \text{if } Q_3 \le x \le Q_4 \\ \omega_1 \left(\dfrac{x - Q_1}{Q_2 - Q_1}\right), & \text{if } Q_4 \le x \le Q_5 \\ 0 & \text{otherwise} \end{cases}$$

2.4. FUZZY SOLID FRACTIONAL TRANSPORTATION PROBLEM

The problem includes constraints on pentagonal fuzzy supply at each source, pentagonal fuzzy demand at each destination, and pentagonal fuzzy capacity constraints on each mode of transportation. Formally, it can be modeled as:

$$\text{Minimize } Z = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\tilde{C}^{P}{}_{ijk}*x_{ijk}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\tilde{D}^{P}{}_{ijk}*x_{ijk}}$$

subject to constraints

$$\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk} = \tilde{a_{\imath}}^{P};\ k = 1,2,\dots,l.$$

$$\sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk} = \tilde{b_{\jmath}}^{P};\ i = 1,2,\dots,m.$$

$$\sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk} = \widetilde{e_{k}}^{P};\ j = 1,2,\dots,n.$$

$$x_{ijk} \geq 0;$$

Where m is the number of sources, n is the number of destinations, k is the number of transportation modes, $\tilde{C}^{P}{}_{ijk}$ is the pentagonal fuzzy profit coefficient, $\tilde{D}^{P}{}_{ijk}$ is the pentagonal fuzzy cost coefficient, $\tilde{a_{\imath}}^{P}$ is the pentagonal fuzzy supply available at source i, $\tilde{b_{\jmath}}^{P}$ is the pentagonal fuzzy demand required at destination j, $\widetilde{e_{k}}^{P}$ is the maximum pentagonal fuzzy capacity for transportation from i to j via k.

Remark: A necessary and sufficient condition for existence of solution is $\sum_{i=1}^{m}\widetilde{a_{\imath}}^{P} = \sum_{j=1}^{n}\widetilde{b_{\jmath}}^{P} = \sum_{k=1}^{m}\widetilde{e_{k}}^{P}$ i.e., the problem must be balanced. If the problem is unbalance, then it must be converted to balanced problem by introducing dummy source or origin.

3. Proposed Ranking method

Fuzzy numbers inherently lack a precise numerical value, as their membership functions describe a spectrum of possible values rather than a single point. Therefore, ranking techniques are essential to compare their magnitudes and transform them into crisp values for effective decision-making. Ranking fuzzy numbers constitutes a fundamental issue in fuzzy arithmetic, especially in decision-making applications. When the parameters of a problem are represented by fuzzy numbers, it becomes necessary to evaluate and compare them quantitatively before arriving at a decision. The use of an appropriate ranking approach ensures accurate and reliable outcomes, while an unsuitable one may lead to misleading conclusions. Consequently, ranking serves as a crucial aspect of the decision-making process. Over the years, numerous ranking methods have been proposed by various researchers, yet comparing and ordering fuzzy numbers remains a difficult task. Unlike real numbers, which possess a natural order, fuzzy numbers require conversion into real numbers to establish an equivalent ordering. Mathematically, a ranking function can be defined as R^{PFN}: F(□) → □, where F(□) denotes the set of fuzzy numbers on the real line □. This function assigns a unique crisp value to each fuzzy number. In this context, we introduce a new ranking function for picture fuzzy numbers (PFNs).

Consider a pentagonal fuzzy number $\widetilde{A^P} = (Q_1, Q_2, Q_3, Q_4, Q_5; \omega_1, \omega_2)$as shown in Figure 1. Extend the line joining A(Q_1, 0) and B(Q_2, w1) and also the line joining E(Q_5, 0) and D(Q_4, w2). Let the intersection of these lines be F(x, y) (see Figure 2). Then,

$$x=\frac{\omega_1(Q_1Q_5-Q_1Q_4)-\omega_2(Q_1Q_5-Q_2Q_5)}{\omega_1(Q_5-\omega_2(Q_1-Q_2))}\ ;\ y=\omega_1\left(\frac{x-Q_1}{Q_2-Q_1}\right),$$

Case 1: If $\omega_1 = \omega_2$, then $x=\frac{Q_1Q_5-Q_1Q_4-Q_1Q_5-Q_2Q_5}{Q_5-Q_4-Q_1+Q_2}$; $y=\frac{x-Q_1}{Q_2-Q_1}$

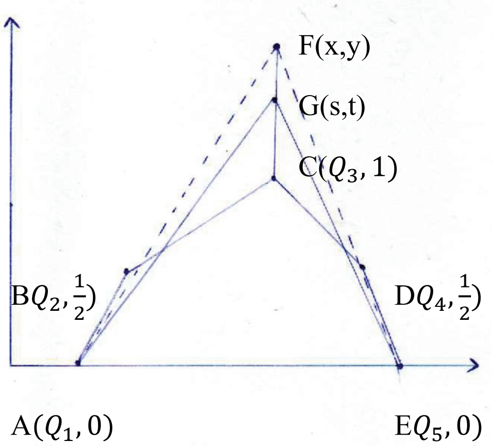

Figure 1

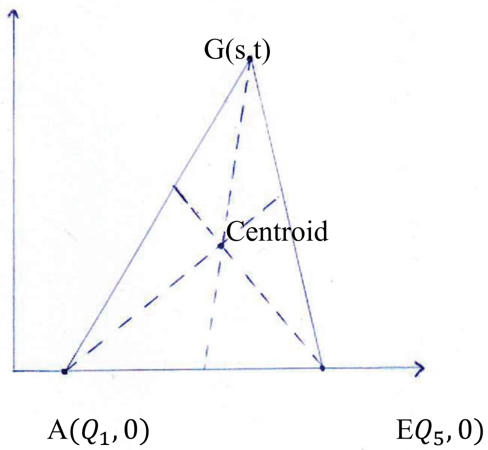

Figure 2

Remark 1: We'll consider $\omega_1 = \omega_2$ in the numerical examples.

Now join F to C let G(s,t) be the midpoint of FC. Then,

$$s = \frac{x+Q_3}{2},\ t = \frac{1+y}{2}$$

Now, the ranking function $R^{PFN}(\widetilde{A^P})$ which is based on the concept of centroid of triangle GAE (see figure 2) is defined as

$$R^{PFN}(\widetilde{A^P})=\frac{Q_1+Q_5+s}{3}$$

Remark 2: Using the above ranking function, comparison of two PFNs $\widetilde{Z_1}^P$ and $\widetilde{Z_2}^P$ can be done in the following way:

(i) If $R^{PFN}(\widetilde{Z_1^P})< R^{PFN}(\widetilde{Z_2^P})$ then $\widetilde{Z_1}^P < \widetilde{Z_2}^P$.

(ii) If $R^{PFN}(\widetilde{Z_1{}^P}) > R^{PFN}(\widetilde{Z_2{}^P})$ then $\widetilde{Z_1}^P > \widetilde{Z_2}^P$.

(iii) If $R^{PFN}(\widetilde{Z_1{}^P}) = R^{PFN}(\widetilde{Z_2{}^P})$ then $\widetilde{Z_1}^P = \widetilde{Z_2}^P$.

3.1 Pentagonal Fuzzy Fractional Solid Transportation Decomposition (PFFSTD) algorithm

Phase I: Preprocessing and Model Preparation

Step 1: Examine whether total source capacities equal total destination requirements over all routing layers. If imbalance exists, introduce suitable dummy sources, destinations, or routing levels with zero transportation cost to achieve equilibrium.

Step 2: Transform all pentagonal fuzzy cost parameters into equivalent crisp values using the proposed ranking technique. The resulting matrices form the crisp representation of the fractional solid transportation model.

Phase II: Numerator Optimization

Step 3: Extract the numerator cost matrix and construct the three-dimensional Source–Destination–Routing (SDR) tableau. Perform row and column reductions by subtracting the smallest element of each row and column, respectively, to generate the reduced cost structure.

Step 4: Decompose the reduced SDR tableau into interconnected two-dimensional projections, Source Capacity–Destination Requirement (SC–DR), Destination Requirement–Routing (DR–R) and Routing–Source Capacity (R–SC). For each sub-tableau, verify that marginal totals satisfy capacity and demand conditions. If inconsistencies exist, apply the stepping-stone adjustment procedure, cover zero entries with the minimum number of lines, modify uncovered entries using the smallest uncovered value and reconstruct and re-evaluate feasibility. Continue the refinement until all three-dimensional constraints are simultaneously satisfied.

Step 5: From the feasible reduced tableau, select the dimension containing the minimum number of zero-cost cells, allocate the maximum permissible quantity to the zero-cell having the smallest original transportation cost, update capacities and requirements accordingly. Repeat the allocation-adjustment cycle until all source capacities, destination requirements, and routing totals are completely satisfied. This yields the optimal numerator solution.

Phase III: Denominator Optimization

Step 6: Extract the denominator cost matrix and repeat the reduction, refinement, and allocation procedure described in Steps 3–5 to determine the optimal denominator solution.

Phase IV: Fractional Solution Construction

Step 7: Combine the optimized numerator and denominator allocations to obtain the complete optimal solution for the Pentagonal Fuzzy Fractional Solid Transportation Problem (PFFSTP).

4. Numerical Illustration:

A numerical example is provided to illustrate the step-by-step implementation of the proposed SDR-based algorithm for the Pentagonal Fuzzy Fractional Solid Transportation Problem. The example demonstrates defuzzification, tableau reduction, optimal allocation, and construction of the final fractional solution.

Table 1: The first criteria, the numerator of the Fuzzy Fractional Solid Transportation Problem.

Routing	R_1			R_1			R_1			$\tilde{e}_1$
		R_2			R_2			R_2		$\tilde{e}_2$
			R_3			R_3			R_3	$\tilde{e}_3$
	DR_1			DR_2			DR_3			Source capacity
SC_1	$\tilde{c}_{111}$	$\tilde{c}_{112}$	$\tilde{c}_{113}$	$\tilde{c}_{121}$	$\tilde{c}_{122}$	$\tilde{c}_{123}$	$\tilde{c}_{131}$	$\tilde{c}_{132}$	$\tilde{c}_{133}$	$\tilde{a}_1$
SC_2	$\tilde{c}_{211}$	$\tilde{c}_{212}$	$\tilde{c}_{213}$	$\tilde{c}_{221}$	$\tilde{c}_{222}$	$\tilde{c}_{223}$	$\tilde{c}_{231}$	$\tilde{c}_{232}$	$\tilde{c}_{233}$	$\tilde{a}_2$
SC_3	$\tilde{c}_{311}$	$\tilde{c}_{312}$	$\tilde{c}_{313}$	$\tilde{c}_{321}$	$\tilde{c}_{322}$	$\tilde{c}_{323}$	$\tilde{c}_{331}$	$\tilde{c}_{332}$	$\tilde{c}_{333}$	$\tilde{a}_3$
Destination requirement	$\tilde{b}_1$			$\tilde{b}_2$			$\tilde{b}_3$			

Table 2: The second criteria, the denominator of the Fuzzy Fractional Solid Transportation Problem.

Routing	R_1			R_1			R_1			$\tilde{e}_1$
		R_2			R_2			R_2		$\tilde{e}_2$
			R_3			R_3			R_3	$\tilde{e}_3$
	DR_1			DR_2			DR_3			Source capacity
SC_1	$\tilde{d}_{111}$	$\tilde{d}_{112}$	$\tilde{d}_{113}$	$\tilde{d}_{121}$	$\tilde{d}_{122}$	$\tilde{d}_{123}$	$\tilde{d}_{131}$	$\tilde{d}_{132}$	$\tilde{d}_{133}$	$\tilde{a}_1$
SC_2	$\tilde{d}_{211}$	$\tilde{d}_{212}$	$\tilde{d}_{213}$	$\tilde{d}_{221}$	$\tilde{d}_{222}$	$\tilde{d}_{223}$	$\tilde{d}_{231}$	$\tilde{d}_{232}$	$\tilde{d}_{233}$	$\tilde{a}_2$
SC_3	$\tilde{d}_{311}$	$\tilde{d}_{312}$	$\tilde{d}_{313}$	$\tilde{d}_{321}$	$\tilde{d}_{322}$	$\tilde{d}_{323}$	$\tilde{d}_{331}$	$\tilde{d}_{332}$	$\tilde{d}_{333}$	$\tilde{a}_3$

Destination requirement	$\tilde{b}_1$	$\tilde{b}_2$	$\tilde{b}_3$	

where $\tilde{c}_{111}$ =(1,3,6,8,13); $\tilde{c}_{112}$ = (2,5,7,11,14); $\tilde{c}_{113}$ = (0,4,5,9,15); $\tilde{c}_{121}$ = (3,4,8,10,17); $\tilde{c}_{122}$=(4,6,9,12,19); $\tilde{c}_{123}$=(5,7,10,13,19); $\tilde{c}_{131}$=(2,6,9,11,16) ; $\tilde{c}_{132}$=(1,5,8,12,17); $\tilde{c}_{133}$=(3,7,11,14,20); $\tilde{c}_{211}$=(6,8,12,15,21); $\tilde{c}_{212}$=(4,9,10,16,22); $\tilde{c}_{213}$=(5,10,13,17,23); $\tilde{c}_{221}$=(7,9,14,18,24); $\tilde{c}_{222}$=(2,7,9,15,19); $\tilde{c}_{223}$=(8,11,13,20,25); $\tilde{c}_{231}$=(6,10,15,19,26); $\tilde{c}_{232}$=(9,12,16,21,27); $\tilde{c}_{233}$=(3,8,12,18,24); $\tilde{c}_{311}$=(4,8,13,17,23); $\tilde{c}_{312}$=(10,12,17,22,28); $\tilde{c}_{313}$=(5,11,16,18,27); $\tilde{c}_{321}$=(7,13,15,21,29); $\tilde{c}_{322}$=(1,9,12,16,26); $\tilde{c}_{323}$=(11,14,18,23,30); $\tilde{c}_{331}$=(8,12,19,25,32); $\tilde{c}_{332}$=(6,13,17,24,31); $\tilde{c}_{333}$=(9,15,20,26,33); $\tilde{d}_{111}$ =(2,4,7,10,13); $\tilde{d}_{112}$= (3,5,9,11,15); $\tilde{d}_{113}$ = (1,6,8,12,14); $\tilde{d}_{121}$ = (4,7,10,13,16); $\tilde{d}_{122}$=(2,8,11,15,18; $\tilde{d}_{123}$=(5,9,12,14,19); $\tilde{d}_{131}$=(3,7,11,14,17) ; $\tilde{d}_{132}$=(6,8,13,17,21); $\tilde{d}_{133}$=(4,9,12,18,22); $\tilde{d}_{211}$=(7,10,14,19,24); $\tilde{d}_{212}$=(2,6,9,15,20); $\tilde{d}_{213}$=(8,11,13,18,25); $\tilde{d}_{221}$=(5,10,15,19,23); $\tilde{d}_{222}$=(9,12,16,20,26); $\tilde{d}_{223}$=(3,8,14,17,21); $\tilde{d}_{231}$=(10,13,17,22,27); $\tilde{d}_{232}$=(4,7,13,18,23); $\tilde{d}_{233}$=(6,9,14,19,25); $\tilde{d}_{311}$=(7,11,16,20,24); $\tilde{d}_{312}$=(8,12,15,19,26); $\tilde{d}_{313}$=(9,14,17,22,28); $\tilde{d}_{321}$=(5,10,13,17,22); $\tilde{d}_{322}$=(6,11,15,20,27); $\tilde{d}_{323}$=(4,8,12,16,21); $\tilde{d}_{331}$=(7,13,18,23,29); $\tilde{d}_{332}$=(3,9,14,20,24); $\tilde{d}_{333}$=(8,12,19,25,30); $\tilde{a}_1$=(6,8,9,10,12); $\tilde{a}_2$=(10,12,14,15,18); $\tilde{a}_3$=(13,15,17,18,20); $\tilde{b}_1$=(8,10,11,12,14); $\tilde{b}_2$=(9,11,12,13,15); $\tilde{b}_3$=(14,15,17,18,20); $\tilde{e}_1$=(7,9,10,11,13); $\tilde{e}_2$=(11,12,13,15,17); $\tilde{e}_3$=(14,15,17,18,20)

Solution:

The total availability from sources is 9+14+17=40, the total demand at destinations is 10+13+17=40, and the aggregate routing capacity is 11+12+17=40. Since all three totals are equal, the problem is balanced. If any mismatch occurs, appropriate dummy sources, destinations, or routing levels with zero transportation cost are introduced to restore equilibrium.

For computational purposes, the numerator cost matrix of the FSTP is taken as the initial working tableau. All pentagonal fuzzy cost coefficients are then transformed into crisp values using the proposed ranking method before proceeding with optimization.

Table 3: The crisp values of the numerator.

Routing	R_1			R_1			R_1			11
		R_2			R_2			R_2		12
			R_3			R_3			R_3	17
	DR_1			DR_2			DR_3			Source capacity
SC_1	7.23	8.66	8.04	10	1061	11.87	10.33	13	13	9
SC_2	14.89	10.5	15.35	14.33	16.77	11.31	17.89	13.02	14.72	14

SC_3	15.53	16.25	18.10	13.41	15.95	12.25	18	13.93	18.79	17
Destination requirement	10			13			17			40

Construct the three-dimensional Source–Destination–Routing (SDR) tableau from the given cost matrix. Normalize the tableau by deducting the smallest element in each row, and subsequently adjust each column by subtracting its minimum value. The resulting matrix represents the reduced cost structure used for further optimization.

Table 4: The reduced three-dimensional source-destination-routing (SDR) table.

Routing	R_1			R_1			R_1			11
		R_2			R_2			R_2		12
			R_3			R_3			R_3	17
	DR_1			DR_2			DR_3			Source capacity
SC_1	0	0	0	0	3.98	0	0	0	2.04	9
SC_2	2.05	3.86	2.19	1.75	0	1.98	2.50	4.48	0	14
SC_3	0.09	3.47	1.52	1.83	0	3.26	3.80	2.96	4.97	17
Destination requirement	10			13			17			40

The reduced SDR matrix is mapped onto the SC–DR projection and its marginal totals are checked against the prescribed supply and demand values. When imbalance persists, a stepping-stone adjustment is performed by covering all zero cells with the minimum number of lines and modifying the uncovered entries using the smallest uncovered value, thereby generating an improved reduced tableau.

Table 5: The source capacity–destination requirement (SC-DR) table.

	DR_1			DR_2			DR_3			Source capacity
SC_1	0	0	0	0	3.98	0	0	0	2.04	9
SC_2	2.05	3.86	2.19	1.75	0	1.98	2.50	4.48	0	14
SC_3	0.09	3.47	1.52	1.83	0	3.26	3.80	2.96	4.97	17
Destination requirement	10			13			17			40

From the optimized DR–R projection, derive the Routing–Source Capacity (R–SC) sub-tableau. The stepping-stone adjustment is then employed by covering all zero elements with the least number of horizontal and vertical lines. The minimum uncovered value is deducted from every uncovered cell and added to the entries at the line intersections to obtain an updated reduced form.

Table 6: The routing–source capacity (R-SC) table.

	R_1			R_2			R_3			Destination requirement
DR_1	2.87	3.17	0	0	2.11	0.51	1.44	1.88	0	10
DR_2	1.13	1.13	0	6.94	1.12	0	0	0.23	0.84	13
DR_3	0.25	0	1.54	0	2.73	0	3.52	0	3.74	17
Routing	11			12			17			40

Within the optimal reduced tableau, select the dimension (source, destination, or routing) containing the least number of zero-cost positions. Assign the maximum permissible quantity to the zero-cell corresponding to the minimum original transportation cost. In case of ties, adopt any consistent selection rule. Subsequently, revise the tableau by removing satisfied rows or columns and updating the remaining capacities and requirements until full allocation is achieved.

Table 7: The allocation table.

Routing	R_1			R_1			R_1			11
		R_2			R_2			R_2		12
			R_3			R_3			R_3	17
	DR_1			DR_2			DR_3			Source capacity
SC_1	2.87	0**(9)**	1.44	1.13	6.94	0	0.25	0	3.52	9
SC_2	3.17	2.11	1.88	1.13	1.12	0.23	0	2.73	0**(14)**	14
SC_3	0	0.51	0**(1)**	0	0**(13)**	0.84	1.54	0**(3)**	3.74	17
Destination requirement	10			13			17			40

Total Minimum cost of the numerator = 9×7.83+1×15.63+13×13.01+3×18.25+14×13.09 = 493.24

Denominator Optimization Phase

Extract the denominator cost matrix from the FSTP and repeat all the steps to obtain the optimal denominator solution.

Table 8: The crisp values of the denominator.

Routing	R_1			R_1			R_1			11
		R_2			R_2			R_2		12
			R_3			R_3			R_3	17
	DR_1			DR_2			DR_3			Source capacity
SC_1	6.40	7.83	7.5	8.79	10.38	9.58	8.87	8.68	11.13	9

SC_2	12.45	15.69	13.69	14.54	10.40	15.56	15.37	17.16	13.09	14
SC_3	13.1	17.91	15.63	17.23	13.01	19.45	19.28	18.25	20.67	17
Destination requirement	10			13			17			40

Consider the three-dimensional Source–Destination–Routing (SDR) cost structure. Standard reduction is carried out by normalizing each row with respect to its smallest element, followed by a similar normalization across columns. The resulting matrix represents the reduced cost tableau for subsequent optimization.

Table 9: The reduced three-dimensional source-destination-routing (SDR) table.

Routing	R_1			R_1			R_1			11
		R_2			R_2			R_2		12
			R_3			R_3			R_3	17
	DR_1			DR_2			DR_3			Source capacity
SC_1	0	0	0	0	0	0.56	0	0	0	9
SC_2	7.66	1.84	7.31	4.33	6.16	0	7.56	0.02	1.72	14
SC_3	8.30	7.59	10.06	3.41	5.34	0.94	7.67	0.93	5.7	17
Destination requirement	10			13			17			40

The reduced SDR matrix is projected onto the Source Capacity–Destination Requirement (SC–DR) plane, and its marginal totals are examined for consistency with the prescribed supply and demand values. If discrepancies arise, a stepping-stone adjustment is executed by covering the zero elements with the minimum number of lines and modifying the tableau using the smallest uncovered value, yielding an improved reduced matrix.

Table 10: The source capacity–destination requirement (SC-DR) table.

	DR_1			DR_2			DR_3			Source capacity
SC_1	0	0	0	0	0	0.56	0	0	0	9
SC_2	7.66	1.84	7.31	4.33	6.16	0	7.56	0.02	1.72	14
SC_3	8.30	7.59	10.06	3.41	5.34	0.94	7.67	0.93	5.7	17
Destination requirement	10			13			17			40

From the optimized SC–DR projection, derive the Destination Requirement–Routing (DR–R) sub-tableau. The reduced form is refined, when necessary, through a stepping-stone adjustment by covering all zero positions with the least number of lines and revising the tableau using the smallest uncovered value, thereby generating an updated reduced matrix.

Table 11: The destination requirement–routing (DR-R) table.

	R_1			R_2			R_3			Destination requirement
DR_1	0	5.82	5.55	0	0	4.84	0	5.47	7.31	10
DR_2	0	2.49	0.66	0	4.32	2.59	2.4	0	0.03	13
DR_3	0	5.72	4.92	1.82	0	0	0.12	0	3.07	17
Routing	11			12			17			40

Based on the optimized DR–R projection, formulate the Routing–Source Capacity (R–SC) sub-tableau. If further improvement is required, perform a stepping-stone refinement by covering all zero-cost positions with the minimum set of lines and adjusting the tableau using the smallest uncovered element, resulting in an enhanced reduced structure.

Table 12: The routing–source capacity (R-SC) table.

	SC_1			SC_2			SC_3			Routing
R_1	0	0	0	5.82	2.49	5.72	5.55	0.66	4.92	11
R_2	0	0	1.82	0	4.32	0	4.84	2.59	0	12
R_3	0	2.4	0.12	5.47	0	0	7.31	0.03	3.07	17
Source capacity	9			14			17			40

From the optimal reduced tableau, select the dimension (source, destination, or routing) containing the minimum number of zero-cost positions. Assign the largest allowable quantity to the zero-cell associated with the smallest original transportation cost. In the event of a tie, any consistent selection criterion may be adopted.

Table 13: The allocation table.

Routing	R_1			R_1			R_1			11
		R_2			R_2			R_2		12
			R_3			R_3			R_3	17
	DR_1			DR_2			DR_3			Source capacity
SC_1	0(**9**)	0.66	0.63	0	0.66	3.03	0	2.48	0.75	9
SC_2	5.16	0(**1**)	5.44	1.86	4.98	0(**13**)	5.09	0.03	0	14
SC_3	4.89	4.84	7.28	0	2.59	0	4.26	0(**17**)	3.04	17
Destination requirement	10			13			17			40

Total Minimum cost of the denominator = 7.23×9+10.5×1+11.31×13+13.93×17= 459.41

Optimum solution of the given Fuzzy Fractional Solid Transportation problem is $\frac{493.24}{459.41}$.

Conclusion

The proposed geometric ranking methodology establishes a consistent and mathematically sound framework for defuzzifying pentagonal fuzzy numbers, effectively resolving ambiguity in fuzzy comparisons within optimization settings. Leveraging this foundation, the PFFSTD algorithm systematically decomposes the fractional objective into independent numerator and denominator optimization phases, thereby simplifying computational complexity without disturbing the inherent three-dimensional source–destination–conveyance architecture. Through iterative tableau refinement and structured adjustment mechanisms, the method guarantees feasibility, balance, and convergence. Consequently, the integrated approach provides a scalable and computationally stable solution strategy for complex transportation systems operating under layered uncertainty and multi-objective environments, contributing significantly to advanced decision-support methodologies in operations research.

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