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Some theorems in Metrically Convex fuzzy metric Spaces

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Abstract

In this paper, we establish a fixed point theorem for generalized set-valued contraction in metrically convex fuzzy metric spaces has been proved which generalizes some existing fixed point theorem in metric spaces.

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1. Introduction and Preliminaries

George and Veeramani introduced the concept of fuzzy metric spaces in different ways. Kramosil and Michalek [7] and later Grabiec [3] obtained the fuzzy version of Banach contraction principle. Many authors proved fixed point theorems for contractive maps in fuzzy metric spaces. In 1986 Jungck [6] generalized the concept of commutativity by introducing compatibility. Mishra et al. [8] proved common fixed point theorems for compatible maps on fuzzy metric spaces. In this paper, we establish a Meir and Keeler type fixed point theorem for set-valued generalized contraction in metrically convex fuzzy metric spaces.

2. Preliminaries

Definition 2.1. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous t-norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever a $\leq$ c and b $\leq$ d for all $a, b, c, d \in [0, 1]$.

Example 2.2. $a * b = \min\{a, b\}$ and $a * b = a \cdot b$ are t-norms.

Definition 2.3. $*$ is said to be continuous if for any sequences $\{a_n\}$, $\{b_n\}$ in [0, 1] with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ implies

$$\lim_{n\to\infty} (a_n * b_n) = (a * b).$$

Definition 2.4. The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(1) M(x, y, 0) = 0,

(2) M(x, y, t) = 1 for all $t > 0$ if and only if x = y,

(3) M(x, y, t) = M(y, x, t)= 1 for all $x, y \in X$,

(4) M(x, z, t + s) $\geq$ M(x, y, t) $*$ M(y, z, s), where x, y, z $\in$ X, s,t $>$ 0.

(5) M(x, y, $\cdot$) : $X^2 \times [0, \infty) \to [0, 1]$ is left continuous.

Example 2.5. Let (X, d) be a metric space. Define a $*$ b = a + b for all a, b $\in$ X. Define $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all x, y $\in$ X and t $>$ 0. Then $(X, M, *)$ is a fuzzy metric space and this fuzzy metric induced by a metric d is called the standard fuzzy metric.

Definition 2.6. Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if

$$\lim_{t\to\infty} \left(\frac{1}{M(x_n, x_{n+p}, t)} - 1 \right) = 0 \ \text{ for all } \ t > 0 \ \text{ and } \ n, p \in N.$$

Definition 2.7. Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be convergent to a point x ∈ X if

$$\lim_{t\to\infty}\left(\frac{1}{M(x_n, x, t)} - 1\right) = 0 \text{ for all } t > 0.$$

Definition 2.8. A fuzzy metric space X is said to be complete if every Cauchy sequence in X converges to some point in X.

Definition 2.9. Let $(X, M, *)$ be a fuzzy metric space. We will say the mapping $T : X \to X$ is fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

for each $x, y \in X$ and $t > 0$. (k is called the contractive constant of T.)

Lemma 2.10. Let $\{x_n\}$ is a sequence in a fuzzy metric space X and if

$$\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) \leq k^n\left(\frac{1}{M(x_0, x_1, t)} - 1\right)$$

where $0 < k < 1,\ \ n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Suppose that

$$\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) \leq k^n\left(\frac{1}{M(x_0, x_1, t)} - 1\right)$$

where $0 < k < 1$, and $t \geq 0$.

Let m, n be two positive integers with $m \geq n$, say $m = n + p$, $p > 0$. Then we have

$$\begin{aligned}\left(\frac{1}{M(x_n, x_{n+p}, t)} - 1\right) &\leq \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) + \left(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1\right) + \cdots \\ &\quad + \left(\frac{1}{M(x_{n+p-1}, x_{n+p}, t)} - 1\right) \\ &\leq k^n\left(\frac{1}{M(x_0, x_1, t)} - 1\right) + k^{n+1}\left(\frac{1}{M(x_0, x_1, t)} - 1\right) + \cdots \\ &\quad + k^{n+p-1}\left(\frac{1}{M(x_0, x_1, t)} - 1\right)\end{aligned}$$

Taking limit as $n \to \infty$ on both sides, we get

$$\lim_{n\to\infty}\left(\frac{1}{M(x_n, x_{n+p}, t)} - 1\right) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. ■

Definition 2.11. Let $(X, M, *)$ be a fuzzy metric space. And the mapping $T : X \to X$ is fuzzy Meir and Keeler contractive if given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon < \left(\frac{1}{M(x, y, t)} - 1\right) < \epsilon + \delta \quad \text{implies} \quad \left(\frac{1}{M(Tx, Ty, t)} - 1\right) < \epsilon.$$

In this paper, we establish a Meir and Keeler type fixed point theorem for set-valued generalized contraction in metrically convex spaces is proved in metrically convex fuzzy metric space.

Note 2.12. In this paper we denote $\left(\frac{1}{M(x, y, t)} - 1\right)$ by $\varphi(x, y, t)$

3. Main results

We now state relevant definition and lemmas which are used in the sequel.

Definition 3.1. A fuzzy metric space $(X, M, *)$ is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$\varphi(x, y, t) = \varphi(x, z, t) + \varphi(z, y, t).$$

Lemma 3.2. Let K be a nonempty closed subset of a metrically convex metric space X. If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that

$$\varphi(x, y, t) = \varphi(x, z, t) + \varphi(z, y, t).$$

In what follows, $CB(X)$ denotes the set of all closed and bounded subsets of $(X, M, *)$, while $C(X)$ for collection of all compact subsets of $(X, M, *)$. Also H denotes the Hausdoraff distance between two sets.

Lemma 3.3. Let $A, B \in CB(X)$. Then for all $\epsilon > 0$ and $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$. If $A, B \in C(X)$, then one can choose $b \in B$ such that $d(a, b) \leq H(A, B)$.

Theorem 3.4. Let $(X, M, *)$ be a complete metrically fuzzy convex metric space and K a nonempty closed subset of X. Let $T : K \to C(X)$ be a set-valued map which satisfies

(i) $\varphi(Tx, Ty, t) \leq \Delta(x, y, t)$ where

$$\Delta(x, y, t) = k \max\left(\frac{\varphi(x, y, t)}{2}, \varphi(x, Tx, t), \varphi(y, Ty, t), \frac{\varphi(x, Ty, t) + \varphi(y, Tx, t)}{q}\right)$$

for all $x, y \in K$, with $x \neq y$, where $0 < k < 1$, $q \geq 1 + 2k$,

(ii) $Tx \in K$ for each $x \in \delta K$.

(iii) for a given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$, $\delta(\epsilon)$ being a nondecreasing function of ϵ such that $\epsilon \leq \Delta(x, y, t) < \epsilon + \delta \;\Rightarrow\; \varphi(Tx, Ty, t) < \epsilon$.

Then T has a fixed point in K.

Proof. Let $x_0 \in K$. Define $x_1^{'} \in Tx_0$. If $x_1^{'} \in K$ then set $x_1^{'} = x_1$. If $x_1^{'}$' $\notin K$ choose $x_1 \in \delta K$ so that

$$\varphi(x_0, x_1, t) + \varphi(x_1, x_1^{'}, t) = \varphi(x_0, x_1^{'}, t).$$

Then $x_1 \in K$. By using above Lemma, select $x_2^{'} \in Tx_1$ such that

$$\varphi(x_1^{'}, x_2^{'}, t) \leq \varphi(Tx_0, Tx_1, t).$$

If $x_2^{'} \in K$ then $x_2^{'} = x_2$. Otherwise choose $x_2 \in \delta K$ such that

$$\varphi(x_1, x_2, t) + \varphi(x_2, x_2^{'}, t) = \varphi(x_1, x_2^{'}, t).$$

Thus by induction, one obtains two sequences $\{x_n\}$ and $\{x_n^{'}\}$ such that

(i) $x_{n+1}^{'} \in Tx_n$;

(ii) $\varphi(x_{n+1}^{'}, x_n^{'}, t) \leq \varphi(Tx_n, Tx_{n-1}, t)$;

(iii) $x_{n+1}^{'} \in K \;\Rightarrow\; x_{n+1}^{'} = x_{n+1}$;

(iv) $x_{n+1}^{'} \notin K \;\Rightarrow\; x_{n+1} \in \delta K$ and

$$\varphi(x_n, x_{n+1}, t) + \varphi(x_{n+1}, x_{n+1}^{'}, t) = \varphi(x_n, x_{n+1}^{'}, t).$$

Now define

$$P = \{x_i \in \{x_n\} : x_i^{'} = x_i, i = 1, 2, 3, \ldots\}$$
$$Q = \{x_i \in \{x_n\} : x_i^{'} \neq x_i, i = 1, 2, 3, \ldots\}.$$

Obviously, the two consecutive terms cannot lie in Q.

Now we distinguish the following three cases.

Case. If $x_n, x_{n+1} \in P$, then

$$\begin{aligned}
\varphi(x_n, x_{n+1}, t) &\leq \varphi(Tx_{n-1}, Tx_n, t)\\
&\leq k \max\left\{\frac{\varphi(x_{n-1}, x_n, t)}{2}, \varphi(x_{n-1}, Tx_{n-1}, t), \varphi(x_n, Tx_n, t),\right.\\
&\qquad \left.\frac{\varphi(x_{n-1}, Tx_n, t) + \varphi(x_n, Tx_{n-1}, t)}{q}\right\}\\
&\leq k \max\left\{\frac{\varphi(x_{n-1}, x_n, t)}{2}, \varphi(x_{n-1}, x_n, t), \varphi(x_n, x_{n+1}, t),\right.\\
&\qquad \left.\frac{\varphi(x_{n-1}, x_{n+1}, t) + \varphi(x_n, x_n, t)}{q}\right\}\\
&\leq k \max\left(\varphi(x_{n-1}, x_n, t), \varphi(x_n, x_{n+1}, t)\right)
\end{aligned}$$

If $\varphi(x_{n-1}, x_n, t) \leq \varphi(x_n, x_{n+1}, t)$ then we get $\varphi(x_n, x_{n+1}, t) \leq \varphi(x_n, x_{n+1}, t)$ which is a contradiction. Otherwise, if $\varphi(x_n, x_{n+1}, t) \leq k\varphi(x_{n-1}, x_n, t)$ then one obtains $\varphi(x_n, x_{n+1}, t) \leq k\varphi(x_{n-1}, x_n, t)$.

Case. If $x_n \in P, x_{n+1} \in Q$, then

$$\varphi(x_n, x_{n+1}, t) + \varphi(x_{n+1}, x'_{n+1}, t) = \varphi(x_n, x'_{n+1}, t),$$

which in turn yields

$$\varphi(x_n, x'_{n+1}, t) \leq \varphi(x_n, x_{n+1}, t).$$

Now, proceeding as in case 3.5.1, we have

$$\varphi(x_n, x_{n+1}, t) \leq k\varphi(x_{n-1}, x_n, t).$$

Case. If $x_n \in Q$ and $x_{n+1} \in P$ then $x_{n-1} \in P$. Since x_n is a convex linear combination of x_{n-1} and x'_n, it follows that

$$\varphi(x_n, x_{n+1}, t) \leq \max\{\varphi(x_{n-1}, x_{n+1}, t), \varphi(x_{n+1}, x'_n, t)\}.$$

Now, if $\varphi(x_{n-1}, x_{n+1}, t) \leq \varphi(x'_n, x_{n+1}, t)$, then proceeding as in case 3.4.1, one obtains

$$\varphi(x_n, x_{n+1}, t) \leq k\varphi(x_{n-1}, x_n, t).$$

Otherwise if $\varphi(x'_n, x_{n+1}, t) \leq \varphi(x_{n-1}, x_{n+1}, t)$, then we have

$$\begin{aligned}
\varphi(x_n, x_{n+1}, t) &\leq \varphi(Tx_{n-2}, Tx_n, t) \\
&\leq k \max\left\{\frac{\varphi(x_{n-2}, x_n, t)}{2}, \varphi(x_{n-2}, Tx_{n-2}, t), \varphi(x_n, Tx_n, t),\right. \\
&\qquad \left.\frac{\varphi(x_{n-2}, Tx_n, t) + \varphi(x_n, Tx_{n-2}, t)}{q}\right\} \\
&\leq k \max\left\{\frac{\varphi(x_{n-2}, x_n, t)}{2}, \varphi(x_{n-2}, x_n, t), \varphi(x_n, x_{n+1}, t),\right. \\
&\qquad \left.\frac{\varphi(x_{n-2}, x_{n+1}, t) + \varphi(x_n, x_{n-1}, t)}{q}\right\}
\end{aligned}$$

Since

$$\frac{\varphi(x_{n-2}, x_n, t)}{2} = \max\{\varphi(x_{n-2}, x_{n-1}, t), \varphi(x_{n-1}, x_n, t)\}.$$

Therefore, one obtains

$$\begin{aligned}
\varphi(x_n, x_{n+1}, t) \leq k \max&\left\{\varphi(x_{n-2}, x_{n-1}, t), d(x_{n-1}, x_n, t), \varphi(x_n, x_{n+1}, t),\right. \\
&\left.\frac{\varphi(x_{n-2}, x_{n+1}, t) + \varphi(x_n, x_{n-1}, t)}{q}\right\}
\end{aligned}$$

which in turn yields

$$\varphi(x_n, x_{n+1}, t) = \begin{cases} k\varphi(x_{n-1}, x_n, t), & \text{if } \varphi(x_{n-1}, x_n, t) \geq \varphi(x_{n-2}, x_{n-1}, t); \\ k\varphi(x_{n-2}, x_{n-1}, t), & \text{if } \varphi(x_{n-1}, x_n, t) \leq \varphi(x_{n-2}, x_{n-1}, t); \end{cases}$$

Thus in all the cases, we have

$$\varphi(x_n, x_{n+1}, t) \leq k \max\{\varphi(x_{n-1}, x_n, t), \varphi(x_{n-2}, x_{n-1}, t)\}.$$

It can be easily shown by induction that for $n \leq 1$, we have

$$\varphi(x_n, x_{n+1}, t) \leq k \max\{\varphi(x_0, x_1, t), \varphi(x_1, x_2, t)\}.$$

Thus $\varphi(x_n, x_{n+1}, t)$ is a decreasing sequence and tending to $s \in [0, '\infty)$ as $n \to \infty$. Let on contrary

$$\varphi(x_n, x_{n+1}, t) > s \;\text{ for }\; n = 0, 1, 2... \tag{1}$$

Suppose $s > 0$. Then there exists a $\delta = \delta(A)$ and a positive integer k such that

$$s \leq \varphi(x_k, x_{k+1}, t) < \delta + s.$$

Hence by (1), one obtains

$$\varphi(x_{k+1}, x_{k+2}, t) = \varphi(Tx_k, Tx_{k+1}, t) < s,$$

which contradicts (2) therefore $\varphi(x_n, x_{n+1}, t) \to 0$ as $n \to \infty$. Now we wish to show that the sequence $\{x_n\}$ is Cauchy. If it is not Cauchy then there exists $2\epsilon > 0$ such that $\varphi(x_m, x_n, t) > 2\epsilon$. Choose $\delta > 0$ with $\delta < \epsilon$ for which (1) is satisfied. Since $\varphi(x_n, x_{n+1}, t) \to 0$ there exists a positive integer $N = N(\delta)$ such that $\varphi(x_i, x_{i+1}, t) \leq \frac{\delta}{6}$ for all $i \leq N$. With this choice of N, let us choose m, n with $m > n > N$ such that

$$\varphi(x_m, x_n, t) \geq 2\epsilon > \epsilon + \delta \tag{2}$$

By (3), $m - n > 6$, otherwise

$$\varphi(x_m, x_n, t) \leq \varphi(x_n, x_{n+1}, t) + \cdots + d(x_{n+4}, x_{n+5}, t) \leq \frac{5\delta}{6} < \delta,$$

a contradiction. Now suppose that $\varphi(x_n, x_{m-1}, t) \leq \epsilon + \frac{\delta}{3}$. Then

$$\varphi(x_n, x_m, t) \leq \varphi(x_n, x_{m-1}, t) + \varphi(x_{m-1}, x_m, t) \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \delta$$

a contradiction. Similarly, suppose $\varphi(x_n, x_{m-2}, t) \leq \epsilon + \frac{\delta}{3}$. Then

$$\begin{aligned} \varphi(x_n, x_m, t) &\leq \varphi(x_n, x_{m-2}, t) + d(x_{m-2}, x_{m-1}, t) + d(x_{m-1}, x_m, t) \\ &\leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} < \epsilon + \delta \end{aligned}$$

Let for the smallest integer $j \in (m, n)$ with $\varphi(x_n, x_j, t) > \epsilon + \frac{\delta}{3}$, whereas

$$\varphi(x_n, x_j, t) \leq \varphi(x_n, x_{j-1}, t) + \varphi(x_{j-1}, x_j, t) \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \frac{2\delta}{3}.$$

Thus there exists a $j \in (n, m)$ such that

$$\epsilon + \frac{\delta}{3} < \varphi(x_n, x_j, t) < \epsilon + \frac{2\delta}{3}.$$

Then

$$\begin{aligned}\varphi(x_n, x_j, t) &\leq \varphi(x_n, x_{n+1}, t) + \varphi(x_{n+1}, x_{j+1}, t) + d(x_{j+1}, x_j, t) \\ &\leq \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3},\end{aligned}$$

which is indeed a contradiction, therefore one may conclude that the sequence $\{x_n\}$ is Cauchy and it converges to a point $z \in X$.

Now, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is contained in P. Using (1), one can write

$$\begin{aligned}\varphi(Tx_{n_{k-1}}, Tz, t) \leq k \max\{&\frac{\varphi(x_{n_{k-1}}, z, t)}{2}, \varphi(x_{n_{k-1}}, Tx_{n_{k-1}}, t), \varphi(z, Tz, t), \\ &\frac{\varphi(z, Tx_{n_{k-1}}, t) + \varphi(x_{n_{k-1}}, Tz, t)}{q}\}\end{aligned}$$

which on letting $k \to \infty$' we get $\varphi(Tz, z, t) \leq k\varphi(Tz, z, t)$, yielding thereby $z = Tz$. This completes the proof. ■

Theorem 3.5. Let $(X, M, *)$ be a complete metrically fuzzy convex metric space and K a nonempty closed subset of X. Let $T : K \to C(X)$ be a set-valued map which satisfies

(i) $\varphi(Tx, Ty, t) \leq \Delta(x, y, t)$ where

$$\Delta(x, y, t) = k \max\left(\frac{\varphi(x, y, t)}{2}, \varphi(x, Tx, t), \varphi(y, Ty, t)\right)$$

for all $x, y \in K$, with $x \neq y$, where $0 < k < 1$,

(ii) $Tx \in K$ for each $x \in \delta K$.

(iii) for a given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$, $\delta(\epsilon)$ being a nondecreasing function of ϵ such that $\epsilon \leq \Delta(x, y, t) < \epsilon + \delta \;\Rightarrow\; \varphi(Tx, Ty, t) < \epsilon$.

Then T has a fixed point in K.

Proof. Let $x_0 \in K$. Define $x_1^{'} \in Tx_0$. If $x_1^{'} \in K$ then set $x_1^{'} = x_1$. If $x_1^{'}$' $\notin K$ choose $x_1 \in \delta K$ so that

$$\varphi(x_0, x_1, t) + \varphi(x_1, x_1^{'}, t) = \varphi(x_0, x_1^{'}, t).$$

Then $x_1 \in K$. By using above Lemma, select $x_2^{'} \in Tx_1$ such that

$$\varphi(x_1^{'}, x_2^{'}, t) \leq \varphi(Tx_0, Tx_1, t).$$

If $x_2^{'} \in K$ then $x_2^{'} = x_2$. Otherwise choose $x_2 \in \delta K$ such that

$$\varphi(x_1, x_2, t) + \varphi(x_2, x_2^{'}, t) = \varphi(x_1, x_2^{'}, t).$$

Thus by induction, one obtains two sequences $\{x_n\}$ and $\{x_n^{'}\}$ such that

(i) $x_{n+1}^{'} \in Tx_n$;

(ii) $\varphi(x_{n+1}^{'}, x_n^{'}, t) \leq \varphi(Tx_n, Tx_{n-1}, t)$;

(iii) $x_{n+1}^{'} \in K \Rightarrow x_{n+1}^{'} = x_{n+1}$;

(iv) $x_{n+1}^{'} \notin K \Rightarrow x_{n+1} \in \delta K$ and

$$\varphi(x_n, x_{n+1}, t) + \varphi(x_{n+1}, x_{n+1}^{'}, t) = \varphi(x_n, x_{n+1}^{'}, t).$$

Now define

$$P = \{x_i \in \{x_n\} : x_i^{'} = x_i, i = 1, 2, 3, ...\}$$

$$Q = \{x_i \in \{x_n\} : x_i^{'} \neq x_i, i = 1, 2, 3, ...\}.$$

Obviously, the two consecutive terms cannot lie in Q.

Now we distinguish the following three cases.

Case. If $x_n, x_{n+1} \in P$, then

$$\begin{aligned}
\varphi(x_n, x_{n+1}, t) &\leq \varphi(Tx_{n-1}, Tx_n, t)\\
&\leq k \max\left(\frac{\varphi(x_{n-1}, x_n, t)}{2}, \varphi(x_{n-1}, Tx_{n-1}, t), \varphi(x_n, Tx_n, t)\right)\\
&\leq k \max\left(\frac{\varphi(x_{n-1}, x_n, t)}{2}, \varphi(x_{n-1}, x_n, t), \varphi(x_n, x_{n+1}, t)\right)\\
&\leq k \max\left(\varphi(x_{n-1}, x_n, t), \varphi(x_n, x_{n+1}, t)\right)
\end{aligned}$$

If $\varphi(x_{n-1}, x_n, t) \leq \varphi(x_n, x_{n+1}, t)$ then we get $\varphi(x_n, x_{n+1}, t) \leq \varphi(x_n, x_{n+1}, t)$ which is a contradiction. Otherwise, if $\varphi(x_n, x_{n+1}, t) \leq k\varphi(x_{n-1}, x_n, t)$ then one obtains $\varphi(x_n, x_{n+1}, t) \leq k\varphi(x_{n-1}, x_n, t)$.

Case. If $x_n \in P, x_{n+1} \in Q$, then

$$\varphi(x_n, x_{n+1}, t) + \varphi(x_{n+1}, x_{n+1}^{'}, t) = \varphi(x_n, x_{n+1}^{'}, t),$$

which in turn yields

$$\varphi(x_n, x^{'}_{n+1}, t) \leq \varphi(x_n, x_{n+1}, t).$$

Now, proceeding as in case 3.5.1, we have

$$\varphi(x_n, x_{n+1}, t) \leq k\varphi(x_{n-1}, x_n, t).$$

Case. If $x_n \in Q$ and $x_{n+1} \in P$ then $x_{n-1} \in P$. Since x_n is a convex linear combination of x_{n-1} and $x^{'}_n$, it follows that

$$\varphi(x_n, x_{n+1}, t) \leq \max\{\varphi(x_{n-1}, x_{n+1}, t), \varphi(x_{n+1}, x^{'}_n, t)\}.$$

Now, if $\varphi(x_{n-1}, x_{n+1}, t) \leq \varphi(x^{'}_n, x_{n+1}, t)$, then proceeding as in case 3.5.1, one obtains

$$\varphi(x_n, x_{n+1}, t) \leq k\varphi(x_{n-1}, x_n, t).$$

Otherwise if $\varphi(x^{'}_n, x_{n+1}, t) \leq \varphi(x_{n-1}, x_{n+1}, t)$, then we have

$$\begin{aligned}\varphi(x_n, x_{n+1}, t) &\leq \varphi(Tx_{n-2}, Tx_n, t) \\ &\leq k \max\left(\frac{\varphi(x_{n-2}, x_n, t)}{2}, \varphi(x_{n-2}, Tx_{n-2}, t), \varphi(x_n, Tx_n, t)\right) \\ &\leq k \max\left(\frac{\varphi(x_{n-2}, x_n, t)}{2}, \varphi(x_{n-2}, x_n, t), \varphi(x_n, x_{n+1}, t)\right)\end{aligned}$$

Since

$$\frac{\varphi(x_{n-2}, x_n, t)}{2} = \max\{\varphi(x_{n-2}, x_{n-1}, t), \varphi(x_{n-1}, x_n, t)\}.$$

Therefore, one obtains

$$\varphi(x_n, x_{n+1}, t) \leq k \max\left(\varphi(x_{n-2}, x_{n-1}, t), d(x_{n-1}, x_n, t), \varphi(x_n, x_{n+1}, t)\right)$$

which in turn yields

$$\varphi(x_n, x_{n+1}, t) = \begin{cases} k\varphi(x_{n-1}, x_n, t), & \text{if } \varphi(x_{n-1}, x_n, t) \geq \varphi(x_{n-2}, x_{n-1}, t); \\ k\varphi(x_{n-2}, x_{n-1}, t), & \text{if } \varphi(x_{n-1}, x_n, t) \leq \varphi(x_{n-2}, x_{n-1}, t); \end{cases}$$

Thus in all the cases, we have

$$\varphi(x_n, x_{n+1}, t) \leq k \max\{\varphi(x_{n-1}, x_n, t), \varphi(x_{n-2}, x_{n-1}, t)\}.$$

It can be easily shown by induction that for $n \leq 1$, we have

$$\varphi(x_n, x_{n+1}, t) \leq k \max\{\varphi(x_0, x_1, t), \varphi(x_1, x_2, t)\}.$$

Thus $\varphi(x_n, x_{n+1}, t)$ is a decreasing sequence and tending to $s \in [0, '\infty)$ as $n \to \infty$. Let on contrary

$$\varphi(x_n, x_{n+1}, t) > s \ \text{ for } \ n = 0, 1, 2... \tag{3}$$

Suppose $s > 0$. Then there exists a $\delta = \delta(A)$ and a positive integer k such that

$$s \leq \varphi(x_k, x_{k+1}, t) < \delta + s.$$

Hence by (1), one obtains

$$\varphi(x_{k+1}, x_{k+2}, t) = \varphi(Tx_k, Tx_{k+1}, t) < s,$$

which contradicts (2) therefore $\varphi(x_n, x_{n+1}, t) \to 0$ as $n \to \infty$. Now we wish to show that the sequence $\{x_n\}$ is Cauchy. If it is not Cauchy then there exists $2\epsilon > 0$ such that $\varphi(x_m, x_n, t) > 2\epsilon$. Choose $\delta > 0$ with $\delta < \epsilon$ for which (1) is satisfied. Since $\varphi(x_n, x_{n+1}, t) \to 0$ there exists a positive integer $N = N(\delta)$ such that $\varphi(x_i, x_{i+1}, t) \leq \frac{\delta}{6}$ for all $i \leq N$. With this choice of N, let us choose m, n with $m > n > N$ such that

$$\varphi(x_m, x_n, t) \geq 2\epsilon > \epsilon + \delta \tag{4}$$

By (3), $m - n > 6$, otherwise

$$\varphi(x_m, x_n, t) \leq \varphi(x_n, x_{n+1}, t) + \cdots + d(x_{n+4}, x_{n+5}, t) \leq \frac{5\delta}{6} < \delta,$$

a contradiction. Now suppose that $\varphi(x_n, x_{m-1}, t) \leq \epsilon + \frac{\delta}{3}$. Then

$$\varphi(x_n, x_m, t) \leq \varphi(x_n, x_{m-1}, t) + \varphi(x_{m-1}, x_m, t) \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \delta$$

a contradiction. Similarly, suppose $\varphi(x_n, x_{m-2}, t) \leq \epsilon + \frac{\delta}{3}$. Then

$$\begin{aligned}\varphi(x_n, x_m, t) &\leq \varphi(x_n, x_{m-2}, t) + d(x_{m-2}, x_{m-1}, t) + d(x_{m-1}, x_m, t)\\ &\leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} < \epsilon + \delta\end{aligned}$$

Let for the smallest integer $j \in (m, n)$ with $\varphi(x_n, x_j, t) > \epsilon + \frac{\delta}{3}$, whereas

$$\varphi(x_n, x_j, t) \leq \varphi(x_n, x_{j-1}, t) + \varphi(x_{j-1}, x_j, t) \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \frac{2\delta}{3}.$$

Thus there exists a $j \in (n, m)$ such that

$$\epsilon + \frac{\delta}{3} < \varphi(x_n, x_j, t) < \epsilon + \frac{2\delta}{3}.$$

Then

$$\begin{aligned}\varphi(x_n, x_j, t) &\leq \varphi(x_n, x_{n+1}, t) + \varphi(x_{n+1}, x_{j+1}, t) + d(x_{j+1}, x_j, t)\\ &\leq \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3},\end{aligned}$$

which is indeed a contradiction, therefore one may conclude that the sequence $\{x_n\}$ is Cauchy and it converges to a point $z \in X$.

Now, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is contained in P. Using (1), one can write

$$\varphi(Tx_{n_{k-1}}, Tz, t) \leq k \max\left(\frac{\varphi(x_{n_{k-1}}, z, t)}{2}, \varphi(x_{n_{k-1}}, Tx_{n_{k-1}}, t), \varphi(z, Tz, t)\right)$$

which on letting $k \to \infty$' we get $\varphi(Tz, z, t) \leq k\varphi(Tz, z, t), \Rightarrow z = Tz$. ■

References

[1] Badard, R., Fixed point theorems for fuzzy numbers, Fuzzy Sets Systems 13 (1984), 291–302.

[2] Butnariu, D., Fixed points for Fuzzy mappings, Fuzzy Sets Systems 7 (1982), 191–207.

[3] Grabiec, M., Fixed points in fuzzy metric spaces, Fuzzy Sets Systems 27 (1988), 385–389.

[4] George, A. and Veeramai, P., on some result in fuzzy metric space, Fuzzy Sets and System, 64 (1994), 395–399.

[5] Iseii, K., Some applications of Banach type contraction principles, Math. Sere. Notes Kobe Univ. 4 (1976), 211–214.

[6] Jungck, G., Compatible mappings and common fixed points, Inter'nat. J. Math. and Math. Sci. 9 (1986), 771–779.

[7] Kramosil, I. and Michalek, J., Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 326–334.

[8] Mishra, S. N., Sharma, N. and Singh, S. L., Common fixed points of maps on fuzzy metric spaces, Int. J. Math. Math. Sci. 17(1994), 253–258.

[9] Pant, R.P., Common fixed points of non commuting mappings, J. Math. Anal. Appl. 188(1994), 436–440.

[10] Rhoades, B.E., A fixed point theorem for some non-self mappings, Math. Japonica. 23 (4) (1978) 457–459.

[11] Weiss, M.D., Fixed points, separation and induced topologies for fuzzy sets, J. Math. Anal. Appl. 50 (1975), 142–150.

[12] Zadeh, L. A., Fuzzy sets, Information & Control 8(1965), 338–353.