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Characterization of Some Fuzzy Subsets of Fuzzy Ideal Topological Spaces and Decomposition of Fuzzy Continuity

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Abstract

In this paper some fuzzy subsets like fuzzy regular-*I*-closed set, fuzzy A_I-set, fuzzy I-locally closed set, fuzzy f_I-set of fuzzy ideal topological spaces are studied and characterized. We also characterized $\mathcal{T}$-codense fuzzy ideals in terms of fuzzy A_I-set, fuzzy regular-*I*-closed set, fuzzy I-locally closed set and we obtain a decomposition of fuzzy continuity.

Keywords: Fuzzy regular-*I*-closed set, fuzzy A_I-set, fuzzy I-locally closed set, $\mathcal{T}$-codense fuzzy ideal.

AMS Mathematics Subject Classification (2000): Primary 54A40; Secondary; 54A05, 54A10, 54C08, 54C10, 54F65

Introduction
The concept of fuzzy sets and fuzzy set operations were first introduced by L.A. Zadeh in his classical paper [31] in the year 1965. Subseqently many researchers have worked on various basic concepts from general topology using fuzzy sets and developed the theory of fuzzy topology. In recent years fuzzy topology has been

found to be very useful in solving many practical problems. The notions of the sets and functions in fuzzy topological spaces are used extensively in many engineering problems, computational topology for geometric design, computer-aided geometric design, engineering design research and mathematical sciences. El-Naschie [9,10] has shown that the notion of fuzzy topology may be applicable to quantum physics in connection with string theory and e^{∞} theory. Shihong Du. et al. [6] are currently working to fuzzify the 9-intersection Egenhofer model [7,8] for describing topological relations in geographic Information System(GIS) query. X. Tang [26] has used a slightly changed version of Chang`s [4] fuzzy topological spaces to model spatial objects for GIS database and Structured Query Language (SQL) for GIS. The concept of an ideal in topological spaces have been considered since 1930. This topic has won its importance by the paper of R. Vaidyanathaswamy [27,28] and K. Kuratowski [18]. Recently extensive studies on the importance of ideal in general topology have been done in this branch by Hamlett, Janković, Natkaniec, Dontchev, Ganster and Rose [5,13,14,17,22,24] and others. In 1997, Mahmoud [19] and Sarkar [25] independently presented some of the ideal concepts in the fuzzy trend and studied many other properties. The notion of continuity is an important concept in general topology as well as in fuzzy topology. Different types of generalizations of continuous functions were introduced and studied by various authors in the recent development of general topology and fuzzy topology. Decomposition of fuzzy continuity is one of the many problems in the fuzzy topology. It becomes very interesting when decomposition is done via fuzzy topological ideals. In this paper, we obtain a decomposition of fuzzy continuity in fuzzy topological spaces by using fuzzy α-I-open sets [30], fuzzy pre-I-open sets [20] and fuzzy I-locally closed set [1].

Preliminaries

Throughout this paper by $(X, \mathcal{T})$ we mean a fuzzy topological space in Chang`s [4] sense. The basic fuzzy sets are the empty set, the whole set and the class of all fuzzy sets in X which will be denoted by 0_X, 1_X and I^X respectively. The collection of all fuzzy open sets containing x will be denoted by $\mathcal{T}(x)$. A fuzzy point [29] in X with support $x \in X$ and value $\alpha(0 < \alpha \leq 1)$ is denoted by x_α. For a fuzzy set A in X, $cl(A)$, $int(A)$ and $1 - A$ will respectively denote the closure, interior and complement of A. A nonempty collection I of fuzzy sets in X is called a fuzzy ideal [19,25] if and only if (1) $B \in I$ and $A \leq B$ then $A \in I$ (heredity), (2) If $A \in I$ and $B \in I$ then $A \vee B \in I$ (finite aditivity). The triple $(X, \mathcal{T}, I)$ denotes a fuzzy topological space with a fuzzy ideal I and a fuzzy topology $\mathcal{T}$. For $(X, \mathcal{T}, I)$, the fuzzy local function of a fuzzy set A in X with respect to $\mathcal{T}$ and I is donoted by $A^*(\mathcal{T},I)$ (briefly A^*) and is defined as $A^*(\mathcal{T},I) = \bigvee\{\, x \in X\colon A \wedge U \notin I$ for every $U \in \mathcal{T}(x)\}$. While $A^*$is the union of the fuzzy points x_α such that if $U \in \mathcal{T}(x)$ and $E \in I$, then there is atleast one $y \in X$ for which $U(y) + A(y)$-1 $> E(y)$. Fuzzy closure operator of a fuzzy set A in $(X, \mathcal{T}, I)$ is defined as $Cl^*(A) = A \vee A^*$. In $(X, \mathcal{T}, I)$ the collection $\mathcal{T}^*(\mathfrak{I})$ means an extension of fuzzy topological space than $\mathcal{T}$ via fuzzy ideal which is constructed by considering the class $\beta = \{U - E\colon U \in \mathcal{T}, E \in I\}$ as a base [25].
First we recall some definitions used in sequel.

***Definition* 2.1.** [2] A fuzzy subset A of a fuzzy topological space $(X, \mathcal{T})$ is said to be fuzzy regular open (resp. fuzzy regular closed) if $A = Int(Cl(A))$(resp. $A = Cl(Int(A))$).

The family of all fuzzy regular open (resp. fuzzy regular closed) sets in X is denoted by FRO(X) (resp.FRC(X)).

From [2, 3,12] we have the following definitions.

***Definition* 2.2.** A fuzzy subset A of a fuzzy topological space $(X, \mathcal{T})$ is said to be fuzzy α-open (resp. fuzzy semiopen, fuzzy preopen, fuzzy β-open) if $A \leq Int(Cl(Int(A)))$ (resp. $A \leq Cl(Int(A))$, $A \leq Int(Cl(A))$, $A \leq Cl(Int(Cl(A)))$).

The family of all fuzzy α-open (resp. fuzzy semiopen, fuzzy preopen) sets in $(X, \mathcal{T})$ is denoted by FαO(X) (resp.FSO(X), FPO(X)).

From [11,23] we have the following definitions.

***Definition* 2.3.** A fuzzy subset A of a fuzzy topological space $(X, \mathcal{T})$ is said to be fuzzy locally closed set (resp. fuzzy A-set) if $A = U \wedge V$, where $U \in \mathcal{T}$ and V is a fuzzy closed (resp. fuzzy regular closed).

The family of all fuzzy locally closed set (resp. fuzzy A-set) is denoted by FLC(X) (resp.FA(X)).

***Definition* 2.4.** [21] A fuzzy subset A of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is said to be fuzzy I-open if $A \leq Int(A^*)$.

The family of all fuzzy I-open sets in X is denoted by FIO(X).

***Definition* 2.5.** [1] A fuzzy subset A of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is said to be

- fuzzy *-dense-in-itself if $A \leq A^*$
- fuzzy $\mathcal{T}^*$-closed if $A^* \leq A$
- fuzzy *-perfect if $A = A^*$.

***Definition* 2.6.** A fuzzy subset A of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is said to be

- fuzzy α-I-open [30] if $A \leq Int(Cl^*(Int(A)))$.
- fuzzy semi-I-open [15] if $A \leq Cl^*(Int(A))$.
- fuzzy pre-I-open [20] if $A \leq Int(Cl^*(A))$.

The family of all fuzzy α-I-open(resp. fuzzy semi-I-open, fuzzy pre-I-open) sets is denoted by FαIO(X)(resp. FSIO(X), FPIO(X)).

***Lemma* 2.1.** [25] Let $(X, \mathcal{T}, I)$ be any fuzzy ideal topological space and A, B be fuzzy subsets of X. Then the following properties hold.

- If $A \leq B$, then $A^* \leq B^*$.
- $A^* = Cl(A^*) \leq Cl(A)$.
- If $U \in \mathcal{T}$, then $U \wedge A^* \leq (U \wedge A)^*$.

- $(A^*)^* \leq A^*$.
- $(A \vee B)^* = A^* \vee B^*$.

***Definition* 2.7.** [21] In a fuzzy ideal topological space $(X, \mathcal{T}, I), I$ is said to be $\mathcal{T}$-codense if $I \wedge \mathcal{T} = \{0_X\}$.

***Lemma* 2.2.** [21] For a fuzzy ideal topological space $(X, \mathcal{T}, I)$, I is $\mathcal{T}$-codense if and only if
$1_X = 1_X{}^*$.

We now introduce the following lemmas.

***Lemma* 2.3.** For a fuzzy ideal topological space $(X, \mathcal{T}, I)$, the following are equivalent.

- I is $\mathcal{T}$-codense.
- $1_X = 1_X{}^*$.
- For every $A \in \mathcal{T}$, $A \leq A^*$.
- For every $A \in$ FSO(X), $A \leq A^*$.
- For every fuzzy regular closed set C, $C = C^*$.

Proof: (1) $\Leftrightarrow$ (2) Follows from lemma 2.2. (2) $\Longrightarrow$ (3) Obvious. (3) $\Longrightarrow$ (4) Let $A \in$ FSO(X). Then there exists a fuzzy open set H in X such that $H \leq A \leq Cl(H)$. For any fuzzy subset H in X, by lemma 2.1, we have $H^* = Cl(H^*) \leq Cl(H)$. Since H is fuzzy open, $H \leq H^*$ and so $H^* = Cl(H^*) = Cl(H)$. Therefore, $A \leq Cl(H) = Cl(H^*) = H^* \leq A^*$ which implies that $A \leq A^*$.

(4) $\Longrightarrow$ (5) If C is fuzzy regular closed, then C is fuzzy semiopen and fuzzy closed. Since C is fuzzy semiopen by hypothesis $C \leq C^*$. Since C is fuzzy closed, C is fuzzy $\mathcal{T}^*$-closed and so $C = Cl^*(C) = C \vee C^*$ which implies that $C^* \leq C$. Hence $C = C^*$. (5) $\Longrightarrow$ (1) Straightforward.

***Lemma* 2.4.** Let $(X, \mathcal{T}, I)$ be a fuzzy ideal topological space. If a fuzzy subset A of X is fuzzy *-dense-in-itself, then $A^* = Cl(A) = Cl^*(A)$.

Proof: As A is fuzzy *-dense-in-itself, $A \leq A^*$. By lemma 2.1, for any fuzzy subset A of X we have $A^* = Cl(A^*) \leq Cl(A)$. As $A \leq A^*$, $Cl(A) \leq Cl(A^*)$ and so $A^* = Cl(A^*) = Cl\ (A)$. Also $Cl(A) = Cl^*(A)$. Hence $A^* = Cl(A) = Cl^*(A)$.

***Lemma* 2.5.** Let $(X, \mathcal{T}, I)$ be a fuzzy ideal topological space and let $\Delta = \{\ A: A$ is fuzzy subset of X and $A \leq A^*\ \}$. Then $\Delta \wedge I = \{0_X\}$.

Proof: Let $A \in \Delta \wedge I$. Then $A \in I$ which implies $A^* = 0_X$ and $A \in \Delta$ which implies $A \leq A^*$. Therefore, $A = 0_X$. Hence $\Delta \wedge I = \{0_X\}$.

Fuzzy regular-I-closed set, fuzzy A_I-set, fuzzy I-locally closed set, fuzzy f_I-set.

In this section fuzzy regular I-closed set, fuzzy A_I-set, fuzzy I-locally closed set,fuzzy f_I-set are studied and characterized. We also characterized $\mathcal{T}$-codense fuzzy ideals in terms of these sets.

***Definition* 3.1.** [1] A fuzzy subset A of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is said to be fuzzy regular-I-closed if $A = (Int(A))*$.

***Definition* 3.2.** [1] A fuzzy subset A of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is said to be fuzzy A_I-set (resp. fuzzy I-locally closed set) if $A = U \wedge V$, where $U \in \mathcal{T}$ and V is a fuzzy regular-I-closed (resp. fuzzy *-perfect) set.

The family of all fuzzy A_I-set (resp. fuzzy I-locally closed set) is denoted by F$A_I(X)$(resp.FILC(X)).

***Lemma* 3.1.** Let $(X, \mathcal{T}, I)$ be a fuzzy ideal topological space. A fuzzy subset A of X is fuzzy I-locally closed if and only if $A = U \wedge A*$ for some $U \in \mathcal{T}$.

Proof: Let A is fuzzy I-locally closed set in $(X, \mathcal{T}, I)$. Then $A = U \wedge V$, where $U \in \mathcal{T}$ and V is a fuzzy *-perfect set. That is, $V = V*$. Then $A \leq V$ and so by lemma 2.1, $A*$ $\leq V*$. Also $A* = (U \wedge V)* \geq\ U \wedge V* = U \wedge V = A$ and so $A = A \wedge A* = (U \wedge V) \wedge A* = U \wedge (V \wedge A*) = U \wedge A*$.

***Theorem* 3.1.** Every fuzzy A_I-set in a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is fuzzy I-locally closed.

Proof: Let A is fuzzy A_I-set in $(X, \mathcal{T}, I)$. Then $A = U \wedge V$, where $U \in \mathcal{T}$ and V is a fuzzy regular-I-closed) set. Then $V = (Int(V))*$. Since $Int(V) \leq V$ by lemma 2.1, $(Int(V))* \leq V*$. Then we have $V = (Int(V))* \leq V*$. Again as $V = (Int(V))*$, $V*$ $=\ ((Int(V))*)* \leq (Int(V))* = V$. Therefore, we obtain $V = V*$ which implies that V is a fuzzy *-perfect set and consequently A is fuzzy I-locally closed.

***Remark* 3.1.** From the theorem 3.1 it is clear that every fuzzy regular-I-closed set is fuzzy *-perfect.

But the converse is not true and is justified by the following example.

***Example* 3.1.** Let $X = \{\, x, y, z \,\}$ and let A, B be fuzzy subsets of X defined as follows:

- $A(x) = 0.3$, $A(y) = 0.5$, $A(z) = 0.6$
- $B(x) = 0.7$, $B(y) = 0.5$, $B(z) = 0.4$

Let $\mathcal{T} = \{\, 0_X,\ B, 1_X \,\}$ and $I = \{0_X\}$. Then A is fuzzy *-perfect set but not fuzzy regular-I-closed.

***Remark* 3.2.** From the example 3.1 it is clear that fuzzy I-locally closed set may not imply fuzzy A_I-set.

***Theorem* 3.2.** Let A is fuzzy open set in X. A is fuzzy A_I-set if and only if A is fuzzy I-locally closed.

Proof: Let A is fuzzy A_I-set in $(X, \mathcal{T}, I)$. Then by theorem 3.1 A is fuzzy I-locally closed. Conversely, let A is fuzzy I-locally closed and fuzzy open. Then by lemma 3.1, $A = U \wedge A*$ for some $U \in \mathcal{T}$ and so $A \leq A*$. Since $Int(A*) \leq A*$, by lemma 2.1, we have $(Int(A*))* \leq (A*)* \leq A*$. Therefore, $(Int(A*))* \leq A*$. On the other hand, as A is fuzzy open and $A \leq A*$, $A*$=$(Int(A))* \leq (Int(A*))*$. Hence $A* = (Int(A*))*$ which implies that $A*$ is fuzzy regular-I-closed. Therefore, A is fuzzy A_I-set.

***Theorem* 3.3.** A fuzzy subset A in a fuzzy topological space $(X,\mathcal{T})$ is fuzzy A-set if and only if A is fuzzy semiopen and fuzzy locally closed.

Proof: Let A be fuzzy A-set in a fuzzy topological space $(X,\mathcal{T})$. Then $A = U \wedge F$, where $U \in \mathcal{T}$ and F is a fuzzy regular closed in X. Obviously, A is fuzzy locally closed. Also $Int(A) = U \wedge Int(F)$. So $A = U \wedge Cl\ (Int(F)) \leq Cl\ (U \wedge Int(F)) = Cl\ (Int(A)$ and hence A is fuzzy semiopen. Conversely, let A is fuzzy semiopen and fuzzy locally closed. Then $A \leq Cl\ (Int(A)$ and $A = U \wedge Cl(A)$, where $U \in \mathcal{T}$. Then $Cl(A) = Cl\ (Int(A))$ and so is fuzzy regular closed. Hence A is fuzzy A-set.

We now introduce the following definition.

***Definition* 3.3.** A fuzzy subset A of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is said to be fuzzy f_I-set if $A \leq (Int(A))*$.

The family of all fuzzy f_I-set in X is denoted by F$f_I(X)$.

***Theorem* 3.4.** If A is a fuzzy A_I-set in fuzzy ideal topological space $(X, \mathcal{T}, I)$ then the following hold.

- A and $Int(A)$ are fuzzy *-dense-in-itself.
- $A* = Cl\ (A) = Cl*(A)$ and $(Int(A))* = Cl\ (Int(A))$.
- A is a fuzzy f_I-set.
- $A* = (Int(A))* = ((Int(A))*)* = (A*)*$.
- $A*$ and $Int(A))*$ are fuzzy *-perfect and fuzzy I-locally closed.
- $A*$ is fuzzy regular-I-closed.

Proof:

1. Let A is a fuzzy A_I-set in fuzzy ideal topological space $(X, \mathcal{T}, I)$. Then $A = U \wedge V$, where $U \in \mathcal{T}$ and V is a fuzzy regular-I-closed. That is, $V = (Int(V))*$. Therefore, $A = U \wedge V = U \wedge (Int(V))* \leq (U \wedge Int(V))* = (Int(U \wedge V))* = (Int(A))* \leq A*$ which implies that $Int(A) \leq A \leq (Int(A))* \leq A*$.Therefore, A and $Int(A)$ are fuzzy *-dense-in-itself.
2. By lemma 2.4, we have $A* = Cl(A) = Cl*(A)$ and $(Int(A))* = Cl\ (Int(A))$.

3. From 1, $A \leq (Int(A))^*$ and so A is a fuzzy f_I-set.
4. From 1, We have $Int(A) \leq A \leq (Int(A))^* \leq A^*$ and so $(Int(A))^* \leq A^* \leq ((Int(A))^*)^* \leq (Int(A))^* \leq A^*$ and hence $A^* = (Int(A))^* = ((Int(A))^*)^* = (A^*)^*$.
5. From 4, It follows that A^* and $(Int(A))^*$ are fuzzy *-perfect and hence are fuzzy I-locally closed.
6. From 4, We have $A^* = (Int(A))^*$. Let $B = (Int(A))^*$. Then $(Int(B))^* = (Int(Int(A))^*)^* = (Int(A^*))^* \geq (Int(A))^* = B$, since A is fuzzy *-dense-in-itself. Therfore, $B \leq (Int(B))^*$. Also $Int(B) \leq B$ implies that $(Int(B))^* \leq B^* = ((Int(A))^*)^* \leq (Int(A))^* = B$. So $(Int(B))^* \leq B$. Therefore, we obtain $B = (Int(B))^*$ which implies that B is fuzzy regular-I-closed. Hence A^* is fuzzy regular-I-closed.

***Theorem* 3.5.** In any fuzzy ideal topological space $(X, \mathcal{T}, I)$, $\mathrm{F}A_I(X) \wedge I = \{0_X\}$.

Proof: Let $A \in \mathrm{F}A_I(X) \wedge I$ Then $A \in \mathrm{F}A_I(X)$. Then by theorem 3.4(1), A is fuzzy *-dense-in-itself. That is, $A \leq A^*$. Then by lemma 2.5, $\mathrm{F}A_I(X) \wedge I = \{0_X\}$.

***Theorem* 3.6.** A fuzzy subset A of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is fuzzy A_I-set if and only if A is both fuzzy f_I-set and fuzzy I-locally closed set.

Proof: Let A be a fuzzy A_I-set. Then by theorem 3.1, A is fuzzy I-locally closed set. Also $A = U \wedge V$, where $U \in \mathcal{T}$ and V is a fuzzy regular-I-closed set. That is, $V = (Int(V))^*$. So $Int(A) = Int(U \wedge V) = U \wedge (Int(V))$. Now $A = U \wedge V$ implies that $A = U \wedge (Int(V))^* \leq (U \wedge Int(V))^* = (Int(A))^*$ which implies that $A \leq (Int(A))^*$. Therefore, A is fuzzy f_I-set. Conversely, let A is both fuzzy f_I-set and fuzzy I-locally closed set. Since A is fuzzy f_I-set, $A \leq (Int(A))^*$ which implies that $A^* \leq ((Int(A))^*)^* \leq (Int(A))^* \leq A^*$ and so $A^* = (Int(A))^*$. Then by theorem 3.4(6), A^* is fuzzy regular-I-closed set. As A is fuzzy I-locally closed, by lemma 3.1, $A = U \wedge A^*$ for some $U \in \mathcal{T}$. Since A^* is fuzzy regular-I-closed, it follows that A is fuzzy A_I-set.

***Theorem* 3.7.** Let $(X, \mathcal{T}, I)$ be a fuzzy ideal topological space and let $A \in \mathcal{T}$. Then A is fuzzy f_I-set if and only if A is fuzzy A_I-set.

Proof: Let $A \in \mathcal{T}$ and let A is fuzzy f_I-set. Then $A \leq (Int(A))^* \leq A^*$ and so $A^* = (Int(A))^*$. Then by theorem 3.4(6), A^* is fuzzy regular-I-closed set. Since $A = A \wedge A^*$, it follows that A is fuzzy A_I-set. Conversely, let A is fuzzy A_I-set. Then by theorem 3.6, A is fuzzy f_I-set.

***Theorem* 3.8.** In any fuzzy ideal topological space $(X, \mathcal{T}, I)$, $\mathrm{F}f_I(X) \wedge I = \{0_X\}$.

Proof: Let $A \in \mathrm{F}f_I(X) \wedge I$. Then $A \in \mathrm{F}f_I(X)$ and $A \in I$. Now $A \in I \Longrightarrow A^* = 0_X$ and $A \in \mathrm{F}f_I(X) \Longrightarrow A \leq (Int(A))^* \leq A^* = 0_X$.Therefore, we obtain $A = 0_X$ and this completes the proof.

***Theorem* 3.9.** A fuzzy subset A of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is fuzzy regular-*I*-closed if and only if A is both fuzzy f_I-set and fuzzy $\mathcal{T}^*$-closed set.

Proof: Let A is fuzzy regular-*I*-closed set in $(X, \mathcal{T}, I)$. Then $A = (Int(A))^*$. Therefore, we have $A \leq (Int(A))^*$ and hence A is fuzzy f_I-set. Since $Int(A) \leq A$, $(Int(A))^* \leq A^*$. Then we have $A = (Int(A))^* \leq A^*$. Since $A = (Int(A))^*$, $A^* = ((Int(A))^*)^* \leq (Int(A))^* = A$. Therefore, we obtain $A = A^*$ and hence $A^* \leq A$. So A is fuzzy $\mathcal{T}^*$-closed set. Conversely, let A is both fuzzy f_I-set and fuzzy $\mathcal{T}^*$-closed set. Then $A \leq (Int(A))^*$ and $A^* \leq A$. Since $Int(A) \leq A$, by lemma 2.1, we obtain $(Int(A))^* \leq A^*$. Hence we have $(Int(A))^* \leq A^* \leq A \leq (Int(A))^*$. Therefore, $A = (Int(A))^*$. So A is fuzzy regular-I-closed.

***Theorem* 3.10.** Every fuzzy f_I-set of a fuzzy ideal topological space $(X, \mathcal{T}, I)$ is fuzzy semi-*I*-open.

Proof: Let A is fuzzy f_I-set. Then we have $A \leq (Int(A))^*$. Therefore, $A \leq (Int(A))^* \leq Cl^*(Int(A))$ and hence A is fuzzy semi-*I*-open.

The following example shows that the converse of the theorem 3.10 is not true.

***Example* 3.2.** Let $X = \{ x, y, z \}$ and let A, B be fuzzy subsets of X defined as follows:

- $A(x) = 0.6$, $A(y) = 0.9$, $A(z) = 0.7$
- $B(x) = 0.3$, $B(y) = 0.8$, $B(z) = 0.1$

Let $\mathcal{T} = \{ 0_X, B, 1_X \}$ and $I = I^X$. Then B is fuzzy semi-*I*-open but not fuzzy f_I-set.

Following theorems give the characterization of codense fuzzy ideals in terms of fuzzy A_I-set, fuzzy regular-*I*-closed set and fuzzy I-locally closed set.

***Theorem* 3.11.** For a fuzzy ideal topological space $(X, \mathcal{T}, I)$, I is $\mathcal{T}$-codense if and only if $A \in \mathcal{T} \Longrightarrow A \in \mathrm{F}A_I(X)$.

Proof: Let I is $\mathcal{T}$-codense. Then $A \in \mathrm{F}A_I(X)$ where $A \in \mathcal{T}$. Conversely, let us suppose that the condition holds. Then by theorem 3.5, $\mathrm{F}A_I(X) \wedge I = \{0_X\}$ and so $I \wedge \mathcal{T} = \{0_X\}$. Hence I is $\mathcal{T}$-codense.

***Corollary* 3.1:** Let $(X, \mathcal{T}, I)$ be a fuzzy ideal topological space. Then the following are equivalent.

- I is $\mathcal{T}$-codense.
- $\mathcal{T} =$ FPIO(X)) $\wedge$ F$A_I(X)$.
- $\mathcal{T} =$ FαIO(X)) $\wedge$ F$A_I(X)$.
- $A \in \mathcal{T} \Longrightarrow A \in \mathrm{F}A_I(X)$.

***Proof*:** Obvious.

***Theorem* 3.12.** Let $(X, \mathcal{T}, I)$ be a fuzzy ideal topological space. Then I is $\mathcal{T}$-codense if and only if FRIC(X) = FRC(X) where FRIC(X) is the collection of all fuzzy regular-*I*-closed set in $(X, \mathcal{T}, I)$.

Proof: Let I is $\mathcal{T}$-codense. Then $A \in$ FRIC(X) if and only if $A = (Int(A))^*$ if and only if $A = Cl(Int(A))$, by lemma 2.3(3) and lemma 2.4, if and only if $A \in$ FRC(X). Conversely, let FRIC(X) = FRC(X). Since 1_X is fuzzy regular closed, 1_X is fuzzy regular-*I*-closed and so $1_X = \left(Int(1_X)\right)^* = 1_X{}^*$.Then by lemma 2.3(2), I is $\mathcal{T}$-codense.

***Theorem* 3.13.** Let $(X, \mathcal{T}, I)$ be a fuzzy ideal topological space. Then I is $\mathcal{T}$-codense if and only if F$A_I(X)$= F$A(X)$ where F$A(X)$ is the collection of all fuzzy A-set in $(X, \mathcal{T}, I)$.

Proof: Let I is $\mathcal{T}$-codense. Let $A \in$ F$A_I(X)$. Then $A = U \wedge V$, where $U \in \mathcal{T}$ and V is a fuzzy regular-*I*-closed set. Therefore, $(Int(V))^* = V$. So $Cl(V) = Cl(Int(V))^* = (Int(V))^* = V$(by lemma 2.1). Again by lemma2.1, we have $(Int(V))^* \leq Cl(Int(V))$ and hence $V = (Int(V))^* \leq Cl(Int(V)) \leq Cl(V) = V$ and hence $V = Cl(Int(V))$ and consequently V is fuzzy regular closed and A is fuzzy A-set. That is, $A \in$ F$A(X)$. Again let $B \in$ F$A(X)$. Then $B = S \wedge T$, where $S \in \mathcal{T}$ and T is a fuzzy regular closed. Then by theorem 3.12, T is a fuzzy regular-*I*-closed set and so B is fuzzy A_I-set. That is, $B \in$ F$A_I(X)$. Conversely, let F$A_I(X)$= F$A(X)$. Since 1_X is fuzzy A-set, 1_X is fuzzy A_I-set. So by theorem 3.4(1), $1_X \leq 1_X{}^*$ Therefore, $1_X = 1_X{}^*$ which implies I is $\mathcal{T}$-codense(by lemma 2.3).

***Theorem* 3.14.** In any fuzzy ideal topological space $(X, \mathcal{T}, I)$, FILC$(X) \wedge I = \{0_X\}$.

Proof: Let $A \in$ FILC(X). Then by lemma 3.1, $A \leq A^*$. So by lemma 2.5, FILC$(X) \wedge I = \{0_X\}$.

***Theorem* 3.15:** Let $(X, \mathcal{T}, I)$ be a fuzzy ideal topological space. Then the following are equivalent.

- I is $\mathcal{T}$-codense.
- $\mathcal{T}$ = FPIO$(X)) \wedge$ FILC(X).
- $\mathcal{T}$ = FαIO$(X)) \wedge$ FILC(X).
- $A \in \mathcal{T} \Longrightarrow A \in$ FILC(X).

Proof: (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (4), (3) $\Longrightarrow$ (4) are obvious. (4) $\Longrightarrow$ (1) Follows from theorem 3.14. We are to establish only (1) $\Longleftrightarrow$ (3) Let I is $\mathcal{T}$-codense. If A is fuzzy open then A is fuzzy α-I-open and $A \leq A^*$. Then by lemma 3.1, it follows that A is fuzzy I-locally closed. Conversely, let A is both fuzzy α-I-open and fuzzy I-locally closed. As A is fuzzy I-locally closed, by lemma 3.1, $A = U \wedge A^*$ for some $U \in \mathcal{T}$. Again as A is fuzzy α-I-open, $A \leq Int(Cl^*(Int(A))) \leq Int(Cl^*(A)) = Int(Cl^*(U \wedge A^*)) \leq Int(Cl^*(A^*)) = Int(A^*)$. Since $A \leq U$, $A \leq U \wedge Int(A^*) = Int(U \wedge A^*) = Int(A)$ and so A is fuzzy open. This completes the proof.

From [1,16] we mention the following definitions.

***Definition* 3.4.** A function *f*: $(X, \mathcal{T}_1, I) \rightarrow (Y, \mathcal{T}_2)$ is said to be fuzzy A_I-continuous (resp. fuzzy A – continuous) if $f^{-I}(V)$ is fuzzy A_I-set (resp. fuzzy A-set) in X for every fuzzy open set V in Y.

***Theorem* 3.16.** Let *f*: $(X, \mathcal{T}_1, I) \rightarrow (Y, \mathcal{T}_2)$ be a mapping and I be $\mathcal{T}_1$-codense. Then *f* is fuzzy A_I-continuous if and only if *f* is fuzzy A-continuous.

Proof: Proof follows from theorem 3.13.

We now introduce the following definition.

***Definition* 3.5.** A function *f*: $(X, \mathcal{T}_1, I) \rightarrow (Y, \mathcal{T}_2)$ is said to be fuzzy f_I-continuous if $f^{-I}(V)$ is fuzzy f_I-set in X for every fuzzy open set V in Y.

From [1,2,23] we have the following definitions.

***Definition* 3.6.** A function *f*: $(X, \mathcal{T}_1, I) \rightarrow (Y, \mathcal{T}_2)$ is said to be FI-LC-continuous (resp. fuzzy semicontinuous, fuzzy locally-continuous) if $f^{-I}(V)$ is fuzzy I-locally closed (resp. fuzzy semiopen, fuzzy locally closed) set in X for every fuzzy open set V in Y.

The following theorem gives a decomposition of fuzzy A_I-continuity.

***Theorem* 3.17.** Let *f*: $(X, \mathcal{T}_1, I) \rightarrow (Y, \mathcal{T}_2)$ be a mapping. Then *f* is fuzzy A_I-continuous if and only if *f* is both fuzzy f_I-continuous and FI-LC-continuous.

Proof: Proof follows from theorem 3.6.

***Theorem* 3.18.** Let *f*: $(X, \mathcal{T}_1, I) \rightarrow (Y, \mathcal{T}_2)$ be a mapping and I be $\mathcal{T}_1$-codense. Then the following are equivalent.

- *f* is fuzzy A-continuous.
- *f* is fuzzy A_I-continuous.
- *f* is both fuzzy f_I-continuous and FI-LC-continuous.
- *f* is both fuzzy semicontinuous and FLC-continuous.

Proof: (1) $\Leftrightarrow$ (2) Follows from theorem 3.16. (2) $\Leftrightarrow$ (3) Follows from theorem 3.17. (4) $\Leftrightarrow$ (1) Follows from theorem 3.3. To establish only (3) $\Longrightarrow$ (4). From lemma 3.1, it follows that every fuzzy I-locally closed set in $(X, \mathcal{T}, I)$ is fuzzy locally closed. Let A is fuzzy f_I-set. Then $A \leq \left(Int(A)\right)^* = Cl(Int(A))$ and so A is fuzzy semiopen. This completes the proof.

From [3,20,30] we mention the following definitions.

***Definition* 3.7.** A function *f*: $(X, \mathcal{T}_1, I) \rightarrow (Y, \mathcal{T}_2)$ is said to be fuzzy α-I-continuous (resp. fuzzy pre-I-continuous, fuzzy α-continuous,fuzzy pre-continuous) if $f^{-I}(V)$ is a fuzzy α-I-open (resp. fuzzy pre-I-open, fuzzy α-open, fuzzy preopen) set in X for every $V \in \mathcal{T}_2$.

The following theorem gives a decomposition of fuzzy continuity.

***Theorem* 3.19.** Let *f*: $(X, \mathcal{T}_1, I) \to (Y, \mathcal{T}_2)$ be any mapping and I be $\mathcal{T}_1$-codense. Then the following conditions are equivalent.

- *f* is fuzzy continuous.
- *f* is fuzzy α-I-continuous and FI-LCcontinuous.
- *f* is fuzzy pre-I-continuous and FI-LC continuous.

Proof: Proof follows from theorem 3.15.

***Corollary* 3.2:** Let *f*: $(X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be any mapping. Then the following conditions are equivalent.

- *f* is fuzzy continuous.
- *f* is fuzzy α-continuous and FI-LC continuous.
- *f* is fuzzy pre-continuous and FI-LC continuous.

Proof: Let us suppose $I = \{ 0_X \}$ for theorem 3.19. Then the corollary follows.

Conclusion

The notions of the sets and functions in fuzzy topological spaces are highly developed and several characterizations have already been obtained. These are used extensively in many practical and engineering problems, computational topology for geometric design, computer-aided geometric design, engineering design research, Geographic Information System (GIS) and mathematical sciences. As the works of professor El-Naschie indicates that fuzzy topology may be relevant to quantum physics particularly in connection with string theory and e^{∞} theory and fuzzy topology may be used to provide information about the elementary particles content of the standard model of high energy physics, the notions and results given in this paper may turn out to be useful in quantum physics. Several characterizations of fuzzy sets and several generalizations of fuzzy continuous functions may also lead to some interesting in-depth analytical study and research from the view point of fuzzy mathematics.

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