t-Intuitionistic Fuzzy Quotient Modules

P.K. Sharma

P.G. Department of Mathematics
D.A.V. College, Jalandhar City, Punjab, India
E-mail: pksharma@davjalandhar.com

Abstract

In this paper, the notion of t-intuitionistic fuzzy cosets of a Intuitionistic fuzzy submodule and t-intuitionistic fuzzy quotient modules are defined and discussed. A homomorphism from a given module onto the set of all t-intuitionistic fuzzy quotient module is established. Some related results has been derived.

Keywords: Intuitionistic fuzzy (IFS), Intuitionistic fuzzy submodule (IFSM), $(\alpha, \beta)$–Cut set, t-intuitionistic fuzzy coset, t-intuitionistic fuzzy quotient module

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Introduction

The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2] as a generalization of that of fuzzy sets and it is a very effective tool to study the case of vagueness. Further many researches applied this notion in various branches of mathematics especially in algebra and defined intuitionistic fuzzy subgroups, intuitionistic fuzzy subrings, intuitionistic fuzzy sublattices, intuitionistic fuzzy submodules and so forth. In this paper we introduce the notion of t-intuitionistic fuzzy cosets and t-intuitionistic fuzzy quotient module as a generalization to the notion of $\alpha$-fuzzy cosets and $\alpha$-fuzzy quotient modules as discussed by Bhambr and Pratiba in [4].

Preliminaries

In this section we recall some definitions and results which will be used later
Definition (2.1)[1]: Let $X$ be a fixed non-empty set. An Intuitionistic fuzzy set (IFS) $A$ of $X$ is an object of the following form $A = \{ < x, \mu_A(x), \nu_A(x) >: x \in X \}$, where $\mu_A: X \to [0, 1]$ and $\nu_A: X \to [0, 1]$ define the degree of membership and degree of non-membership of the element $x \in X$ respectively and for any $x \in X$, we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Remark (2.2):

i. When $\mu_A(x) + \nu_A(x) = 1$, i.e. when $\nu_A(x) = 1 - \mu_A(x) = \mu^c_A(x)$. Then $A$ is called fuzzy set.

ii. We write $A = (\mu_A, \nu_A)$ to denote the IFS $A = \{ < x, \mu_A(x), \nu_A(x) >: x \in X \}$.

Definition (2.3)[9]: Let $A$ be an intuitionistic fuzzy set of a universe set $X$. Then $(\alpha, \beta)$-cut of $A$ is a crisp subset $C_{\alpha, \beta}(A)$ of the IFS $A$ is given by $C_{\alpha, \beta}(A) = \{ x \in X: \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \}$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

Definition (2.4)[8]: Let $M$ be a module over a ring $R$. An IFS $A = (\mu_A, \nu_A)$ of $M$ is called intuitionistic fuzzy (left) submodule (IFSM) if

i. $\mu_A(0) = 1$, $\nu_A(0) = 0$

ii. $\mu_A(x + y) \geq \mu_A(x) \land \mu_A(y)$ and $\nu_A(x + y) \leq \nu_A(x) \lor \nu_A(y)$, $\forall x, y \in M$

iii. $\mu_A(rx) \geq \mu_A(x)$ and $\nu_A(rx) \leq \nu_A(x)$, $\forall x \in M, r \in R$

If we replace the condition (iii) with $\mu_A(xr) \geq \mu_A(x)$ and $\nu_A(xr) \leq \nu_A(x)$, $\forall x \in M, r \in R$, it is called intuitionistic fuzzy (right) submodule. When $R$ is a commutative ring, then these two modules coincide. From this onward, $R$ will be a commutative ring with unity.

Theorem (2.5)[8]: Let $A = (\mu_A, \nu_A)$ be an IFS of an $R$-module $M$. Then $A$ is an IFSM of $M$ if and only if $A$ satisfies the following conditions:

(i) $\mu_A(0) = 1$, $\nu_A(0) = 0$

(ii) $\mu_A(rx + sy) \geq \mu_A(x) \land \mu_A(y)$ and $\nu_A(rx + sy) \leq \nu_A(x) \lor \nu_A(y)$, $\forall x, y \in M, r, s \in R$

$t$-Intuitionistic fuzzy quotient modules

Proposition (3.1)[9]: If $A = (\mu_A, \nu_A)$ be IFS of an $R$-module $M$, then $A$ is an IFSM of $M$ if and only if $C_{\alpha, \beta}(A)$ is submodule of $M$, for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, where $\mu_A(0) \geq \alpha$, $\nu_A(0) \leq \beta$.

Definition (3.2): Let $A$ be a IFSM of an $R$-module $M$. Let $t \in [0, 1]$. For any $x \in M$

Define a IFS $A^t_x$ of $M$ called $t$ – Intuitionistic fuzzy coset of $A$ in $M$ as follows

$A^t_x(m) = (\mu_{A^t_x}(m), \nu_{A^t_x}(m))$, where

$\mu_{A^t_x}(m) = \min\{\mu_A(x + m), t\}$ and $\nu_{A^t_x}(m) = \max\{\nu_A(x + m), 1 - t\}$.
for all \(x, m \in M\).

**Proposition (3.3):** Let \(S\) be the set of all \(t\)-Intuitionistic fuzzy cosets of \(A\) in \(M\) i.e. \(S = \{ A_x : x \in M \}\). Then the two operations \(\oplus\) and \(\ominus\) defined on the set \(S\) as follows are well defined

\[
(i) \quad A_x^t \oplus A_y^t = A_{x+y}^t, \quad \text{for all } x, y \in M
\]

\[
(ii) \quad r \ominus A_x^t = A_{rx}^t, \quad \text{for all } x \in M, r \in R
\]

**Proof:** Let \(A_x^t = A_x^{t'}\) and \(A_y^t = A_y^t\), for some \(x, y, x', y' \in M\).

Let \(m \in M\) be any element, then

\[
(i) \quad [A_x^t \oplus A_y^t](m) = (A_{x+y}^t)(m) = (\mu_{A_{x+y}^t}(m), \nu_{A_{x+y}^t}(m))
\]

Now \(\mu_{A_{x+y}^t}(m) = \min\{\mu_A(x + y + m), t\} = \mu_{A_{x'}^t}(y + m) = \mu_{A_{x'}^t}(y + m) = \min\{\mu_A(y' + x' + m), t\}
\]

\[
= \mu_{A_{x'}^t}(x' + m) = \mu_{A_{x'}^t}(x' + m) = \min\{\mu_A(y' + x' + m), t\}
\]

\[
= \mu_{A_{x'}^t}(m)
\]

Similarly, we can show that \(\nu_{A_{x+y}^t}(m) = \nu_{A_{x'}^t}(m)\), for all \(m \in M\)

Thus \(A_x^t \oplus A_y^t = A_{x'}^t \oplus A_{y'}^t\).

Therefore \(\oplus\) is well defined

(ii) Let \(A_x^t = A_{x'}^t\) for \(x, x' \in M\), then \(A_x^t(-x') = A_{x'}^t(-x')\)

\[
(\min\{\mu_A(x - x'), t\}, \max\{\nu_A(x-x'), 1 - t\}) = (\min\{\mu_A(x' - x'), t\}, \max\{\nu_A(x' - x'), 1 - t\})
\]

\[
= (\min\{\mu_A(0), t\}, \max\{\nu_A(0), 1 - t\}) = (t, 1 - t)
\]

\[
\Rightarrow \min\{\mu_A(x - x'), t\} = t \quad \text{and} \quad \max\{\nu_A(x-x'), 1 - t\} = 1 - t
\]

\[
\Rightarrow \mu_A(x - x') \geq t \quad \text{and} \quad \nu_A(x-x') \leq 1 - t \quad \text{and so} \quad x-x' \in C_{t,1-t}(A) = N(say)
\]

\[
\Rightarrow N + x = N + x' \quad \text{.................................(*)}
\]

Now, we show that if \(N + x = N + x'\), then \(A_x^t = A_{x'}^t\).

Suppose \(\mu_A(x + y) < t\) and \(\mu_A(x' + y) \geq t\)

Therefore \(\nu_A(x + y) \leq 1-t\) and \(\nu_A(x' + y) \leq 1-t\)

\[
\Rightarrow x + y \in C_{t,1-t}(A) = N
\]

\[
\Rightarrow N + x + y = N \Rightarrow N + x + y = N \text{ (Using (*))}
\]

\[
\Rightarrow x + y \in N \text{ and so } \mu_A(x + y) \geq t, \text{ a contradiction}
\]

Similarly, if \(\mu_A(x + y) \geq t\) and \(\mu_A(x' + y) < t\) also leads to contradiction.
Therefore either $\mu_A(x + y) \geq t$ and $\mu_A(x' + y) \geq t$ i.e. $v_A(x + y) \leq 1-t$ and $v_A(x' + y) \leq 1-t$ or $\mu_A(x + y) < t$ and $\mu_A(x' + y) < t$ i.e. $v_A(x + y) \leq 1-t$ and $v_A(x' + y) \leq 1-t$

In the first case

$\text{Min} \{ \mu_A(x + y), t \} = t$ and $\text{max} \{ v_A(x + y), 1-t \} = 1-t$

And so $A'_x(y) = (t, 1-t)$ and also

$\text{Min} \{ \mu_A(x' + y), t \} = t$ and $\text{max} \{ v_A(x' + y), 1-t \} = 1-t$

And so $A'_{x'}(y) = (t, 1-t)$. Thus $A'_x(y) = A'_{x'}(y)$, for all $y \in M$

Therefore $A'_x = A'_{x'}$

In the second case

$\text{Min} \{ \mu_A(x + y), t \} = \mu_A(x + y) < t$ and $\text{max} \{ v_A(x + y), 1-t \} = 1-t$

And also

$\text{Min} \{ \mu_A(x' + y), t \} = \mu_A(x' + y) < t$ and $\text{max} \{ v_A(x' + y), 1-t \} = 1-t$

Now since $N + x = N + x'$, therefore let $x = n + x'$, where $n \in N$

So that $\mu_A(n) \geq t$ and $v_A(n) \leq 1-t$

$A'_x(y) = (\text{min} \{ \mu_A(x' + y), t \}, \text{max} \{ v_A(x' + y), 1-t \})$

$= (\mu_A(x' + y), 1-t)$

$\geq (\mu_A(n) \land \mu_A(x + y), 1-t)$

$= (\mu_A(x + y), 1-t)$

$= (\text{min} \{ \mu_A(x + y), t \}, \text{max} \{ v_A(x + y), 1-t \})$

$= A'_{x'}(y)$

Thus $A'_x(y) \geq A'_{x'}(y)$, for all $y \in M$

Similarly $A'_x(y) = (\text{min} \{ \mu_A(x + y), t \}, \text{max} \{ v_A(x + y), 1-t \})$

$= (\mu_A(x + y), 1-t)$

$= (\mu_A(n + x' + y), 1-t)$

$\geq (\mu_A(n) \land \mu_A(x' + y), 1-t)$

$= (\mu_A(x' + y), 1-t)$

$= (\text{min} \{ \mu_A(x' + y), t \}, \text{max} \{ v_A(x' + y), 1-t \})$

$= A'_{x'}(y)$

Thus $A'_x(y) \geq A'_{x'}(y)$, for all $y \in M$

Therefore $A'_x = A'_{x'}$
Consequently \( A'_x = A'_{x'} \) if and only if \( N + x = N + x' \)

if and only if \( N + rx = N + rx' \)

if and only if \( A'_{rx} = A'_{rx'} \)

if and only if \( r \circ A'_{x} = r \circ A'_{x'} \)

Hence \( \bigcirc \) is well defined.

**Proposition (3.4):** The set \( S \) of all \( t \)-Intuitionistic fuzzy cosets of a IFSM \( A \) of a module \( M \), form a module under the well-defined operations \( \oplus \) and \( \bigcirc \).

**Proof:** An easy result

**Proposition (3.5):** The IFS \( B \) of \( S \) defined by \( B(A'_a) = \left( \mu_b(A'_a), v_b(A'_a) \right) \)

where \( \mu_b(A'_a) = \sup_{A'_a \in A'_b} \{ \mu_d(x) : x \in M \} \) and \( v_b(A'_a) = \inf_{A'_a \in A'_b} \{ v_d(x) : x \in M \} \) is a IFSM of \( S \), called \( t \)-Intuitionistic fuzzy quotient module.

**Proof:** As \( 0 \in M \), therefore

\( A'_x = A'_0 \Rightarrow N + x = N + 0 = N \Rightarrow x \in N \) and so \( \mu_d(x) \geq t \) and \( v_d(x) \leq 1 - t \)

Thus \( \mu_b(A'_0) = \sup_{A'_a \in A'_b} \{ \mu_d(x) : x \in M \} = 1 \) and \( v_b(A'_0) = \inf_{A'_a \in A'_b} \{ v_d(x) : x \in M \} = 0 \)

Again, let \( a, b \in M \) and let \( B(A'_a) = (\theta_1, \theta_2) \) and \( B(A'_b) = (\phi_1, \phi_2) \), where

\( \theta_1 = \sup_{A'_a \in A'_b} \{ \mu_d(x) : x \in M \} \), \( \theta_2 = \inf_{A'_a \in A'_b} \{ v_d(x) : x \in M \} \) and

\( \phi_1 = \sup_{A'_a \in A'_b} \{ \mu_d(x) : x \in M \} \), \( \phi_2 = \inf_{A'_a \in A'_b} \{ v_d(x) : x \in M \} \)

\( \exists \ x, y \in M \) such that \( \theta_1 - \varepsilon < \mu_A(x) \), \( N + x = N + a \) and \( \phi_1 - \varepsilon < \mu_A(y) \), \( N + y = N + b \)

Therefore \( N + x + y = N + a + b \Rightarrow A'_{x+y} = A'_{a+b} \Rightarrow A'_x \oplus A'_y = A'_{a+b} \oplus A'_b \)

So that \( \mu_A(x+y) \leq \mu_b(A'_x \oplus A'_y) = \mu_b(A'_a \oplus A'_b) \)

Now \( \mu_d(x+y) \geq \mu_d(x) \wedge \mu_d(y) = \mu_d(x) \) (say) \( \theta_1 - \varepsilon \)

\( \because \theta_1 - \varepsilon < \mu_A(x+y) \leq \mu_b(A'_a \oplus A'_b) \), \( \forall \epsilon > 0 \)

so that \( \theta_1 \leq \mu_b(A'_a \oplus A'_b) \)

Now two cases arises

**Case (i) When \( \theta_1 \geq \phi_1 \), then \( \theta_1 - \varepsilon \geq \phi_1 - \varepsilon \)**


\[\phi - \varepsilon \leq \theta_i - \varepsilon \leq \mu_b(A'_a \oplus A'_b), \forall \varepsilon > 0\]

so that \[\phi_i \leq \mu_b(A'_a \oplus A'_b)\]

Therefore \[\mu_b(A'_a) \land \mu_b(A'_b) = \theta_i \land \phi_i = \phi_i \leq \mu_b(A'_a \oplus A'_b)\]

Case (ii) When \(\theta_i < \phi_i\), then

\[\mu_b(A'_a) \land \mu_b(A'_b) = \theta_i \land \phi_i = \theta_i \leq \mu_b(A'_a \oplus A'_b)\]

Thus in any case we get \[\mu_b(A'_a \oplus A'_b) \geq \mu_b(A'_a) \land \mu_b(A'_b)\]

Similarly, we can show that \[\nu_b(A'_a \oplus A'_b) \leq \nu_b(A'_a) \lor \nu_b(A'_b)\]

Next, let \(\theta_i = \mu_b(A'_a) = \text{Sup}\{\mu_\lambda(y) : N + y = N + a\}\)

Let \(\varepsilon > 0\) be given, then \(\exists y \in M\) such that \(\theta_i - \varepsilon < \mu_\lambda(y)\)

Where \(N + y = N + a\) i.e. \(N + ry = N + ra\)

But \(\mu_b(r \circ A'_{a}) = \mu_b(A'_{ra}) = \text{Sup}\{\mu_\lambda(x) : N + x = N + ra\}\)

\[\therefore \theta_i - \varepsilon < \mu_\lambda(y) \leq \mu_\lambda(ry) \leq \mu_b(A'_{ra})\]

Thus \(\theta_i - \varepsilon \leq \mu_b(A'_{ra}), \forall \varepsilon > 0\)

Hence \(\theta_i \leq \mu_b(A'_{ra})\) consequently \(\mu_b(r \circ A'_{a}) = \mu_b(A'_{ra}) \geq \mu_b(A'_{a})\)

Similarly, we can show that \(\nu_b(r \circ A'_{a}) = \nu_b(A'_{ra}) \leq \nu_b(A'_{a})\)

Whence \(B\) is IFSM of \(S\)

**Proposition (3.6):** A mapping \(f : M \to S\), where \(M\) is an R-module and \(S\) is the set of all \(t\)-Intuitionistic fuzzy cosets of the IFSM \(A\) of \(M\) defined by \(f(x) = A'_x\), is an onto homomorphism with \(\ker f = N (= C_{t,1-t}(A))\), where \(t \in [0,1]\)

**Proof:** Clearly \(f\) is an onto homomorphism

Let \(x \in \ker f\), then \(f(x) = \text{zero element of } S = A'_0\)

Therefore \(A'_x = A'_0\) so \(N + x = N + 0 = N \Rightarrow x \in N\)

\(\Rightarrow \ker f \subseteq N \) ……………………………………….(1)

Conversely, let \(x \in N \Rightarrow N + x = N\) so that

\(N + x + m = N + m \forall m \in M\)

Suppose \(\mu_\lambda(x + m) < t\) and \(\mu_\lambda(m) \geq t\), i.e. \(\nu_\lambda(x + m) \leq 1-t\) and \(\nu_\lambda(m) \leq 1-t\)

Therefore \(\mu_\lambda(m) \geq t\) and \(\nu_\lambda(m) \leq 1-t \Rightarrow m \in N\) so \(N + m = N\)

i.e. \(N + x + m = N \Rightarrow x + m \in N\) and so \(\mu_\lambda(x + m) \geq t\), a contradiction

Similarly, \(\mu_\lambda(x + m) \geq t\) and \(\mu_\lambda(m) < t\) is not possible.

\(\therefore\) either \(\mu_\lambda(x + m) \geq t\), \(\mu_\lambda(m) \geq t\), i.e. \(\nu_\lambda(x + m) \leq 1-t\) and \(\nu_\lambda(m) \leq 1-t\)

or \(\mu_\lambda(x + m) < t\), \(\mu_\lambda(m) < t\), i.e. \(\nu_\lambda(x + m) \leq 1-t\) and \(\nu_\lambda(m) \leq 1-t\)

In the first case
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Min \{ μ_A(x + m), t \} = t and Max \{ ν_A(x + m), 1-t \} = 1-t and so \( A'_x(m) = (t, 1-t) \)
similarly we get \( A'_0(m) = (t, 1-t) \)
Thus \( A'_x(m) = A'_0(m) \)

In the second case
Min \{ μ_A(x + m), t \} = μ_A(x + m) < t and Max \{ ν_A(x + m), 1-t \} = 1-t
\( A'_x(m) = (\min\{ μ_A(x + m), ,t\}, \max\{ ν_A(x + m), ,1-t\}) \)
\( = (\mu_A(m), 1-t) \)
\( \geq (\mu_A(x) \wedge \mu_A(m), 1-t) \)
\( = (\mu_A(m), 1-t) \quad [\because x \in N, \mu_A(x) \geq t \text{ and } \mu_A(m) < t] \)
\( = (\min\{ μ_A(0 + m), ,t\}, \max\{ ν_A(0 + m), ,1-t\}) \)
\( = A'_0(m) \)
\( = (\mu_A(m), 1-t) \)
\( = (\mu_A(x + m - x), 1-t) \)
\( \geq (\mu_A(x + m) \wedge \mu_A(x), 1-t) \)
\( = (\mu_A(x + m), 1-t) \)
\( = (\min\{ μ_A(x + m), ,t\}, \max\{ ν_A(x + m), ,1-t\}) \)
\( = A'_x(m) \)

Thus \( A'_x(m) = A'_0(m) \). Therefore, we get \( A'_x = A'_0 \)

i.e., \( f(x) = \text{zero of } S \) and so \( x \in \ker f \)
\( \therefore N \subseteq \ker f \) so \( \ker f = N. \)

**Proposition (3.7):** If \( f: M \rightarrow S \), is an onto homomorphism, then \( f(A) = B \), where \( A \) is IFS of \( M \) and \( B \) is IFS of \( S \).

**Proof:** Let \( A'_a \in S \) be any element of \( S \), where \( a \in M \) such that \( f(a) = A'_a \)
Let \( A \) be IFS of \( M \), then
\[
f(A)(A'_a) = \begin{cases} 
\sup\{\mu_A(x) : x \in f^{-1}(A'_a)\}, \inf\{\nu_A(x) : x \in f^{-1}(A'_a)\} & (0, 1) \\
\sup\{\mu_A(x) : A'_x = A'_a\}, \inf\{\nu_A(x) : A'_x = A'_a\} & (0, 1) 
\end{cases}
\]
\( = B(A'_a) \)
Hence \( f(A) = B \)
**Theorem (3.8):** Let $A$ be a IFSM of $M$ and $B$ be a IFSM of $S$, then

$$C_{t,1-t}(B) = \{ A'_0 \}$$

**Proof:**

Now $B(A'_0) = \left( \mu_B(A'_0), \nu_B(A'_0) \right)$, where

$$\mu_B(A'_0) = \sup \{ \mu_A(x) : N + x = N \}$$
$$= \sup \{ \mu_A(x) : x \in N \}$$
$$\geq \mu_A(n), \text{ for all } n \in N = C_{t,1-t}(A)$$
$$\geq t$$

Similarly, we can show that $\nu_B(A'_0) \leq 1 - t$. Thus $A'_0 \in C_{t,1-t}(B)$

Let $A'_a \in C_{t,1-t}(B) \Rightarrow \mu_B(A'_a) \geq t$ and $\nu_B(A'_a) \leq 1 - t$

Let $\theta_1 = \mu_B(A'_a) = \sup \{ \mu_A(x) : N + x = N + a \}$ and

$$\theta_2 = \nu_B(A'_a) = \inf \{ \nu_A(x) : N + x = N + a \}$$

Therefore $\theta_1 \geq t$ and $\theta_2 \leq 1 - t$. Let $\varepsilon > 0$ be given, $\exists$'s $x, y \in M$ such that

$N + x = N + a$ so that $x - a = n_1 \in N$ and $\mu_A(x) > \theta_1 - \varepsilon \geq t - \varepsilon$ and

$N + y = N + a$ so that $y - a = n_2 \in N$ and $\nu_A(y) < \theta_2 + \varepsilon \leq (1 - t) + \varepsilon$

$$\mu_A(a) = \mu_A(a + n_1 - n_2) \geq \mu_A(a + n_1) \wedge \mu_A(n_2) = \begin{cases} \geq t & \text{if } \mu_A(a + n_1) \geq t \\
 = \mu_A(a + n_1) & \text{if } \mu_A(a + n_1) < t \end{cases}$$

and

$$\nu_A(a) = \nu_A(a + n_2 - n_1) \leq \nu_A(a + n_2) \vee \nu_A(n_1) = \begin{cases} \leq 1 - t & \text{if } \nu_A(a + n_2) \leq 1 - t \\
 = \nu_A(a + n_2) & \text{if } \nu_A(a + n_2) > 1 - t \end{cases}$$

Thus in any case $\mu_A(a) > t - \varepsilon$ and $\nu_A(a) \leq (1 - t) + \varepsilon$, for all $\varepsilon > 0$

$\Rightarrow \mu_A(a) \geq t$ and $\nu_A(a) \leq (1 - t)$ implies that $a \in C_{t,1-t}(A)$

$\Rightarrow N + a = N$ so $A'_a = A'_0$ Hence $C_{t,1-t}(B) = \{ A'_0 \}$.

**References**
