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Application of the Intuitionistic Fuzzy Laplace Transform Method for Resolution of One Dimensional Wave Equations

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Abstract

In this article, our objective is to employ the intuitionistic fuzzy laplace transform to solve one-dimensional intuitionistic fuzzy wave equations under strongly H-differentiability. The associated theorems and properties and also the solution procedure are demonstrated in detail, an example is provided to show the applicability and soundness of the presented technique.

Keywords: Intuitionistic fuzzy number, Intuitionistic fuzzy wave equations, intuitionistic fuzzy Laplace Transform, strongly generalized hukuhara differentiability, intuitionistic fuzzy valued function.

1. INTRODUCTION

In 1965, Lotfi Zadeh [16] suggested the notion of fuzzy sets in order to model the uncertainty and inaccuracy and to progressively evaluate the membership of elements to a set the fuzzy set generalize classical sets, Dubois in 1978 presented the results for the algebraic combination of fuzzy sets with infinite supports [17], who used the extension principe in their approach, the H-derivative of a fuzzy-numbers-valued function was introduce by M. Puri and D. Ralescu [4]. In 1980s Atanassov generalized the concept of fuzzy sets he added the non-membership function and defined the idea of intuitionistic fuzzy set [2], the concept of intuitionistic fuzzy partial differential

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equations is introduced by S. Melliani and L. S. Chadli [5] and the authors in [8] presented the theory of intuitionistic fuzzy metric space in other areas of mathematics and computer sciences, more references on the theory of intuitionistic fuzzy differential equations and their resolution by numerical methods can be found in [9-14].

The modeling of scientific phenomena (electromagnetics, fluid mechanics, quantum mechanics, electricity) is relies on the resolution of partial differential equations, the laplace transform is a very practical tool for the resolution of this type of equations, the fuzzy laplace transform (FLT) has been first introduced by Allahviranloo and Ahmadi [1], in [3] N. Ahmad, M. Mamat, J.K. Kumar, N.S. Amir, Hamzah are used the laplace transform method to resolve the Duffing's equation, the authors in [10, 12] applied the fuzzy laplace transform in second-order fuzzy differential equation, Salahshour, Soheil, and Elnaz Haghi used fuzzy laplace transform method in fuzzy heat equation, and in [19] the authors applied the intuitionistic fuzzy laplace transform electrical circuit.

The objective of this research is to find solutions to intuitionistic fuzzy wave equations by intuitionistic fuzzy laplace transform method under strongly generalized H-differentibility, it will enable us to transform the partial differential equations into algebraic equation, then by solving these algebraic equations, we may find the unkown function by using the inverse laplace.

This paper is organizationally as shown below : In section 2, we give the basic notion of intuitionistic fuzzy sets, intuitionistic fuzzy numbers, derivative au sens de hukuhara, intuitionistic fuzzy integral and a basic definition that we will use during this research, In section 3, we start with the presentation of the fuzzy laplace transform method, we recall its fundamental properties. Then, we introduce and demonstrate our main result, we developed a technique to solve one-dimensional wave equations with intuitionistic fuzzy values using the intuitionistic fuzzy Laplace transform method. A numerical example in section 4 to prove the capability of the proposed method, and the conclusion in the last section 5.

2. PRELIMINARIES

An intuitionistic fuzzy set $A \in X$ is given by

$$A = \{(x, u_A(x), v_A(x))|x \in X\}$$

Where the function $u_A(x), v_A(x) : X \to [0, 1]$ define respectively the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every $x \in X$,

$$0 \leq u_A(x) + v_A(x) \leq 1$$

Obviously, every fuzzy set has the form

$$\{(x, u_A(x), u_{A^c}(x))|x \in X\}$$

Let us $J = [a, b] \subset \mathbb{R}$ be a compact interval. We denote by

$$\mathbb{IF}_1 = \{\langle u, v\rangle \mid \mathbb{R} \to [0,1]^2, \forall x \in \mathbb{R}; 0 \leq u(x) + v(x) \leq 1\}$$

the collection of all intuitionistic fuzzy number.

An element $\langle u, v\rangle$ of $\mathbb{IF}_1$ is said an intuitionistic fuzzy number if it satisfies the following conditions :

(i) $\langle u, v\rangle$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) The membership function u is fuzzy convex i.e $u(\lambda x_1 + (1 - \lambda)x_2) \geq min(u(x_1), u(x_2))$.

(iii) The non-membership function v is fuzzy concave

i.e $v(\lambda x_1 + (1 - \lambda)x_2) \leq max(v(x_1), v(x_2))$.

(iv) u is upper semi-continuous and v is lower semi-continuous

(v) $Supp\langle u, v\rangle = cl\{x \in \mathbb{R} : |v(x) < 1\}$ is bounded.

Definition 1. *An intuitionistic fuzzy number in parametric form is a pair of functions*

$$\langle u, v\rangle = ((\underline{\langle u, v\rangle}^+, \overline{\langle u, v\rangle}^+), (\underline{\langle u, v\rangle}^-, \overline{\langle u, v\rangle}^-))$$

wich satisfy the following requirements:

(i) $\underline{\langle u, v\rangle}^+(\alpha)$ is a bounded monotonic increasing continuous function,

(ii) $\overline{\langle u, v\rangle}^+(\alpha)$ is a bounded monotonic decreasing continuous function,

(iii) $\underline{\langle u, v\rangle}^-(\alpha)$ is a bounded monotonic increasing continuous function,

(iv) $\overline{\langle u, v\rangle}^-(\alpha)$ is a bounded monotonic decreasing continuous function,

(v) $\underline{\langle u, v\rangle}^-(\alpha) \leq \overline{\langle u, v\rangle}^-(\alpha)$ and $\underline{\langle u, v\rangle}^+(\alpha) \leq \overline{\langle u, v\rangle}^+(\alpha)$ $\forall \alpha \in [0, 1]$.

For $\alpha \in [0, 1]$ and $\langle u, v\rangle \in \mathbb{F}_1$, the upper and lower α-cuts of $\langle u, v\rangle$ are defined by :

$$[\langle u, v\rangle]^\alpha = \{x \in \mathbb{R} : v(x) \leq 1 - \alpha\} \text{ and } [\langle u, v\rangle]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$$

Remark 1. *If $\langle u, v\rangle \in \mathbb{IF}_1$, so we can see $[\langle u, v\rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v\rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.*

We define $0_{(1,0)} \in \mathbb{IF}_1$ as :

$$0_{(1,0)}(t) = \begin{cases} (1,0) & t = 0 \\ (0,1) & t \neq 0 \end{cases}$$

Definition 2. *Let $\langle u, v\rangle$, $\langle u', v'\rangle \in \mathbb{IF}_1$ and $\lambda \in \mathbb{R}$, we define the following operations by :*

$$(\langle u, v\rangle \oplus \langle u', v'\rangle)(z) = (sup_{z=x+y} min(u(x), u'(y)), inf_{z=x+y} max(v(x), v'(y)))$$

$$\lambda\langle u, v\rangle = \begin{cases} \langle \lambda u, \lambda v\rangle & if \;\; \lambda \neq 0 \\ 0_{(1,0)} & if \;\; \lambda = 0 \end{cases}$$

For $\langle u, v\rangle$, $\langle z, w\rangle \in \mathbb{IF}_1$ and $\lambda \in \mathbb{R}$, according to Zadeh's extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space are defined as follows :

$[\langle u, v\rangle \oplus \langle z, w\rangle]^\alpha = [\langle u, v\rangle]^\alpha + [\langle z, w\rangle]^\alpha$, $\qquad [\lambda\langle z, w\rangle]^\alpha = \lambda[\langle z, w\rangle]^\alpha$,

$[\langle u, v\rangle \oplus \langle z, w\rangle]_\alpha = [\langle u, v\rangle]_\alpha + [\langle z, w\rangle]_\alpha$, $\qquad [\lambda\langle z, w\rangle]_\alpha = \lambda[\langle z, w\rangle]_\alpha$.

Definition 3. *Let $\langle u, v\rangle$ an element of $\mathbb{IF}_1$ and $\alpha \in [0.1]$, we define the following sets:*

$[\langle u, v\rangle]_l^+(\alpha) = inf\{x \in \mathbb{R} | u(x) \geq \alpha\}$, $\quad [\langle u, v\rangle]_r^+(\alpha) = sup\{x \in \mathbb{R} | u(x) \geq \alpha\}$,

$[\langle u, v\rangle]_l^-(\alpha) = inf\{x \in \mathbb{R} | v(x) \leq 1 - \alpha\}$, $[\langle u, v\rangle]_r^-(\alpha) = sup\{x \in \mathbb{R} | v(x) \leq 1 - \alpha\}$.

Remark 2. $[\langle u, v\rangle]_\alpha = [[\langle u, v\rangle]_l^+(\alpha), [\langle u, v\rangle]_r^+(\alpha)]$,
$[\langle u, v\rangle]^\alpha = [[\langle u, v\rangle]_l^-(\alpha), [\langle u, v\rangle]_r^-(\alpha)]$.

On the space $\mathbb{IF}^1$ we will consider the following metric,

$$\begin{aligned} d_\infty(\langle u, v\rangle, \langle z, w\rangle) = & \frac{1}{4} \sup_{0<\alpha\leq 1} \parallel [\langle u, v\rangle]_r^+(\alpha) - [\langle z, w\rangle]_r^+(\alpha) \parallel \\ & + \frac{1}{4} \sup_{0<\alpha\leq 1} \parallel [\langle u, v\rangle]_l^+(\alpha) - [\langle z, w\rangle]_l^+(\alpha) \parallel \\ & + \frac{1}{4} \sup_{0<\alpha\leq 1} \parallel [\langle u, v\rangle]_r^-(\alpha) - [\langle z, w\rangle]_r^-(\alpha) \parallel \\ & + \frac{1}{4} \sup_{0<\alpha\leq 1} \parallel [\langle u, v\rangle]_l^-(\alpha) - [\langle z, w\rangle]_l^-(\alpha) \parallel \end{aligned}$$

Where $\parallel . \parallel$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Theorem 1 (8)**.** *The metric space $(\mathbb{IF}_1, d_\infty)$ is complete.*

Proposition 1. *For all* $\alpha, \beta \in [0, 1]$ *and* $\langle u, v\rangle \in \mathbb{F}_1$
(i) $[\langle u, v\rangle]_\alpha \subset [\langle u, v\rangle]^\alpha$,
(ii) $[\langle u, v\rangle]_\alpha$ *and* $[\langle u, v\rangle]^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}$,
(iii) If $\alpha \leq \beta$ *then* $[\langle u, v\rangle]_\beta \subset [\langle u, v\rangle]_\alpha$ *and* $[\langle u, v\rangle]^\beta \subset [\langle u, v\rangle]^\alpha$,
(iv) If $\alpha_n \nearrow \alpha$ *then* $[\langle u, v\rangle]_\alpha = \cap_n[\langle u, v\rangle]_{\alpha_n}$ *and* $[\langle u, v\rangle]^\alpha = \cap_n[\langle u, v\rangle]^{\alpha_n}$.

Let M any set and $\alpha \in [0, 1]$ *we denote by :*

$$M_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\} \text{ and } M^\alpha = \{x \in \mathbb{R} : v(x) \leq 1 - \alpha\}$$

Lemma 1 (8). *Let* $\{M_\alpha, \alpha \in [0, 1]\}$ *and* $\{M^\alpha, \alpha \in [0, 1]\}$ *two families of subsets of* $\mathbb{R}$ *satifies (i)-(iv) in Proposition 1, if u and v define by :*

$$u(x) = \begin{cases} 0 & if x \notin M_0 \\ sup\{\alpha \in [0, 1] : x \in M_\alpha\} & if x \in M_0 \end{cases}$$

$$v(x) = \begin{cases} 1 & if x \notin M_0 \\ 1 - sup\{\alpha \in [0, 1] : x \in M^\alpha\} & if x \in M_0 \end{cases}$$

then $\langle u, v\rangle \in \mathbb{F}_1$

Definition 4. *Let* $\langle u, v\rangle$, $\langle u', v'\rangle \in \mathbb{F}_1$, *if there exists* $\langle w, z\rangle \in \mathbb{F}_1$ *such that*

$$\langle u, v\rangle = \langle u', v'\rangle + \langle w, z\rangle,$$

then $\langle w, z\rangle$ $\rangle$ *is called Hukuhara difference of* $\langle u, v\rangle$ *and* $\langle u', v'\rangle$ *denote by* $\langle u, v\rangle \ominus_H \langle u', v'\rangle$.

Definition 5. *The generalized Hukuhara difference of two intuitionistic fuzzy number* $\langle u, v\rangle$ *and* $\langle u', v'\rangle \in \mathbb{F}_1$ *is as follows:*

$$\langle u', v'\rangle \ominus_{gH} \langle u, v\rangle = \langle u", v"\rangle \Longleftrightarrow \begin{cases} i)\langle u', v'\rangle = \langle u, v\rangle + \langle u", v"\rangle \\ or \\ ii)\langle u, v\rangle = \langle u', v'\rangle + (-1)\langle u", v"\rangle. \end{cases}$$

Definition 6 (10). *Let be* $F : J \to \mathbb{IF}_1$ *and* $x_0 \in (a, b)$. *It is said that F is strongly generalized differentiable on* x_0, *if* $\exists$ $F'(x_0) \in E^1$, *such that :*
(i) for all $h > 0$ *sufficiently small,* $\exists$ $F(x_0 + h) \ominus_{gH} F(x_0)$, $F(x_0) \ominus_{gH} F(x_0 - h)$

and the limits (in the metric D)

$$\lim_{h\to 0}\frac{F(x_0+h)\ominus_{gH}F(x_0)}{h} = \lim_{h\to 0}\frac{F(x_0)\ominus_{gH}F(x_0-h)}{h} = F'(x_0)$$

Or

(ii) for all $h>0$ *sufficiently small,* $\exists\, F(x_0)\ominus_{gH}F(x_0+h)$, $F(x_0-h)\ominus_{gH}F(x_0)$ *and the limits*

$$\lim_{h\to 0}\frac{F(x_0)\ominus_{gH}F(x_0+h)}{-h} = \lim_{h\to 0}\frac{F(x_0-h)\ominus_{gH}F(x_0)}{-h} = F'(x_0)$$

Or

(iii) for all $h>0$ *sufficiently small,* $\exists\, F(x_0+h)\ominus_{gH}F(x_0)$, $F(x_0-h)\ominus_{gH}F(x_0)$ *and the limits*

$$\lim_{h\to 0}\frac{F(x_0+h)\ominus_{gH}F(x_0)}{h} = \lim_{h\to 0}\frac{F(x_0-h)\ominus_{gH}F(x_0)}{-h} = F'(x_0)$$

Or

(iv) for all $h>0$ *sufficiently small,* $\exists\, F(x_0)\ominus_{gH}F(x_0+h)$, $F(x_0)\ominus_{gH}F(x_0-h)$ *and the limits*

$$\lim_{h\to 0}\frac{F(x_0)\ominus_{gH}F(x_0+h)}{-h} = \frac{F(x_0)\ominus_{gH}F(x_0-h)}{h} = F'(x_0)$$

where, h and (-h) at denominators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$, respectively.

Theorem 2 (10)**.** *Let* $F(t)$ *and* $F'(t)$ *are differentiable functions. Denote*

$$[F'(t)]_\alpha = [\underline{F'}^+(t;\alpha), \overline{F'}^+(t;\alpha)],\ [F'(t)]^\alpha = [\underline{F'}^-(t;\alpha), \overline{F'}^-(t;\alpha)],\ 0\le\alpha\le 1.$$

Then $\underline{F'}^+(t;\alpha), \overline{F'}^+(t;\alpha), \underline{F'}^-(t;\alpha), \overline{F'}^-(t;\alpha)$ *are differentiable and we have,*

$$[F''(t)]_\alpha = [\underline{F''}^+(t;\alpha), \overline{F''}^+(t;\alpha)],\ [F''(t)]^\alpha = [\underline{F''}^-(t;\alpha), \overline{F''}^-(t;\alpha)]$$

Definition 7 (10)**.** *A mapping* $F : J \to \mathbb{IF}_1$ *is differentiale of second order on* x_0 *if there exist* $F''(x_0)\in\mathbb{IF}_1$ *such that:*

1) For all $h>0$ $\exists\, F'(x_0+h)\ominus_{gH}F'(x_0)$, $\exists F'(x_0)\ominus_{gH}F'(x_0-h)$ *and the limits (in the metric D)*

$$\lim_{h\to 0^+}\frac{F'(x_0+h)\ominus_{gH}F'(x_0)}{h} = \lim_{h\to 0^+}\frac{F'(x_0)\ominus_{gH}F'(x_0-h)}{h} = F''(x_0)$$

2) For all $h<0$ $\exists\, F'(x_0+h)\ominus_{gH}F'(x_0)$, $\exists F'(x_0)\ominus_{gH}F'(x_0-h)$ *and the limits (in the metric D)*

$$\lim_{h\to 0^-}\frac{F'(x_0+h)\ominus_{gH}F'(x_0)}{h} = \lim_{h\to 0^-}\frac{F'(x_0)\ominus_{gH}F'(x_0-h)}{h} = F''(x_0)$$

3. THE INTUITIONISTIC FUZZY LAPLACE TRANSFORM METHOD

Theorem 3. *Let $u(x,t)$ be an intuitionistic fuzzy-valued function on $[a,\infty)$ represented by $(\underline{u}(x,t,\alpha),\overline{u}(x,t,\alpha),$
$\underline{\underline{u}}(x,t,\alpha),\overline{\overline{u}}(x,t,\alpha))$, for all fixed $\alpha \in [0,1]$, assume that $\underline{u}(x,t,\alpha),\overline{u}(x,t,\alpha),\underline{\underline{u}}(x,t,\alpha),\overline{\overline{u}}(x,t,\alpha)$ are Riemann-integrabl on [a,b] for every $b \geq a$, and assume that for positive constants $\underline{M}(\alpha)$, $\overline{M}(\alpha)$, $\underline{\underline{M}}(\alpha)$ and $\overline{\overline{M}}(\alpha)$ such that*

$$\int_a^\infty \mid (\underline{u}(x,t,\alpha) \mid dt \leq \underline{M}(\alpha), \quad \int_a^\infty \mid (\overline{u}(x,t,\alpha) \mid dt \leq \overline{M}(\alpha),$$
$$\int_a^\infty \mid (\underline{\underline{u}}(x,t,\alpha) \mid dt \leq \underline{\underline{M}}(\alpha), \quad \int_a^\infty \mid (\overline{\overline{u}}(x,t,\alpha) \mid dt \leq \overline{\overline{M}}(\alpha).$$

Then $u(x,t)$ is an improper intuitionistic fuzzy Riemann-integrable on $[a,\infty)$ and the improper intuitionistic fuzzy Riemann-integrable is an intuitionistic fuzzy number.

Furthermore, we have for all $\alpha \in [0,1]$:

$\int_a^\infty u(x,t,\alpha)dt$= $(\int_a^\infty \underline{u}(x,t,\alpha)dt, \int_a^\infty \overline{u}(x,t,\alpha)dt, \int_a^\infty \underline{\underline{u}}(x,t,\alpha)dt, \int_a^\infty \overline{\overline{u}}(x,t,\alpha)dt)$.

Definition 8. *Let $u = u(x,t)$ be continuous intuitionistic fuzzy-valued function on $[0,\infty)$, suppose that $u(x,t)e^{-st}$ is intuitionistic fuzzy Riemann-integrable on $[0,\infty)$, then we denote by $U(x,s) = \int_0^\infty e^{-st}u(x,t)dt$ the intuitionistic fuzzy laplace transform, such as : $U(x,s) = L\{u(x,t)\}$ = $\int_0^\infty e^{-st}u(x,t)dt \quad s > 0$.*

We have for all $\alpha \in [0,1]$:

$\int_a^\infty u(x,t,\alpha)e^{-st}dt$=
$(\int_a^\infty \underline{u}(x,t,\alpha)e^{-st}dt, \int_a^\infty \overline{u}(x,t,\alpha)e^{-st}dt, \int_a^\infty \underline{\underline{u}}(x,t,\alpha)e^{-st}dt, \int_a^\infty \overline{\overline{u}}(x,t,\alpha)e^{-st}dt)$.

Therefore by using the definition of classical Laplace transform, we can present this definition of the intuitionistic fuzzy laplace transform based on the $\alpha - cut$ of the intuitionistic fuzzy valued function as follows :

$$U(x,s) = L\{u(x,t)\} = ([\underline{\ell}\{u(x,t,\alpha)\}, \overline{\ell}\{u(x,t,\alpha)\}]; [\underline{\underline{\ell}}\{u(x,t,\alpha)\}, \overline{\overline{\ell}}\{u(x,t,\alpha)\}])$$
$$= ([\int_0^\infty e^{-st}\underline{u}(x,t,\alpha)dt, \int_0^\infty e^{-st}\overline{u}(x,t,\alpha)dt]; [\int_0^\infty e^{-st}\underline{\underline{u}}(x,t,\alpha)dt, \int_0^\infty e^{-st}\overline{\overline{u}}(x,t,\alpha)dt])$$

Remark 3. *For the application of the intuitionistic fuzzy laplace transform method we remark that the transform of dervative is the derivative of the transform :*

$$L[\frac{\partial^2 u}{\partial x^2}(x,t)] = \int_0^\infty e^{-st}\frac{\partial^2}{\partial x^2}u(x,t)dt$$
$$= \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}u(x,t)dt$$
$$= \frac{\partial^2}{\partial x^2}U(x,s)$$

Theorem 4. *Let $u'(x,t)$ be an integrable fuzzy-valued function, and $u(x,t)$ is the primitive of $u'(x,t)$ on $[0,\infty)$. Then*

$$L[\frac{\partial u}{\partial t}(x,t)] = s.L[u(x,t)] \ominus u(x,0) \tag{1}$$

Where u is (i)-differentiable.

Or

$$L[\frac{\partial u}{\partial t}(x,t)] = -u(x,0) \ominus (-s.L[u(x,t)]) \tag{2}$$

Where u is (ii)-differentiable.

Proof. We denote by $(\underline{u}(x,t,\alpha), \overline{u}(x,t,\alpha), \underline{\underline{u}}(x,t,\alpha), \overline{\overline{u}}(x,t,\alpha))$ the parametric forms of the $u(x,t)$ and $u'(x,t)$ respectively.

$$\begin{aligned} L[\tfrac{\partial u}{\partial t}(x,t)] &= \textstyle\int_0^\infty e^{-st}\tfrac{\partial u}{\partial t}(x,t)dt \\ &= s\textstyle\int_0^\infty u(x,t)e^{-st}dt \ominus u(x,0) \\ &= sL\{u(x,t)\} \ominus u(x,0) \end{aligned}$$

since u is (i)-differentiable.

Now we assume that u is (ii)-differentiable, for $\alpha \in [0,1]$ we have :
$L[\frac{\partial u}{\partial t}(x,t)] = -u(x,0) \ominus (-s\int_0^\infty u(x,t)e^{-st}dt)$
$= -u(x,0) \ominus (-s.L[u(x,t)])$ □

Theorem 5. *Let $u(x,t)$ be a continuous intuitionistic fuzzy-valued function, such that $e^{-st}u(x,t)$, $e^{-st}\frac{\partial u}{\partial t}(x,t)$ and $e^{-st}\frac{\partial^2 u}{\partial t^2}(x,t)$ are continuous and Riemann integrable on $[0,\infty)$. We distinguish between the following cases :*

1) If $u(x,t)$ and $\frac{\partial u}{\partial t}(x,t)$ are (i)-differentiable, then

$$L[\frac{\partial^2 u}{\partial t^2}(x,t)] = \{s^2L[u(x,t)] \ominus su(x,0)\} \ominus \frac{\partial u}{\partial t}(x,0) \tag{3}$$

2) If $u(x,t)$ is (i)-differentiable and $\frac{\partial u}{\partial t}(x,t)$ is (ii)-differentiable, then

$$L[\frac{\partial^2 u}{\partial t^2}(x,t)] = (-\frac{\partial u}{\partial t}(x,0)) \ominus \{-s^2L[u(x,t)] \ominus (-su(x,0))\} \tag{4}$$

3) If $u(x,t)$ is (ii)-differentiable and $\frac{\partial u}{\partial t}(x,t)$ is (i)-differentiable, then

$$L[\frac{\partial^2 u}{\partial t^2}(x,t)] = \{(-su(x,0)) \ominus (-s^2L[u(x,t))]\} \ominus \frac{\partial u}{\partial t}(x,0) \tag{5}$$

4) If $u(x,t)$ and $\frac{\partial u}{\partial t}(x,t)$ are (ii)-differentiable, then

$$L[\frac{\partial^2 u}{\partial t^2}(x,t)] = s^2 L[u(x,t)] \ominus su(x,0) - \frac{\partial u}{\partial t}(x,0) \tag{6}$$

Proof. 1) Suppose that $u(x,t)$ and $\frac{\partial u}{\partial t}(x,t)$ are (i)-differentiable, then by appying 1) to $u(x,t)$ and $\frac{\partial u}{\partial t}(x,t)$ respectively, we have :

$L[\frac{\partial u}{\partial t}(x,t)] = sL[u(x,t)] \ominus u(x,0)$ and $L[\frac{\partial^2 u}{\partial t^2}(x,t)] = sL\{\frac{\partial u}{\partial t}(x,t)\} \ominus \frac{\partial u}{\partial t}(x,0)$

then we find that :

$$\begin{aligned} L[\tfrac{\partial^2 u}{\partial t^2}(x,t)] &= s\{sL[u(x,t)] \ominus u(x,0)\} \ominus \tfrac{\partial u}{\partial t}(x,0) \\ &= \{s^2 L[u(x,t)] \ominus su(x,0)\} \ominus \tfrac{\partial u}{\partial t}(x,0) \end{aligned}$$

Then for any $\alpha \in [0,1]$, if we use the parametric form and the classical transform we get:

$$\begin{cases} \mathcal{L}[\frac{\partial^2 \underline{u}}{\partial t^2}(x,t,\alpha)] = s\{s\mathcal{L}[\underline{u}(x,t,\alpha)] \ominus \underline{u}(x,0,\alpha)\} \ominus \frac{\partial \underline{u}}{\partial t}(x,0,\alpha) \\ \mathcal{L}[\frac{\partial^2 \overline{u}}{\partial t^2}(x,t,\alpha)] = s\{s\mathcal{L}[\overline{u}(x,t,\alpha)] \ominus \overline{u}(x,0,\alpha)\} \ominus \frac{\partial \overline{u}}{\partial t}(x,0,\alpha) \\ \mathcal{L}[\frac{\partial^2 \underline{\underline{u}}}{\partial t^2}(x,t,\alpha)] = s\{s\mathcal{L}[\underline{\underline{u}}(x,t,\alpha)] \ominus \underline{\underline{u}}(x,0,\alpha)\} \ominus \frac{\partial \underline{\underline{u}}}{\partial t}(x,0,\alpha) \\ \mathcal{L}[\frac{\partial^2 \overline{\overline{u}}}{\partial t^2}(x,t,\alpha)] = s\{s\mathcal{L}[\overline{\overline{u}}(x,t,\alpha)] \ominus \overline{\overline{u}}(x,0,\alpha)\} \ominus \frac{\partial \overline{\overline{u}}}{\partial t}(x,0,\alpha) \end{cases}$$

2) Suppose that $u(x,t)$ is (i)-differentiable and $\frac{\partial u}{\partial t}(x,t)$ is (ii)-differentiable then by appying (1) to $u(x,t)$ and (2) to $\frac{\partial u}{\partial t}(x,t)$, we have :

$L[\frac{\partial u}{\partial t}(x,t)] = sL[u(x,t)] \ominus u(x,0)$ and $L[\frac{\partial^2 u}{\partial t^2}(x,t)] = -\frac{\partial u}{\partial t}(x,0) \ominus (-sL[\frac{\partial u}{\partial t}(x,t)])$

therefore

$$\begin{aligned} L[\tfrac{\partial^2 u}{\partial t^2}(x,t)] &= -\tfrac{\partial u}{\partial t}(x,0) \ominus (-s\{sL[u(x,t)] \ominus u(x,0)\}) \\ &= -\tfrac{\partial u}{\partial t}(x,0) \ominus \{-s^2 L[u(x,t)] \ominus (-su(x,0))\} \end{aligned}$$

Then for any $\alpha \in [0,1]$, if we use the parametric form and the classical transform we get:

$$\begin{cases} \mathcal{L}[\frac{\partial^2 \underline{u}}{\partial t^2}(x,t,\alpha)] = -\frac{\partial \underline{u}}{\partial t}(x,0,\alpha) \ominus \{-s^2\mathcal{L}[\underline{u}(x,t,\alpha)] \ominus (-s\underline{u}(x,0,\alpha))\} \\ \mathcal{L}[\frac{\partial^2 \overline{u}}{\partial t^2}(x,t,\alpha)] = -\frac{\partial \overline{u}}{\partial t}(x,0,\alpha) \ominus \{-s^2\mathcal{L}[\overline{u}(x,t,\alpha)] \ominus (-s\overline{u}(x,0,\alpha))\} \\ \mathcal{L}[\frac{\partial^2 \underline{\underline{u}}}{\partial t^2}(x,t,\alpha)] = -\frac{\partial \underline{\underline{u}}}{\partial t}(x,0,\alpha) \ominus \{-s^2\mathcal{L}[\underline{\underline{u}}(x,t,\alpha)] \ominus (-s\underline{\underline{u}}(x,0,\alpha))\} \\ \mathcal{L}[\frac{\partial^2 \overline{\overline{u}}}{\partial t^2}(x,t,\alpha)] = -\frac{\partial \overline{\overline{u}}}{\partial t}(x,0,\alpha) \ominus \{-s^2\mathcal{L}[\overline{\overline{u}}(x,t,\alpha)] \ominus (-s\overline{\overline{u}}(x,0,\alpha))\} \end{cases}$$

In the same way we can demonstrate 3) and 4) □

Theorem 6. *Let $u(x,t)$ et $v(x,t)$ be a continuous intuitionistic fuzzy-valued functions suppose that C_1, C_2 are constant, then*

$$L[(C_1 \odot u(x,t)) \oplus (C_2 \odot v(x,t))] = (C_1 \odot L[u(x,t)]) \oplus (C_2 \odot L[v(x,t)]).$$

Proof. Let $L[(C_1 \odot u(x,t)) \oplus (C_2 \odot v(x,t))] = \int_0^\infty ((C_1 \odot u(x,t)) \oplus (C_2 \odot v(x,t))) \odot e^{-sx} dx$

$$= \int_0^\infty C_1 \odot u(x,t) \odot e^{-sx} dx \oplus \int_0^\infty C_2 \odot v(x,t) \odot e^{-sx} dx$$
$$= (C_1 \odot \int_0^\infty u(x,t) \odot e^{-sx} dx) \oplus (C_2 \odot \int_0^\infty v(x,t) \odot e^{-sx} dx)$$
$$= C_1 \odot L[u(x,t)] \oplus C_2 \odot L[v(x,t)]$$

Hence

$L[(C_1 \odot u(x,t)) \oplus (C_2 \odot v(x,t))] = (C_1 \odot L[u(x,t)]) \oplus (C_2 \odot L[v(x,t)]).$

□

Remark 4. *Let $u(x,t)$ be continuous intuitionistic fuzzy-valued function on $[0,\infty)$ and $\lambda \geq 0$, then*

$$L[\lambda \odot u(x,t)] = \lambda \odot L[u(x,t)].$$

Proof. Intuitionistic fuzzy transform $\lambda \odot u(x,t)$ is denoted as $L[\lambda \odot u(x,t)] = \int_0^\infty \lambda \odot u(x,t) \odot e^{-sx} dx$ and also we have

$$\int_0^\infty \lambda \odot u(x,t) \odot e^{-sx} dx = \lambda \odot \int_0^\infty u(x,t) \odot e^{-sx} dx$$

Then $L[\lambda \odot u(x,t)] = \lambda \odot L[u(x,t)]$ □

Theorem 7. *Let $u(x,t)$ be continuous intuitionistic fuzzy-valued function and $L[u(x,t)] = F(t)$, then $L[e^{sx} \odot u(x,t)] = F(t-x)$ where e^{sx} is real value function and $t - x > 0$.*

Proof. $L[e^{sx} \odot u(x,t)] = \int_0^\infty e^{sx-tx} \odot u(x,t) dx$

$$= \int_0^\infty e^{-(t-s)x} \odot u(x,t) dx$$
$$= F(t-s).$$

□

4. INTUITIONISTIC FUZZY LAPLACE TRANSFORM FOR INTUITIONISTIC FUZZY WAVE EQUATIONS

In this section, our aim is to solve the one dimensional intuitionistic fuzzy wave equation by using the intuitionistic fuzzy Laplace transform method under strongly generalized differentiability, then the problem studied is of the form :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad 0 < x < 1, \quad t > 0, \\ u(x,0) = ([\alpha - 1, 1 - \alpha], [-2\beta, 2\beta]) \\ u_t(x,0) = ([5 + 2\alpha, 9 - 2\alpha], [3 + 4\beta, 11 - 4\beta]) \end{cases} \tag{7}$$

the parametric form of the IFIVP $\forall \alpha, \beta \in [0,1]$ is written as follows:

$$\begin{cases} \frac{\partial^2 \underline{u}}{\partial t^2}(x,t,\alpha) = a^2 \frac{\partial^2 \underline{u}}{\partial x^2}(x,t,\alpha), \\ \frac{\partial^2 \overline{u}}{\partial t^2}(x,t,\alpha) = a^2 \frac{\partial^2 \overline{u}}{\partial x^2}(x,t,\alpha), \\ \frac{\partial^2 \underline{\underline{u}}}{\partial t^2}(x,t,\beta) = a^2 \frac{\partial^2 \underline{\underline{u}}}{\partial x^2}(x,t,\beta), \\ \frac{\partial^2 \overline{\overline{u}}}{\partial t^2}(x,t,\beta) = a^2 \frac{\partial^2 \overline{\overline{u}}}{\partial x^2}(x,t,\beta). \end{cases} \tag{8}$$

By using the fuzzy Laplace transformation method, we obtain :

$$L[\frac{\partial^2 u}{\partial t^2}(x,t)] = a^2 L[\frac{\partial^2 u}{\partial x^2}(x,t)]$$

i.e

$$\begin{cases} \mathcal{L}[\frac{\partial^2 \underline{u}}{\partial t^2}(x,t,\alpha)] = a^2 \mathcal{L}[\frac{\partial^2 \underline{u}}{\partial x^2}(x,t,\alpha)], \\ \mathcal{L}[\frac{\partial^2 \overline{u}}{\partial t^2}(x,t,\alpha)] = a^2 \mathcal{L}[\frac{\partial^2 \overline{u}}{\partial x^2}(x,t,\alpha)], \\ \mathcal{L}[\frac{\partial^2 \underline{\underline{u}}}{\partial t^2}(x,t,\beta)] = a^2 \mathcal{L}[\frac{\partial^2 \underline{\underline{u}}}{\partial x^2}(x,t,\beta)], \\ \mathcal{L}[\frac{\partial^2 \overline{\overline{u}}}{\partial t^2}(x,t,\beta)] = a^2 \mathcal{L}[\frac{\partial^2 \overline{\overline{u}}}{\partial x^2}(x,t,\beta)]. \end{cases} \tag{9}$$

Then

$$\begin{cases} s^2 \mathcal{L}[\underline{u}(x,t,\alpha)] - s\underline{u}(x,0,\alpha) - \underline{u}_t(x,0,\alpha) = a^2 \mathcal{L}[\frac{\partial^2 \underline{u}}{\partial x^2}(x,t,\alpha)] \\ s^2 \mathcal{L}[\overline{u}(x,t,\alpha)] - s\overline{u}(x,0,\alpha) - \overline{u}_t(x,0,\alpha) = a^2 \mathcal{L}[\frac{\partial^2 \overline{u}}{\partial x^2}(x,t,\alpha)] \\ s^2 \mathcal{L}[\underline{\underline{u}}(x,t,\beta)] - s\underline{\underline{u}}(x,0,\beta) - \underline{\underline{u}}_t(x,0,\beta) = a^2 \mathcal{L}[\frac{\partial^2 \underline{\underline{u}}}{\partial x^2}(x,t,\beta)] \\ s^2 \mathcal{L}[\overline{\overline{u}}(x,t,\beta)] - s\overline{\overline{u}}(x,0,\beta) - \overline{\overline{u}}_t(x,0,\beta) = a^2 \mathcal{L}[\frac{\partial^2 \overline{\overline{u}}}{\partial x^2}(x,t,\beta)] \end{cases} \tag{10}$$

therefore

$$\begin{cases} \frac{\partial^2}{\partial x^2}\underline{U}(x,t,\alpha) - \frac{s^2}{a^2}\underline{U}(x,t,\alpha) = \frac{1}{a^2}(-s\underline{u}(x,0,\alpha) - \underline{u}_t(x,0,\alpha)) \\ \frac{\partial^2}{\partial x^2}\overline{U}(x,t,\alpha) - \frac{s^2}{a^2}\overline{U}(x,t,\alpha) = \frac{1}{a^2}(-s\overline{u}(x,0,\alpha) - \overline{u}_t(x,0,\alpha)) \\ \frac{\partial^2}{\partial x^2}\underline{\underline{U}}(x,t,\beta) - \frac{s^2}{a^2}\underline{\underline{U}}(x,t,\beta) = \frac{1}{a^2}(-s\underline{\underline{u}}(x,0,\beta) - \underline{\underline{u}}_t(x,0,\beta)) \\ \frac{\partial^2}{\partial x^2}\overline{\overline{U}}(x,t,\beta) - \frac{s^2}{a^2}\overline{\overline{U}}(x,t,\beta) = \frac{1}{a^2}(-s\overline{\overline{u}}(x,0,\beta) - \overline{\overline{u}}_t(x,0,\beta)) \end{cases} \tag{11}$$

hence $\mathcal{L}[\frac{\partial^2 \underline{u}}{\partial x^2}(x,t,\alpha)] = \frac{\partial^2}{\partial x^2}\underline{U}(x,t,\alpha)$, $\mathcal{L}[\frac{\partial^2 \overline{u}}{\partial x^2}(x,t,\alpha)] = \frac{\partial^2}{\partial x^2}\overline{U}(x,t,\alpha)$, $\mathcal{L}[\frac{\partial^2 \underline{\underline{u}}}{\partial x^2}(x,t,\beta)] = \frac{\partial^2}{\partial x^2}\underline{\underline{U}}(x,t,\beta)$ and $\mathcal{L}[\frac{\partial^2 \overline{\overline{u}}}{\partial x^2}(x,t,\beta)] = \frac{\partial^2}{\partial x^2}\overline{\overline{U}}(x,t,\beta)$ After that we replace the initiale value of parametric form $u_t(x,0,\alpha)$ and $u(x,0,\alpha)$ in the system last, we get the following:

$$\begin{cases} \frac{\partial^2}{\partial x^2}\underline{U}(x,t,\alpha) - \frac{s^2}{a^2}\underline{U}(x,t,\alpha) = \frac{1}{a^2}(-s(\alpha-1) - (5+2\alpha)) \\ \frac{\partial^2}{\partial x^2}\overline{U}(x,t,\alpha) - \frac{s^2}{a^2}\overline{U}(x,t,\alpha) = \frac{1}{a^2}(-s(1-\alpha) - (9-2\alpha)) \\ \frac{\partial^2}{\partial x^2}\underline{\underline{U}}(x,t,\beta) - \frac{s^2}{a^2}\underline{\underline{U}}(x,t,\beta) = \frac{1}{a^2}(-s(-2\beta) - (3+4\beta)) \\ \frac{\partial^2}{\partial x^2}\overline{\overline{U}}(x,t,\beta) - \frac{s^2}{a^2}\overline{\overline{U}}(x,t,\beta) = \frac{1}{a^2}(-s(2\beta) - (11-4\beta)) \end{cases} \tag{12}$$

the solution of this problem is :

$$U(x,s) = U_h(x,s) \oplus U_p(x,s)$$

Wherein $U_h(x,s)$ is general solution of the homogeneous equation and $U_p(x,s)$ is the particular solution of the non-homogeneous equation such that

$$U_h(x,s) = Ce^{\frac{s}{a}x} + Ce^{-\frac{s}{a}x}, \quad U_p(x,s) = \frac{u(x,0)}{s} \oplus \frac{u_t(x,0)}{s^2}$$

$$\begin{cases} \underline{U}(x,s,\alpha) = C_1 e^{\frac{s}{a}x} + C_2 e^{-\frac{s}{a}x} + \frac{(\alpha-1)}{s} + \frac{(5+2\alpha)}{s^2} \\ \overline{U}(x,s,\alpha) = C_3 e^{\frac{s}{a}x} + C_4 e^{-\frac{s}{a}x} + \frac{(1-\alpha)}{s} + \frac{(9-2\alpha)}{s^2} \quad \forall \alpha,\beta \in [0,1] \\ \underline{\underline{U}}(x,s,\beta) = C_5 e^{\frac{s}{a}x} + C_6 e^{-\frac{s}{a}x} + \frac{(-2\beta)}{s} + \frac{(3+4\beta)}{s^2} \\ \overline{\overline{U}}(x,s,\beta) = C_7 e^{\frac{s}{a}x} + C_8 e^{-\frac{s}{a}x} + \frac{(2\beta)}{s} + \frac{(11-4\beta)}{s^2} \end{cases} \tag{13}$$

now, assume that $C_1 = C_3 = C_5 = C_7 = 0$ and after finding the values of the constants C_2, C_4, C_6 and C_8 by using the boundary conditions we find the following system :

$$\begin{cases} \mathcal{L}[\underline{u}(x,t,\alpha)] = \frac{1-\alpha}{s}e^{-\frac{s}{a}x} - \frac{(5+2\alpha)}{s^2}e^{-\frac{s}{a}x} + \frac{(\alpha-1)}{s} + \frac{(5+2\alpha)}{s^2} \\ \mathcal{L}[\overline{u}(x,t,\alpha)] = \frac{\alpha-1}{s}e^{-\frac{s}{a}x} - \frac{(2\alpha-9)}{s^2}e^{-\frac{s}{a}x} + \frac{(1-\alpha)}{s} + \frac{(9-2\alpha)}{s^2} \\ \mathcal{L}[\underline{\underline{u}}(x,t,\beta)] = \frac{(2\beta)}{s}e^{-\frac{s}{a}x} - \frac{(3+4\beta)}{s^2}e^{-\frac{s}{a}x} - \frac{(2\beta)}{s} + \frac{(3+4\beta)}{s^2} \\ \mathcal{L}[\overline{\overline{u}}(x,t,\beta)] = -\frac{(2\beta)}{s}e^{-\frac{s}{a}x} + \frac{(4\beta-11)}{s^2}e^{-\frac{s}{a}x} + \frac{(2\beta)}{s} + \frac{(11-4\beta)}{s^2} \end{cases} \tag{14}$$

By the inverse Laplace transform we deduce:

$$\begin{cases} \underline{u}(x,t,\alpha) = (1-\alpha)H(t-\frac{x}{a}) - (5+2\alpha)(t-\frac{x}{a})H(t-\frac{x}{a}) + (\alpha-1) + (5+2\alpha)(t-\frac{x}{a}) \\ \overline{u}(x,t,\alpha) = (\alpha-1)H(t-\frac{x}{a}) + (2\alpha-9)(t-\frac{x}{a})H(t-\frac{x}{a}) + (1-\alpha) + (9-2\alpha)(t-\frac{x}{a}) \\ \underline{\underline{u}}(x,t,\beta) = (2\beta)H(t-\frac{x}{a}) - (3+4\beta)(t-\frac{x}{a})H(t-\frac{x}{a}) - (2\beta) + (3+4\beta)(t-\frac{x}{a}) \\ \overline{\overline{u}}(x,t,\beta) = -(2\beta)H(t-\frac{x}{a}) + (4\beta-11)(t-\frac{x}{a})H(t-\frac{x}{a}) + (2\beta) + (11-4\beta)(t-\frac{x}{a}) \end{cases} \tag{15}$$

$$\text{Where } H(t-\tfrac{x}{a}) = \begin{cases} 0, & t < \frac{x}{a} \\ 1, & t \geq \frac{x}{a} \end{cases}$$

We note that $\underline{u}(x,t,\alpha) \leq \overline{u}(x,t,\alpha)$, $\underline{\underline{u}}(x,t,\beta) \leq \overline{\overline{u}}(x,t,\beta)$ as well as the functions $\underline{u}(x,t,\alpha)$, $\underline{\underline{u}}(x,t,\beta)$ are increasing with respect to α and β respectively and the functions $\overline{u}(x,t,\alpha)$, $\overline{\overline{u}}(x,t,\beta)$ are decreasing with respect to α and β respectively, $\forall\alpha,\beta \in [0,1]$.

Hence, we have proved that $(\underline{u}(x,t,\alpha), \overline{u}(x,t,\alpha), \underline{\underline{u}}(x,t,\beta), \overline{\overline{u}}(x,t,\beta))$ is the parametric form of the solution of the problem (7).

5. CONCLUSION

In this work, the main objective is to solve the one dimensional intuitionistic fuzzy wave equations under strongly generalized H-differentiability by using the Intuitionistic fuzzy laplace transform method, the accuracy and efficiency of the proposed method are shown with numerical example.

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