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# Variational Analysis of a Quasi-Static Thermo-Electro-Elastic Contact Problem with Normal Compliance

**Lahcen OUMOUACHA** [*1], **Rachid FAKHAR**[2], **EL-Hassan BENKHIRA**[3] , **Abdelhadi HACHLAF**[4]

[1]*University Sultan Moulay Slimane, Laboratory LS3M, 25000 Khouribga, Morocco*
[2]*University Sultan Moulay Slimane, Laboratory LS3M, 25000 Khouribga, Morocco.*
[3]*University Moulay Ismaïl, Laboratory MACS, ESTM, BP 3103, Toulal-Meknes, Morocco.*
[4]*University Cadi Ayyad, Faculty of Semlalia, B.P. 2390 , 40000, Marrakech, Morocco.*

**Abstract**

In this work we present the existence of weak solution for a quasi-static thermo-piezoelectric problem which describes frictional contact problem between a deformable body and a thermally-electrically conductive foundation. We model the material's behavior with a thermo-electro-elastic constitutive law and the contact with a normal compliance conditions and a version of Coulomb's law of dry friction including the electrical and thermal conductivity conditions in which the frictional heat generated in the process is taken into account. The model is in the form of a coupled system for displacements, electric potential, and temperature. The existence of solutions is obtained from an elliptic quasi-variational inequalities, time discretization and Banach fixed point argument.

**Keywords:** Thermopiezoelectric material, quasi-static process, Coulomb's friction law, frictional heat generation, Variational inequality.

## 1. INTRODUCTION

In this paper we consider the quasi-static frictional contact between a thermo-piezoelectric body and a conductive foundation. The body is assumed to satisfy the piezoelectric constitutive equations with added thermal effects. Our main interest lies

*Corresponding author's E-mail ID: lahcen.oumouacha@gmail.com

in coupling between thermal effects and piezoelectrical properties and in electrical and thermal boundary conditions describing the contact. More precisely, it is supposed that the contact is modeled with normal compliance, a version of Coulomb's law of dry friction, a regularized electrical conductivity condition and a condition of thermal contact in which the frictional heat generated in the process is taken into account.

Recently piezoelectric frictional contact problems with or without conductivity of foundation have been investigated in a large number of papers, see e.g. [2, 10, 11, 12, 13] and the references therein. Contact problems involving thermo-piezoelectric materials with conductive foundation in the static case have been studied in [3, 4, 5]. This paper extends the results provided in [5] to the quasi-static case. We deals with a mathematical model for the quasi-static process of frictional contact between a thermo-piezoelectric body and a conductive foundation, under small deformations hypothesis, wherein the material's behavior is modeled by a linear thermo-electro-elastic constitutive law and the contact is described with the normal compliance, the quasi-static of nonlocal Coulomb friction law of dry friction, a regularized electrical conductivity condition and the condition of thermal contact.

The paper is structured as follows. In Section 3 we introduce the notation, we list the assumptions on problem's data, we derive the variational formulation of the problem and we present our main result stated in Theorem 3.1. The proof of this theorem is provided in sections 4-6. It is carried out in several steps and is based on arguments of compactness, time discretization and Banach fixed point theorem.

## 2. MATHEMATICAL MODEL AND ITS MECHANICAL FORMULATION

We consider a piezoelectric body occupies a bounded domain $\Omega$ in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary $\Gamma$. We suppose that $\Gamma$ is divided in three disjoint parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$ on the one hand and that $\Gamma_D \cup \Gamma_N$ is partitioned into two open parts $\Gamma_a$ and $\Gamma_b$ on the other hand, such that $\Gamma_D$ and $\Gamma_a$ being on nonzero measure. The body is supposed to be stress free at a free temperature and the temperature variations, accompanying the deformations, produce changes in the material parameters which are considered as depending on temperature. Let $[0, T]$ be the time interval of interest, the body is clamped on $\Gamma_D \times (0, T)$, the displacement field vanishes there. In $\Omega \times (0, T)$, it is subjected to a volume force $f_0$, a volume electric charge $\phi_0$ and heat source $q_0$. Surface tractions $f_N$ are applied on $\Gamma_N \times (0, T)$, surface electric charges $\phi_b$ act on $\Gamma_b \times (0, T)$ and the body is subjected to heat source $q_0$ on $\Omega \times (0, T)$. Moreover, we suppose that is at null electric potential on $\Gamma_a \times (0, T)$ and null variation of temperature on $\Gamma_D \cup \Gamma_N \times (0, T)$. We assume that body is in frictional contact with an obstacle over the part $\Gamma_C$.

Here and below, to simplify the notation, we do not indicate the dependence of various functions on the spatial variable $x \in \bar{\Omega}$, the indices $i, j, k, l$ take values between $1$ and $d$, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. We also denote by $\operatorname{Div} \sigma = (\sigma_{ij,j})$ and $\operatorname{div} D = (D_{j,j})$ the divergence operator for tensor and vector valued functions, respectively.

Let $\mathbb{S}^d$ be the space of second order symmetric tensors on $\mathbb{R}^d$. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$
\begin{aligned}
u \cdot v = u_i \cdot v_i\,, &\quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \quad \forall\, u,\ v \in \mathbb{R}^d,\\
\sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij}\,, &\quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}, \quad \forall \sigma, \tau = (\tau_{ij}) \in \mathbb{S}^d.
\end{aligned}
$$

Moreover, $\nu$ represents the unit exterior normal on $\Gamma$. Then the normal and the tangential components of the displacement vector $v$ and the stress $\sigma$ on the boundary are

$$
v_\nu = v \cdot \nu\,,\ v_\tau = v - v_\nu\, \nu \quad \text{and} \quad \sigma_\nu = \sigma\, \nu \cdot \nu\,,\ \sigma_\tau = \sigma\, \nu - \sigma_\nu\, \nu\,.
$$

The governing equations of thermo-electro-elasticity consist of the equilibrium equation, constitutive relations, strain-mechanical displacement, electric field-potential and thermal field-temperature change relations.

The elastic strain-displacement, electric field-potential and the thermal field-temperature change relations are given by

$$
\begin{aligned}
&\varepsilon(u) = (\frac{1}{2}(u_{i,j} + u_{j,i})) \quad \text{in } \Omega \times (0, T),\\
&E(\varphi) = -(\varphi_{,i}) \quad \text{in } \Omega \times (0, T),\\
&q = -\mathcal{K}\, \nabla\theta \quad \text{in } \Omega \times (0, T),
\end{aligned}
$$

where $\varepsilon(u) = (\varepsilon_{ij}(u)$, $E(\varphi) = (E_i(\varphi))$, $\mathcal{K} = (k_{ij})$, $u = (u_i)$, $\varphi$, $\theta$ are respectively, the linear strain tensor, quasi-static electric field vector, conductivity tensor, displacement vector field, electric potential and temperature change.

We suppose that the process is quasi-static. The equations of stress equilibrium, the equation of quasi-stationary electric field and the heat conduction equation are, respectively, given by

$$
\operatorname{Div} \sigma + f_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.1}
$$

$$
\operatorname{div} D = \phi_0 \quad \text{in } \Omega \times (0, T), \tag{2.2}
$$

$$
\dot{\theta} + \operatorname{div} q = -\mathcal{M}\, \dot{\varepsilon}(u) - \mathcal{P}\, \dot{E}(\varphi) + q_0 \quad \text{in } \Omega \times (0, T), \tag{2.3}
$$

where $\sigma = (\sigma_{ij})$, $D = (D_i)$ and $q = (q_i)$ represent the stress tensor, the electric displacement field and the heat flux vector, respectively.

The constitutive equations of a piezoelectric material including the effect thermal expansion can be written as

$$\sigma = \mathfrak{F}\,\varepsilon(u) - \mathcal{E}^* E(\varphi) - \mathcal{M}\,\theta \quad \text{in } \Omega \times (0,T), \tag{2.4}$$

$$D = \mathcal{E}\,\varepsilon(u) + \beta\,E(\varphi) + \mathcal{P}\,\theta \quad \text{in } \Omega \times (0,T), \tag{2.5}$$

where $\mathfrak{F} = (\mathfrak{F}_{ijkl})$, $\mathcal{E} = (e_{ijk})$, $\mathcal{M} = (m_{ij})$, $\beta = (\beta_{ij})$ and $\mathcal{P} = (p_i)$ are respectively, the linear elasticity operator, piezoelectric tensor, thermal expansion tensor, electric permittivity tensor, pyroelectric tensor and $\mathcal{E}^*$ is the transpose of $\mathcal{E}$. We note that $\mathcal{E}^* = (e_{ijk})$.

Finally, to complete the mechanical model according to the description of the physical setting, we have

$$u = 0 \quad \text{on } \Gamma_D \times (0,T), \tag{2.6}$$

$$\sigma\nu = f_N \quad \text{on } \Gamma_N \times (0,T), \tag{2.7}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{2.8}$$

$$D \cdot \nu = \phi_b \quad \text{on } \Gamma_b \times (0,T), \tag{2.9}$$

$$\theta = 0 \quad \text{on } \Gamma_D \cup \Gamma_N \times (0,T). \tag{2.10}$$

On the contact surface $\Gamma_C$, we consider

$$\sigma_\nu(u) = -p_\nu(u_\nu - g) \quad \text{on } \Gamma_C \times (0,T), \tag{2.11}$$

$$\left.\begin{array}{l} \|\sigma_\tau\| \leqslant p_\tau(u_\nu - g) \\ \|\sigma_\tau\| < p_\tau(u_\nu - g) \Longrightarrow \dot{u}_\tau = 0 \\ \|\sigma_\tau\| = p_\tau(u_\nu - g) \Longrightarrow (\exists\,\lambda \in \mathbb{R}^+),\ \dot{u}_\tau = -\lambda\,\sigma_\tau \end{array}\right\} \quad \text{on } \Gamma_C \times (0,T), \tag{2.12}$$

$$D \cdot \nu = \psi(u_\nu - g)\,\phi_L(\varphi - \varphi_F) \quad \text{on } \Gamma_C \times (0,T), \tag{2.13}$$

$$q \cdot \nu = -\mu\,p_\nu(u_\nu - g)\,S_c(\,\cdot\,,\|\dot{u}_\tau\|\,) + k_c(u_\nu - g)\,\phi_L(\theta - \theta_F) \quad \text{on } \Gamma_C \times (0,T), \tag{2.14}$$

$$u(\cdot,0) = u_0,\ \theta(\cdot,0) = \theta_0,\ \varphi(\cdot,0) = \varphi_0 \quad \text{in } \Omega. \tag{2.15}$$

Then, the condition (2.11) represent the normal compliance contact condition where in which $p_\nu$ is prescribed function. Relations (2.12) are a version of coulomb's law of dry friction, $p_\tau$ is prescribed nonnegative function, the so-called friction bound. The condition (2.13) is the electric boundary condition. Relation (2.14) (as in [1]) is the boundary condition for the temperature on $\Gamma_C$, we assume that there is heat exchange between the surface and the foundation, which is at temperature $\theta_F$, Moreover, the flux of heat generated by the frictional contact forces is proportional to the tangential velocity $\dot{u}_\tau$, The inclusion of such a term is required if the effects of the frictional heat generation are to be taken into account.

Finally, the initial conditions $u_0$, $\varphi_0$ and $\theta_0$ in (2.15) are given and the functions $\psi$ and $\phi_L$ used in (2.13) and (2.14) represent, respectively, the truncation function and a given positive function

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L \\ s & \text{if } -L \leq s \leq L \\ L & \text{if } s > L \end{cases} \quad , \quad \psi(r) = \begin{cases} 0 & \text{if } r < 0 \\ k\,\delta r & \text{if } 0 \leq r \leq 1/\delta\,, \\ k & \text{if } r > 1/\delta \end{cases}$$

where $L > 0$ is a sufficiently large constant, $\delta > 0$ is a small given parameter and $k \geq 0$ is the electrical conductivity coefficient.

With these assumptions, the mechanical problem of frictional contact of the piezoelectric body under thermal environment, may be formulated as follows

**Problem** $(P)$**.** *Find a displacement field* $u : \Omega \times [0,T] \to \mathbb{R}^d$*, an electric potential* $\varphi : \Omega \times [0,T] \to \mathbb{R}$ *and temperature field* $\theta : \Omega \times [0,T] \to \mathbb{R}$ *such that* (2.1)-(2.15) *hold.*

## 3. VARIATIONAL PROBLEM AND MAIN RESULT

First, Let us consider the following real Hilbert spaces

$$H = \{u = (u_i)\,,\ u_i \in L^2(\Omega)\} = \left[L^2(\Omega)\right]^d, \quad \mathcal{H} = \{\sigma = (\sigma_{ij})\,,\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

$$H_1 = \{u \in H\,,\ \varepsilon(u) \in \mathcal{H}\} = \left[H^1(\Omega)\right]^d, \qquad \mathcal{H}_1 = \{\sigma \in \mathcal{H}\,,\ \mathrm{Div}\,\sigma \in H\},$$

endowed with their natural norms and inner products as follows

$$(u,v)_H = \int_\Omega u_i\, v_i\, dx, \qquad (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\, \tau_{ij}\, dx, \qquad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\mathrm{Div}\,\sigma, \mathrm{Div}\,\tau)_H\,.$$

The notation $\langle \cdot, \cdot \rangle_{X',X}$ always denotes the duality pairing between a space $X$ and its dual $X$. We consider the trace map $\gamma : H_1 \to H_\Gamma = H^{1/2}(\Gamma)^d$, then for every $\sigma \in \mathcal{H}_1$, there exists $\sigma\nu \in H'_\Gamma = H^{-1/2}(\Gamma)^d$ satisfying the following Green formula

$$\langle \sigma\nu, \gamma v \rangle_{H'_\Gamma, H_\Gamma} = (\sigma, \varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}\,\sigma, v)_H\,, \quad \forall\, v \in\, H_1\,. \tag{3.1}$$

Moreover, if $\sigma$ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle_{H'_\Gamma, H_\Gamma} = \int_\Gamma \sigma\nu \cdot \gamma v \, da \,, \quad \forall\, v \in H_1 \,, \tag{3.2}$$

where $da$ is the surface element. Let us also introduce the following subspace $H^{1/2}_{\Gamma_C}$ of $L^2(\Gamma_C)$ given by

$$H^{1/2}_{\Gamma_C} = \big\{\, v_\nu \in L^2(\Gamma_C) \,, \ \ \exists\, v \in H_1 \,,\ v_\nu = \gamma v \cdot \nu \,\big\},$$

and called the space of normal traces on $\Gamma_C$. The spaces $H^{1/2}_{\Gamma_C}$ and its dual $H^{-1/2}_{\Gamma_C}$ endowed with the norms

$$\|v_\nu\|_{H^{1/2}_{\Gamma_C}} = \inf_{v \in H_1} \big\{\, \|v\|_{H_1} \,, \ \ v_\nu = \gamma v \cdot \nu \,\big\}\,, \qquad \forall\, v_\nu \in H^{1/2}_{\Gamma_C} \,,$$

$$\|\sigma_\nu\|_{H^{-1/2}_{\Gamma_C}} = \sup_{v_\nu \in H^{1/2}_{\Gamma_C}} \left\{ \frac{\langle \sigma_\nu, v_\nu\rangle_{\Gamma_C}}{\|v_\nu\|_{H^{1/2}_{\Gamma_C}}} \right\}, \qquad \forall\, \sigma_\nu \in H^{-1/2}_{\Gamma_C} \,,$$

where $\langle \cdot, \cdot \rangle_{\Gamma_C}$ denote the duality pairing between $H^{-\frac{1}{2}}_{\Gamma_C}$ and $H^{\frac{1}{2}}_{\Gamma_C}$.

In the sequel, for every $v \in H_1$, we denote the trace map $\gamma v$ of $v$ on $\Gamma$, again by $v$ to simplify notations. In order to prove existence results concerning our problem, we first give equivalent variational formulation and to this aim, we need to introduce the following closed subspace of $H_1$ defined by

$$V = \{\, v \in H_1 \,, \ \ v = 0 \ \text{ on } \ \Gamma_D \,\} \,.$$

Since $meas(\Gamma_D) > 0$, the following Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}} \geqslant c_k \, \|v\|_{H_1} \,, \quad \forall\, v \in V \,, \tag{3.3}$$

where $c_k > 0$ is a constant which depends only on $\Omega$ and $\Gamma_D$. Therefore the space $V$ endowed with the inner product $(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}$ is a real Hilbert space and its associated norm $\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$ is equivalent on $V$ to the usual norm $\|\cdot\|_{H_1}$. Moreover, using the Sobolev's trace theorem, there exists a constant $c_0$ which depends only on $\Omega$, $\Gamma_C$ and $\Gamma_D$ such that

$$\|v\|_{L^2(\Gamma_C)^d} \leqslant c_0 \, \|v\|_V \,, \quad \forall\, v \in V \,. \tag{3.4}$$

We also need to introduce the spaces

$$Q = \{\, \eta \in H^1(\Omega) \,,\ \ \eta = 0 \ \text{ on } \ \Gamma_D \cup \Gamma_N \,\} \ , \quad W = \{\, \psi \in H^1(\Omega) \,,\ \ \psi = 0 \ \text{ on } \ \Gamma_a \,\},$$

endowed with the following inner products and associated norms

$$(\theta, \eta)_Q = (\theta, \eta)_{H^1(\Omega)}, \qquad \|\eta\|_Q = \|\eta\|_{H^1(\Omega)}, \quad \forall\, \theta,\, \eta \in Q,$$

$$(\varphi, \psi)_W = (\varphi, \psi)_{H^1(\Omega)}, \quad \|\psi\|_W = \|\psi\|_{H^1(\Omega)}, \quad \forall\, \varphi, \psi \in W.$$

Since $\Gamma_D$ and $\Gamma_a$ are on nonzero measure, it follows from the Friedrichs-Poincaré inequalities that the two spaces $(Q, \|\cdot\|_Q)$ and $(W, \|\cdot\|_W)$ are real Hilbert. Moreover, using the Sobolev trace theorem, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\|\eta\|_{L^2(\Gamma_C)} \leqslant c_1 \|\eta\|_Q, \quad \forall\, \eta \in Q, \tag{3.5}$$

$$\|\xi\|_{L^2(\Gamma_C)} \leqslant c_2 \|\xi|_W, \quad \forall\, \xi \in W. \tag{3.6}$$

Recall that, for a regular vector fields $q, D \in \{\varpi \in H,\, \operatorname{div} \varpi \in L^2(\Omega)\}$, the below Green formulas hold

$$(q, \nabla\eta)_{L^2(\Omega)^d} + (\operatorname{div} q, \eta)_{L^2(\Omega)} = \int_\Gamma q \cdot \nu\, \eta\, da, \quad \forall\, \eta \in H^1(\Omega), \tag{3.7}$$

$$(D, \nabla\xi)_{L^2(\Omega)^d} + (\operatorname{div} D, \nabla\xi)_{L^2(\Omega)} = \int_\Gamma D \cdot n\, \xi\, da, \quad \forall\, \xi \in H^1(\Omega). \tag{3.8}$$

Let $(X, \|\cdot\|_X)$ be a real Banach space. For every $1 \leqslant p \leqslant \infty$, we will use the spaces $L^p(0,T;X)$ endowed with its standard norm and the sobolev space $W^{1,\infty}(0,T;X)$ equipped with the norm

$$\|x\|_{W^{1,\infty}(0,T;X)} = \|x\|_{L^\infty(0,T;X)} + \|\dot{x}\|_{L^\infty(0,T;X)}.$$

Finally, we will also use $C^1([0,T];X)$ the spaces of continuous functions from $[0,T]$ to $X$ endowed with

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

In ordre to study of the problem $(P)$, we need hypotheses on the data's problem. Hence, we assume

$\underline{(H_1)}$ : The elasticity tensor $\mathfrak{F} = (f_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is symmetric, continuous and definite positive

$$f_{ijkl} = f_{jikl} = f_{klij} = f_{ijlk} \in L^\infty(\Omega),$$

$$f_{ijkl}\, \xi_{kl}\, \xi_{ij} \geqslant \alpha_0 \|\xi\|^2, \quad \forall\, \xi = (\xi_{ij}) \in \mathbb{S}^d.$$

$\underline{(H_2)}$ : The piezoelectric tensor $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ is partial symmetric and continuous

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega).$$

We recall here that the transposite tensor $\mathcal{E}^* = (e^*_{ijk})$ is given by $e^*_{ijk} = e_{kij}$ and we have

$$\mathcal{E}\sigma \cdot v = \sigma \cdot \mathcal{E}^* v\,, \quad \forall\, \sigma \in \mathbb{S}^d\,,\, \forall\, v \in \mathbb{R}^d\,. \tag{3.9}$$

The thermal expansion tensor $\mathcal{M} = (m_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is symmetric and continuous

$$m_{ij} = m_{ji} \,\in\, L^\infty(\Omega)\,.$$

The pyroelectric vector field $\mathcal{P} = (p_i) : \Omega \to \mathbb{R}^d$ is continuous

$$p_i \,\in\, L^\infty(\Omega)\,.$$

Notice that the two conditions above, allows us to define $M_{\mathcal{M}} = \sup_{ij} \|m_{ij}\|_{L^\infty(\Omega)}$ and $M_{\mathcal{P}} = \sup_i \|p_i\|_{L^\infty(\Omega)}$.

$\underline{(H_3)}$ : The electric permittivity $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is symmetric, continuous and definite positive

$$\beta_{ij} = \beta_{ji} \in L^\infty(\Omega), \qquad \beta_{ij}\, b_i\, b_j \;\geqslant\; m_\beta\, \|b\|^2\,, \quad \forall\, b = (b_i) \in \mathbb{R}^d\,.$$

The thermal conductivity $\mathcal{K} = (k_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is symmetric, continuous and definite positive

$$k_{ij} = k_{ji} \in L^\infty(\Omega), \qquad k_{ij}\, z_i\, z_j \;\geqslant\; m_{\mathcal{K}}\, \|z\|^2\,, \quad \forall\, z = (z_i) \in \mathbb{R}^d\,.$$

$\underline{(H_4)}$ : The coefficient of friction satisfies $\mu \geqslant 0 \in L^\infty(\Gamma_C)$ a.e on $\Gamma_C$. Then, we can pose

$$\bar{\mu} = \sup_{t\in[0,T]} \|\mu\|_{L^\infty(\Gamma_C)}\,.$$

$\underline{(H_5)}$ : The mapping $R : H^{'}_{\Gamma_C} \to L^2(\Gamma_C)$ is linear compact and continuous with $c_R = \|R\|$.

$\underline{(H_6)}$ : The surface electrical conductivity and the coefficient of heat exchange satisfies for $\pi = \psi, k_c$

$(a)$ $(\exists\, L_\pi > 0)\,,\, (\forall\, a\,, b \in \mathbb{R}^+)\,, \quad |\pi(\cdot, a) - \pi(\cdot, b)| \,<\, L_\pi\, |a - b|$ a.e. on $\Gamma_C$ ,

$(b)$ For all $a \in \mathbb{R}^+$, the mapping $x \mapsto \pi(x, a)$ is measurable on $\Gamma_C$ ,

$(c)$ For all $a \in \mathbb{R}^+$, the mapping $x \mapsto \pi(x, a)$ is $M_\pi$-bounded a.e. on $\Gamma_C$ ,

$(d)$ $x \to \pi(x, u) = 0$ for $u \leqslant 0$, a.e.$x \in \Gamma_C$.

$\underline{(H_7)}$ : The normal compliance function $p_r : \Gamma_C \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies for $r = \nu, \tau$

$(a)$ $(\exists\, L_r > 0)\,,\ (\forall\, a\,, b \in\, \mathbb{R}^+)\,,\quad |p_r(\cdot, a) - p_r(\cdot, b)| \,<\, L_r\,|a - b|$ a.e. on $\Gamma_C$ ,

$(b)$ For all $a \in \mathbb{R}^+$, the mapping $x \mapsto p_r(x, a)$ is measurable on $\Gamma_C$ ,

$(c)$ For all $a \in \mathbb{R}^+$, the mapping $x \mapsto p_\nu(x, a)$ is $L_{p_\nu}$-bounded a.e. on $\Gamma_C$ ,

$(d)$ $x \to p_r(x, u) = 0$ for $u \leqslant 0$, a.e.$x \in \Gamma_C$.

$\underline{(H_8)}$ : The function $S_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfies

$(a)$ $(\exists\, L_{S_c} > 0)\,,\ (\forall\, r_1\,, r_2 \in\, \mathbb{R}^+)\,,\quad |S_c(\cdot, r_1) - S_c(\cdot, r_2)| \,<\, L_{S_c}\,|r_1 - r_2|$ a.e. on $\Gamma_C$ ,

$(b)$ The mapping $S_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ is Borel measurable ,

$(c)$ For all $r \in \mathbb{R}$, the mapping $x \mapsto S_c(x, r)$ is $S^*$-bounded a.e. on $\Gamma_C$ .

$\underline{(H_9)}$ : The forces, tractions, charges, heat source densities and the foundation's temperature satisfy

$$f_0 \in W^{1,\infty}(0, T; L^2(\Omega)^d), \qquad \phi_0\,,\, q_0 \in W^{1,\infty}(0, T; L^2(\Omega)),$$

$$f_N \in W^{1,\infty}(0, T; L^2(\Gamma_N)^d), \qquad \phi_b \in W^{1,\infty}(0, T; L^2(\Gamma_b)), \qquad \theta_F \in L^2(\Gamma_C).$$

Using Riesz's representation theorem, we define $f : [0, T] \to V$, $q_e : [0, T] \to W$, $\Theta : [0, T] \to Q$ by

$$(f, v)_V = \int_\Omega f_0\, v\, dx + \int_{\Gamma_N} f_N\, v\, da\,, \quad \forall\, v \in V, \tag{3.10}$$

$$(q_e, \xi)_W = \int_\Omega \phi_0\, \xi\, dx - \int_{\Gamma_b} \phi_b\, \xi\, da\,, \quad \forall\, \xi \in W, \tag{3.11}$$

$$(\Theta, \eta)_{L^2(\Omega)} = \int_\Omega q_0\, \eta\, dx\,, \quad \forall\, \eta \in Q. \tag{3.12}$$

We consider $j : V \times V \to \mathbb{R}$, $\chi : V \times V \times Q \to \mathbb{R}$, $\omega : V \times Q \times Q \to \mathbb{R}$ and $l : V \times W \times W \to \mathbb{R}$ such that

$$j(u; v) = \int_{\Gamma_C} p_\nu(u_\nu - g) v_\nu\, da + \int_{\Gamma_C} p_\tau(u_\tau - g)\, \|v_\tau\|\, da\,, \tag{3.13}$$

$$\chi(u; v; \eta) = \int_{\Gamma_C} \mu\, p_\nu(u_\nu - g)\, S_c(\cdot, \|v_\tau\|)\, \eta\, da\,, \tag{3.14}$$

$$\omega((u, \theta); \eta) = \int_{\Gamma_C} k_c(u_\nu - g)\, \phi_L(\theta - \theta_F)\, \eta\, da\,, \quad \forall\, \theta \in Q\,,\ \forall\, \eta \in Q, \tag{3.15}$$

$$l((u, \varphi); \xi) = \int_{\Gamma_C} \psi(u_\nu - g)\, \phi_L(\varphi - \varphi_F)\, \xi\, da\,, \quad \forall\, \theta \in Q\,,\ \forall\, \eta \in Q, \tag{3.16}$$

for all $(u, \varphi, \theta) \in X$, $v \in V$, $\eta \in Q$. We consider some assumptions on the initial data

$$\left\{\begin{array}{l} u_0 \in V \text{ and } \theta_0 \in Q \text{ such that for all } (v,\xi) \in V \times W, \text{ we have} \\ \quad (\mathfrak{F}\,\varepsilon(u_0), \varepsilon(v))_{\mathcal{H}} + (\mathcal{E}^*\,\nabla\varphi_0, \varepsilon(v))_H - (\mathcal{M}\,\theta_0, \varepsilon(v))_H + j(u_0; v) \geqslant (f(0), v)_V, \\ \quad (\beta\,\nabla\varphi_0, \nabla\xi)_H = (\mathcal{E}\varepsilon(u_0), \nabla\xi)_H + (\mathcal{P}\,\theta_0, \nabla\xi)_H + (q_e(0), \xi)_W - l((u_0, \varphi_0); \xi), \\ \text{and there exists } T_0 \in L^2(\Omega) \text{ such that} \\ \quad (\mathcal{K}\,\nabla\theta_0, \nabla\eta)_H + \omega(u_0, \theta_0, \eta) - (\Theta(0), \eta)_{L^2(\Omega)} = (T_0, \eta)_{L^2(\Omega)}. \end{array}\right. \tag{3.17}$$

According to this notation and by a standard procedure based on Green's formulas, We can state the variational formulations of the problem $P$, in the terms of displacement, electric potential and temperature.

**Problem** $(PV)$**.** *Find a displacement field* $u : [0,T] \to V$*, an electric potential* $\varphi : [0,T] \to W$ *and a temperature field* $\theta : [0,T] \to Q$ *such that*

$$(\mathfrak{F}\,\varepsilon(u), \varepsilon(v - \dot u))_{\mathcal{H}} + (\mathcal{E}^*\,\nabla\varphi, \varepsilon(v - \dot u))_H - (\mathcal{M}\,\theta, \varepsilon(v - \dot u))_H + j(u, v) - j(u, \dot u) \tag{3.18}$$

$$\geqslant (f, v - \dot u)_V, \quad \forall\, v \in V.$$

$$(\beta\,\nabla\varphi, \nabla\xi)_H - (\mathcal{E}\varepsilon(u), \nabla\xi)_H - (\mathcal{P}\,\theta, \nabla\xi)_H + l((u, \varphi); \xi) = (q_e, \xi)_W, \quad \forall\, \xi \in W. \tag{3.19}$$

$$(\dot\theta, \eta)_{L^2(\Omega)} + (\mathcal{K}\,\nabla\theta, \nabla\eta)_H + (\mathcal{M}\,\dot\varepsilon(u), \eta)_{L^2(\Omega)} - (\mathcal{P}\,\nabla\dot\varphi, \eta)_{L^2(\Omega)} \tag{3.20}$$

$$- \chi(u; \dot u; \eta) + \omega(u, \theta; \eta) = (\Theta, \eta)_{L^2(\Omega)}, \quad \forall\, \eta \in Q.$$

$$u(\cdot, 0) = u_0, \ \ \varphi(\cdot, 0) = \varphi_0, \ \ \theta(\cdot, 0) = \theta_0 \quad \text{in } \Omega. \tag{3.21}$$

Our main existence result that we state now and prove in the next sections, is the following.

**Theorem 3.1.** *Assume that* $(H_1)$*-*$(H_9)$ *and* (3.17) *hold. If there exists a positive constant* $L^*$ *such that* $\bar\mu + L_\tau + L_\psi L + M_\psi + L_{k_c} L + M_{k_c} \leqslant L^*$*, then the problem* $(PV)$ *has at least a solution.*

## 4. AN ABSTRACT RESULT

The existence of solutions of problems $(PV)$ will be obtained by using the following abstract problem.

Let $X$ be a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. We consider the problem of finding $x \in X$ such that

$$(Ax, y-x)_X + (Bx, y-x)_X + j(x,y) - j(x,x) \geqslant (f, y-x)_X\,, \quad \forall\, y \in X. \quad (4.1)$$

In the study the problem (4.1), we make the following assumptions :
1. The operator $A : X \to X$ is strongly monotone and Lipschitz continuous ,

$$(h_1) : \begin{cases} (a) \text{ There exists a constant } m > 0 \text{ such that for all } x, y \in X, \text{ one has} \\ \qquad\qquad (Ax - Ay, x - y)_X \geqslant m\,\|x-y\|_X^2\,. \\ (b) \text{ There exists a constant } M > 0 \text{ such that for all } x, y \in X, \text{ one has} \\ \qquad\qquad \|Ax - Ay\|_X \leqslant M\,\|x-y\|_X\,. \end{cases}$$

2. The operator $B : X \to X$ satisfies

$$(h_2) : \text{There exists } m > 0 \text{ such that } \quad (B\,x, x)_X \;\geqslant\; -m\,\|x\|_X^2\,, \quad \forall\, x \in X.$$

$$(h_3) : \begin{cases} \text{For every sequence } \{\eta_n\} \subset X \text{ such that } \eta_n \to \eta \in X, \text{ then there} \\ \text{exist a subsequence } \{\eta_{n'}\} \subset X \text{ such that } B\eta_{n'} \to B\eta \text{ strongly in } X. \end{cases}$$

$$(h_4) : \big(Bx - By, y - x)_X < (m-\alpha)\,\|x-y\|_X^2\,, \quad \forall\, x \neq y \in X.$$

$$(h_5) : \begin{cases} \text{There exists } 0 < \beta < (m-\alpha) \text{ such that} \\ \qquad (Bx - By, y-x)_X \leqslant \beta\,\|x-y\|_X^2\,, \quad \forall\, x,\, y \in X. \end{cases}$$

3. The functional $j : X \times X \to \mathbb{R}$ satisfies

$$(h_6) : j(\eta, \cdot) : X \to \mathbb{R} \;\text{ is a convex functional on } X\,, \quad \forall\, \eta \in X.$$

It is well known that there exists the directional derivative $j_2^{'}$ given by

$$(h_7) : j_2^{'}(\eta, x; y) = \lim_{\lambda \to 0} \frac{1}{\lambda}\big[j(\eta, x + \lambda y) - j(\eta, y)\big]\,, \;\; \forall\, \eta,\, x,\, y \in X.$$

4. The following assumptions are satisfied

$$(J_1) : \begin{cases} \text{For every sequence } \{x_n\} \subset X \text{ with } \|x_n\|_X \to \infty \\ \text{and for every sequence } \{t_n\} \subset [0,1] \text{ one has} \\ \liminf_{n\to\infty} \big[\frac{1}{\|x_n\|_X^2}\, j_2^{'}(t_n x_n, x_n; -x_n)\big] < m - C. \end{cases}$$

$$(J_2) : \begin{cases} \text{For every sequence } \{x_n\} \subset X \text{ with } \|x_n\|_X \to \infty \\ \text{and for every bounded sequence } \{\eta_n\} \subset X \text{ one has} \\ \liminf_{n\to\infty} \big[\frac{1}{\|x_n\|_X^2} j_2'(\eta_n, x_n; -x_n)\big] < m. \end{cases}$$

$$(J_3) : \begin{cases} \text{For every sequence } \{x_n\} \subset X \text{ and } \{\eta_n\} \subset X \text{ such that} \\ x_n \rightharpoonup x \in X, \eta_n \rightharpoonup \eta \in X \text{ and for every } y \in X \text{ one has} \\ \limsup_{n\to\infty} \big[j(\eta_n, y) - j(\eta_n, x_n)\big] < j(\eta, y) - j(\eta, x). \end{cases}$$

$$(J_4) : j(x,y) - j(x,x) + j(y,x) - j(y,y) \leqslant m\|x-y\|_X^2\,, \quad \forall\, x \neq y\ \in X.$$

$$(J_5) : \begin{cases} \text{There exists } \alpha < m \text{ such that for all } x, y \in X \text{ one has} \\ j(x,y) - j(x,x) + j(y,x) - j(y,y) \leqslant \alpha\|x-y\|_X^2. \end{cases}$$

Under all these assumptions, we have the following result.

**Theorem 4.1.** *Assume the conditions* $(h_1)$*-*$(h_3)$ *and* $(h_6)$ *hold, then :*

1. *Under the assumptions* $(J_1)$*-*$(J_3)$ *and* $(J_5)$*, the problem* (4.1) *has at least one solution* $x \in X$*.*
2. *Under the assumptions* $(J_1)$*-*$(J_3)$*,* $(J_5)$ *and* $(h_4)$*, the problem* (4.1) *has a unique solution.*
3. *Under the assumptions* $(J_1)$*-*$(J_3)$*,* $(J_5)$ *and* $(h_5)$*, the problem* (4.1) *has a unique solution* $x = x_f$ *which depends Lipschitz continuously on* $f$ *with the Lipschitz constant* $(m - \alpha - \beta)^{-1}$*.*

Theorem 4.1 has been obtained in [8] and therefore we do not provide here the details of the proof. We just specify that it was obtained in several step and it is based on topological degree theory as well as on convexity, monotonicity, compactness and fixed point arguments.

## 5. EXISTENCE OF THE DISCRETE SOLUTION

The problem $(PV)$ can be integrated in time by an implicit scheme. We use a uniform subdivision $0 = t_0 < t_1 < \cdots < t_n = T$ of the time interval $[0,T]$ where $t_i = i\,\Delta t$ for $i = 0, \cdots, n$ and $\Delta t = \frac{T}{n}$. For any function $\zeta$, we denote by $\zeta^i$ the approximation of the value $\zeta(t_i)$, by $\Delta\zeta^i$ the backward difference $\zeta^{i+1} - \zeta^i$ and if $\zeta$ is continuous, we use the notation $\zeta^i = \zeta(t_i)$. In each time step, we then obtain the following incremental problems $(PV^i)$ defined for $u(\cdot,0) = u_0$, $\varphi(\cdot,0) = \varphi_0$ and $\theta(\cdot,0) = \theta_0$ by

**Problem** $(PV^i)$. *Find a displacement* $u^{i+1} \in V$*, an electric potential* $\varphi^{i+1} \in W$ *and a temperature* $\theta^{i+1} \in Q$ *such that :*

$$(\mathfrak{F}\,\varepsilon(u^{i+1}), \varepsilon(v - u^{i+1}))_{\mathcal{H}} + (\mathcal{E}^*\,\nabla\varphi^{i+1}, \varepsilon(v - u^{i+1}))_H - (\mathcal{M}\,\theta^{i+1}, \varepsilon(v - u^{i+1}))_H$$
$$+ j(u^{i+1}, v - u^i) - j(u^{i+1}, u^{i+1} - u^i) \geqslant (f^{i+1}, v - u^{i+1})_V\,, \quad \forall\, v \in V, \tag{5.1}$$

$$(\beta\,\nabla\varphi^{i+1}, \nabla\xi)_H - (\mathcal{E}\varepsilon(u^{i+1}), \nabla\xi)_H - (\mathcal{P}\,\theta^{i+1}, \nabla\xi)_H + l((u^{i+1}, \varphi^{i+1}); \xi)$$
$$= (q_e^{i+1}, \xi)_W\,, \quad \forall\, \xi \in W, \tag{5.2}$$

$$(\theta^{i+1}, \eta)_{L^2(\Omega)} + \Delta t\,(\mathcal{K}\,\nabla\theta^{i+1}, \nabla\eta)_H + (\mathcal{M}\,\varepsilon(u^{i+1}), \eta)_{L^2(\Omega)} - (\mathcal{P}\,\nabla\varphi^{i+1}, \eta)_{L^2(\Omega)}$$
$$- \Delta t\,\chi(u^{i+1}; \frac{u^{i+1} - u^i}{\Delta t}; \eta) + \Delta t\,\omega(u^{i+1}, \theta^{i+1}, \eta) \tag{5.3}$$
$$= \Delta t\,(\Theta^{i+1}, \eta)_{L^2(\Omega)} + (\theta^i, \eta)_{L^2(\Omega)} + (\mathcal{M}\,\varepsilon(u^i), \eta)_{L^2(\Omega)} - (\mathcal{P}\,\nabla\varphi^i, \eta)_{L^2(\Omega)}\,, \quad \forall\, \eta \in Q.$$

For every $z_1 \in L^2(\Gamma_C)$, $z_2 \in L^2(\Gamma_C)$ and $z_3 \in L^2(\Gamma_C)$ we define the functions, $\forall\, \eta \in Q, \forall\, \xi \in W$

$$\chi_{z_1}(\eta) = \int_{\Gamma_C} z_1\,\eta\,da, \qquad \omega_{z_2}(\eta) = \int_{\Gamma_C} z_2\,\eta\,da, \qquad \ell_{z_3}(\xi) = \int_{\Gamma_C} z_3\,\eta\,da. \tag{5.4}$$

and we construct the following intermediate problem in which $z = (z_1, z_2, z_3)$ is supposed to be known.

**Problem** $(PV_z^i)$. *Find a displacement* $u_z^{i+1} \in V$*, an electric potential* $\varphi_z^{i+1} \in W$ *and a temperature* $\theta_z^{i+1} \in Q$ *such that :*

$$(\mathfrak{F}\,\varepsilon(u_z^{i+1}), \varepsilon(v - u_z^{i+1}))_{\mathcal{H}} + (\mathcal{E}^*\,\nabla\varphi_z^{i+1}, \varepsilon(v - u_z^{i+1}))_H - (\mathcal{M}\,\theta_z^{i+1}, \varepsilon(v - u_z^{i+1}))_H$$
$$+ j((u_z^{i+1}, \varphi_z^{i+1}, \theta_z^{i+1}), v - u^i) - j((u_z^{i+1}, \varphi_z^{i+1}, \theta_z^{i+1}), u_z^{i+1} - u^i) \tag{5.5}$$
$$\geqslant (f^{i+1}, v - u_z^{i+1})_V\,, \quad \forall\, v \in V.$$

$$(\beta\,\nabla\varphi_z^{i+1}, \nabla\xi)_H - (\mathcal{E}\varepsilon(u_z^{i+1}), \nabla\xi)_H - (\mathcal{P}\,\theta_z^{i+1}, \nabla\xi)_H + \ell_{z_3}(\xi) = (q_e^{i+1}, \xi)_W\,, \quad \forall\, \xi \in W. \tag{5.6}$$

$$(\theta_z^{i+1}, \eta)_{L^2(\Omega)} + \Delta t\,(\mathcal{K}\,\nabla\theta_z^{i+1}, \nabla\eta)_H + (\mathcal{M}\,\varepsilon(u_z^{i+1}), \eta)_{L^2(\Omega)} - (\mathcal{P}\,\nabla\varphi_z^{i+1}, \eta)_{L^2(\Omega)}$$
$$+ \Delta t\,\omega_{z_2}(\eta) - \Delta t \chi_{z_1}(\eta) \tag{5.7}$$
$$= \Delta t\,(\Theta^{i+1}, \eta)_{L^2(\Omega)} + (\theta^i, \eta)_{L^2(\Omega)} + (\mathcal{M}\,\varepsilon(u^i), \eta)_{L^2(\Omega)} - (\mathcal{P}\,\nabla\varphi^i, \eta)_{L^2(\Omega)}\,, \quad \forall\, \eta \in Q.$$

We consider an element $\Theta_z^{i+1}$ of $Q$ defined for all $\eta \in Q$ by

$$(\Theta_z^{i+1},\eta)_Q = (\Delta t\,\Theta^{i+1},\eta)_{L^2(\Omega)} + (\theta^i,\eta)_{L^2(\Omega)} + (\mathcal{M}\,\varepsilon(u^i),\eta)_{L^2(\Omega)} - (\mathcal{P}\,\nabla\varphi^i,\eta)_{L^2(\Omega)} + \Delta t\,\chi_{z_1}(\eta) - \Delta t\,\omega_{z_2}(\eta).$$

To solve the problem $(PV_z^i)$, we consider the spaces $X = V \times W \times Q$ and $Y = L^2(\Gamma_C)^d \times \big(L^2(\Gamma_C)\big)^2$ endowed with the following inner product and the associated norm $\|.\|_X$ and $\|.\|_Y$ such that

$$(x,y)_X = (u,v)_V + (\varphi,\xi)_W + (\theta,\eta)_Q, \tag{5.8}$$

$$(z,z')_Y = (z_1,z_1')_{L^2(\Gamma_C)^d} + (z_2,z_2')_{L^2(\Gamma_C)} + (z_3,z_3')_{L^2(\Gamma_C)}, \tag{5.9}$$

for all $x = (u,\varphi,\theta),\, y = (v,\xi,\eta) \in X$ and $z = (z_1,z_2,z_3),\, z' = (z_1',z_2',z_3') \in Y$.

Next, we define the operators $A : X \to X$ and $B : X \to X$ given by

$$\begin{aligned}(Ax,y)_X \;=\;& (\mathfrak{F}\,\varepsilon(u),\varepsilon(v))_{\mathcal{H}} + (\Delta t\,\mathcal{K}\,\nabla\theta,\nabla\eta)_H + (\beta\nabla\varphi,\nabla\xi)_H \\ &+ (\mathcal{E}^*\,\nabla\varphi,\varepsilon(v))_H - (\mathcal{E}\varepsilon(u),\nabla\xi)_H\,,\end{aligned} \tag{5.10}$$

$$\begin{aligned}(Bx,y)_X \;=\;& -(\mathcal{P}\,\theta,\nabla\xi)_H - (\mathcal{P}\,\nabla\varphi,\eta)_{L^2(\Omega)} + (\theta,\eta)_{L^2(\Omega)} \\ &- (\mathcal{M}\,\theta,\varepsilon(v))_H + (\mathcal{M}\,\varepsilon(u),\eta)_{L^2(\Omega)}\,, \qquad \forall x = (u,\varphi,\theta)\,,\; y = (v,\xi,\eta) \in X,\end{aligned} \tag{5.11}$$

the functionals $J : X \times X \to \mathbb{R}$ defined by

$$J(x,y) = j(u;v)\,, \qquad \forall\, x = (u,\varphi,\theta)\,,\; y = (v,\xi,\eta) \in X. \tag{5.12}$$

and the element $f_z^{i+1}$ of $X$ given by

$$f_z^{i+1} = (f^{i+1}, q_e^{i+1} - \ell_{z_3}(\xi), \Theta_z^{i+1})\,. \tag{5.13}$$

We have the following lemma.

**Lemma 5.1.** The problem $(PV_z^i)$ is equivalent to the following problem

$$\begin{cases} \textit{Find } x_z^{i+1} = (u_z^{i+1},\, \varphi_z^{i+1},\, \theta_z^{i+1}) \in X \textit{ such that :} \\ (Ax_z^{i+1}, y - x_z^{i+1})_X + (Bx_z^{i+1}, y - x_z^{i+1})_X + J(x_z^{i+1}, y - x^i) \\ \qquad -J(x_z^{i+1}, x_z^{i+1} - x^i) \geqslant (f_z^{i+1}, y - x_z^{i+1})_X\,, \quad (\forall\, y \in X)\,. \end{cases} \tag{5.14}$$

**Proof.** Let $x_z^{i+1} = (u_z^{i+1}, \varphi_z^{i+1}, \theta_z^{i+1}) \in X$ be a solution to Problem $(PV_z^i)$ and let $y = (v,\xi,\eta) \in X$. We use the test function $\xi - \varphi_z^{i+1}$ in (5.6) and $\eta - \theta_z^{i+1}$ in (5.7). We add the corresponding inequalities to(5.5) and we use (5.10)-(5.13) to obtain (5.14).

Conversely, let $x_z^{i+1} = (u_z^{i+1}, \varphi_z^{i+1}, \theta_z^{i+1}) \in X$ be a solution to the quasivariational inequality (5.14). If we take $y = (v, \varphi_z^{i+1}, \theta_z^{i+1})$ in (5.14) where $v$ is an arbitrary element of $V$, we obtain (5.5) and if we take successively $y = (v, \varphi_z^{i+1} + \xi, \theta_z^{i+1})$ and $y = (v, \varphi_z^{i+1} - \xi, \theta_z^{i+1})$ in (5.14) where $\xi$ is an arbitrary element of $W$, we deduce (5.6). In the same way, we take successively $y = (v, \varphi_z^{i+1}, \theta_z^{i+1} + \eta)$ and $y = (v, \varphi_z^{i+1}, \theta_z^{i+1} - \eta)$ in (5.14) where $\eta$ is an arbitrary element of $Q$ to get (5.7).Thus, the lemma is proved. □

Using the previous lemma, we obtain the following existence and uniqueness result of $(PV_z^i)$.

**Lemma 5.2.** *For every $z$ of $(L^2(\Gamma_C))^3$, the problem $(PV_z^i)$ has a unique solution*

$$x_z^{i+1} = (u_z^{i+1}, \varphi_z^{i+1}, \theta_z^{i+1}) \in V \times W \times Q\,.$$

In order to proving this lemma, we use theorem 4.1 by taking to the account the mathematical inequality $c_0^2(L_\nu + L_\tau) + 2M_{\mathcal{P}} \leqslant m$. We start by investigating the proprieties of the operators $A$, $B$ and the functionals $J$ given in (5.12)-(3.13). Thus, we have the following lemmas.

**Lemma 5.3.** *The operator $A : X \to X$ is strongly monotone and Lipschitz continuous.*

**Proof.** We consider two elements $x_1 = (u_1, \varphi_1, \theta_1)$ and $x_2 = (u_2, \varphi_2, \theta_2)$ of $X$. Using (4.1) we can get

$$\begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X &= (\mathfrak{F}\,\varepsilon(u_1) - \mathfrak{F}\,\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}}\\ &+(\beta\nabla\varphi_1 - \beta\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_H + (\Delta t\,\mathcal{K}\,\nabla\theta_1 - \Delta t\,\mathcal{K}\,\nabla\theta_2, \nabla\theta_1 - \nabla\theta_2)_H\\ &+(\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \varepsilon(u_1) - \varepsilon(u_2))_H - (\mathcal{E}\varepsilon(u_1) - \mathcal{E}\varepsilon(u_2), \nabla\varphi_1 - \nabla\varphi_2)_H.\end{aligned}$$

Taking to the account that $(\mathcal{E}^*\,\nabla\varphi, \varepsilon(u))_H = (\mathcal{E}\varepsilon(u), \nabla\varphi)_H$, we find

$$\begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X &= (\mathfrak{F}\,\varepsilon(u_1) - \mathfrak{F}\,\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}}\\ &+(\beta\nabla\varphi_1 - \beta\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_H + (\Delta t\,\mathcal{K}\,\nabla\theta_1 - \Delta t\,\mathcal{K}\,\nabla\theta_2, \nabla\theta_1 - \nabla\theta_2)_H.\end{aligned}$$

Combining with $(H_1)$ and $(H_3)$, there exists $m > 0$ depending on $\mathfrak{F}$, $\beta$, $\mathcal{K}$, $\Omega$, $\Gamma_D$, $\Gamma_N$, $\Gamma_a$ such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geqslant m\left(\|u_1 - u_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2 + \|\theta_1 - \theta_2\|_Q^2\right).$$

Thus, it follows from (5.8) that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geqslant m\,\|x_1 - x_2\|_X^2\,. \tag{5.15}$$

In the same way, $H_1$ and $H_3$ imply that there exists $c_4 > 0$ such that

$$(Ax_1 - Ax_2, y)_X \leqslant c_4 \left( \|u_1 - u_2\|_V \|v\|_V + \|u_1 - u_2\|_V \|\xi\|_W \right.$$
$$\left. + \|\theta_1 - \theta_2\|_Q \|\eta\|_Q + \|\varphi_1 - \varphi_2\|_W \|\xi\|_W + \|\varphi_1 - \varphi_2\|_W \|v\|_V \right).$$

Then, we obtain

$$(Ax_1 - Ax_2, y)_X \leqslant 5\, c_4 \|x_1 - x_2\|_X \|y\|_X\,, \quad \forall y \in X\,.$$

However, if we set $y = Ax_1 - Ax_2$ and $M = 5\, c_4$, we find

$$\|Ax_1 - Ax_2\|_X \leqslant M \|x_1 - x_2\|_X \tag{5.16}$$

and thus the lemma 5.3 is established. □

Next, we recall that the functional $J$ defined by (5.12) and (3.13), satisfies $(h_6)$. Moreover, we have

**Lemma 5.4.** *The functional $J$ satisfies the assumptions $(J_1), (J_2)$ and $(J_3)$.*

**Proof.** The proof of this lemma can be found in [5].

**Lemma 5.5.** *If the assumption $(H_6)$ hold, then the functional $J$ satisfies the inequality*

$$J(x, y) - J(x, x) + J(y, x) - J(y, y) \leqslant \alpha \|x - y\|_X^2\,, \quad \forall\, x,\, y \in X. \tag{5.17}$$

**Proof.** The proof of this lemma can be found in [5] with $\alpha = c_0^2(L_\nu + L_\tau)$.

**Lemma 5.6.** *The operator $B : X \to X$ satisfies the conditions $(h_2), (h_3)$ and $(h_5)$.*

**Proof.** Using the asumptions on $\mathcal{P}$ (see $(H_2)$), we obtain

$$\begin{aligned}(\mathcal{P}\,\theta, \nabla\varphi)_H + (\mathcal{P}\,\nabla\varphi, \theta)_{L^2(\Omega)} &= 2\,(\mathcal{P}\,\theta, \nabla\varphi)_H \\ &\leqslant 2\, M_{\mathcal{P}} \|\nabla\varphi\|_H \|\theta\|_{L^2(\Omega)} \\ &\leqslant 2\, M_{\mathcal{P}} \|\varphi\|_W \|\theta\|_Q \\ &\leqslant C \|x\|_X^2\end{aligned}$$

where $C = 2M_{\mathcal{P}}$. Then

$$-(\mathcal{P}\,\theta, \nabla\varphi)_H - (\mathcal{P}\,\nabla\varphi, \theta)_{L^2(\Omega)} \geqslant -C \|x\|_X^2. \tag{5.18}$$

Combined with $(\mathcal{M}\,\theta, \varepsilon(u))_H = (\mathcal{M}\,\varepsilon(u), \theta)_{L^2(\Omega)}$ and $(\theta, \theta)_{L^2(\Omega)} \geqslant 0$, we get

$$(Bx, x)_X \geqslant -\, C \|x\|_X^2\,, \quad \forall\, x \in X \tag{5.19}$$

and then $(h_2)$ is satisfied. Let us now consider the sequence $x_n = (u_n, \varphi_n, \theta_n)$ of $X$ such that

$$x_n = (u_n, \varphi_n, \theta_n) \rightharpoonup x = (u, \varphi, \theta) \in X.$$

It follows from (5.8) and (5.11) that there exists a constant $c_6 \geqslant 0$ such that

$$\begin{aligned}(Bx_n - Bx, y)_X &= -(\mathcal{P}\,\theta_n - \mathcal{P}\,\theta, \nabla\xi)_H - (\mathcal{P}\,\nabla\varphi_n - \mathcal{P}\,\nabla\varphi, \eta)_{L^2(\Omega)} + (\theta_n - \theta, \eta)_{L^2(\Omega)} \\ &\quad -(\mathcal{M}\,\theta_n - \mathcal{M}\,\theta, \varepsilon(v))_H + (\mathcal{M}\,\varepsilon(u_n) - \mathcal{M}\,\varepsilon(u), \eta)_{L^2(\Omega)} \\ &\leqslant c_6\,\|x_n - x\|_X\,\|y\|_X\,, \quad \forall\, y = (v, \xi, \eta) \in X.\end{aligned}$$

Taking $y = Bx_n - Bx$ in the previous inequality, we have

$$\|Bx_n - Bx\|_X \leqslant c_6\,\|x_n - x\|_X. \tag{5.20}$$

On the other hand, it comes from (5.14) where $x_z^{i+1} \in X$ is still denoted $x \in X$ that

$$(Ax_n - Ax, x_n - x)_X + (Bx_n - Bx, x_n - x)_X \leq J(x, x_n) - J(x, x) + J(x_n, x) - J(x_n, x_n),$$

Applying (5.15), (5.17) and (5.19), we obtain

$$(m - C)\,\|x_n - x\|_X^2 \leqslant (L_\nu + L_\tau)\,\|x_n - x\|_Y^2$$

and since $\alpha \leqslant (m - C)$, we have

$$\alpha\,\|x_n - x\|_X^2 \leqslant (L_\nu + L_\tau)\,\|x_n - x\|_Y^2.$$

Then

$$\begin{aligned}\|x_n - x\|_X &\leqslant \sqrt{\frac{(L_\nu + L_\tau)}{\alpha}}\,\|x_n - x\|_Y \\ &\leqslant \frac{1}{c_0}\,\|x_n - x\|_Y\end{aligned} \tag{5.21}$$

Combining (5.23) and (5.24), we deduce

$$\|Bx_n - Bx\|_X \leqslant \frac{1}{c_0}\,\|x_n - x\|_Y. \tag{5.22}$$

Since the trace map $\gamma : X \to Y$ is compact operator, the weak convergence $x_n \rightharpoonup x$ leads to the strong convergence $x_n \to x$ in $Y$. So, we conclude that $Bx_n \to Bx$ in $X$ strongly for a subsequence and that prove the condition $(h_3)$. Finally, for all $x_1 = (u_1, \varphi_1, \theta_1)$, $x_2 = (u_2, \varphi_2, \theta_2)$ of $X$, (5.11) implies that

$$\begin{aligned}(Bx_1 - Bx_2, x_2 - x_1)_X &= 2\,(\mathcal{P}\,\theta_1 - \mathcal{P}\,\theta_2, \nabla\varphi_1 - \nabla\varphi_2)_H \\ &\leqslant C\,\|x_1 - x_2\|_X \\ &\leqslant \beta\,\|x_1 - x_2\|_X.\end{aligned}$$

where $\beta = C > 0$ and since $\alpha + C \leqslant m$ we have $0 \leqslant \beta \leqslant m - \alpha$. Thus, the condition $(h_5)$ is satisfied.

At the end, using the lemmas 5.1-5.6 and the theorem 4.1 to deduce that problem $PV_z^i$ has a unique solution and it is easy to show that this solution depends Lipschitz continuously on $z^{i+1} \in L^2(\Gamma_C)^3$. In the next step, we consider the following operator

$$\Lambda : L^2(\Gamma_C) \times L^2(\Gamma_C) \times L^2(\Gamma_C) \to L^2(\Gamma_C) \times L^2(\Gamma_C) \times L^2(\Gamma_C)$$

such that for all $z = (z_1, z_2, z_3) \in L^2(\Gamma_C)^3$, we have

$$\Lambda z = \Bigg[ p_\tau(u^{i+1}_{z_1\nu} - g)\, S_C(\,\cdot\,, \frac{1}{\Delta t}\|u^{i+1}_{z_1\tau} - u^i\|)\,; \\ \psi(u_{z_2\nu} - g)\, \phi_L(\varphi^{i+1}_{z_2} - \varphi_F)\,;\, k_c(u_{z_3\nu} - g)\, \phi_L(\theta^{i+1}_{z_3} - \theta_F) \Bigg].$$

Now, let's majorate the quantity $I = \left\|\Lambda z_1^{i+1} - \Lambda z_2^{i+1}\right\|_{L^2(\Gamma_C)^3}$, we have

$$\begin{aligned} I \leqslant & \left\| \left[p_\tau(u^{i+1}_{z_1\nu} - g) - p_\tau(u^{i+1}_{z'_1\nu} - g)\right] S_C(\cdot, \tfrac{1}{\Delta t}\|u^{i+1}_{z_1\tau} - u^i\|)\right\|_{L^2(\Gamma_C)} \\ & + \left\|\mu\, p_\nu(u^{i+1}_{z'_1\nu} - g)\, \left[S_C(\,\cdot\,, \tfrac{1}{\Delta t}\|u^{i+1}_{z_1\tau} - u^i\|) - S_C(\,\cdot\,, \tfrac{1}{\Delta t}\|u^{i+1}_{z'_1\tau} - u^i\|)\right]\right\|_{L^2(\Gamma_C)} \\ & + \left\| \left[\psi(u^{i+1}_{z_2\nu} - g) - \psi(u^{i+1}_{z'_2\nu} - g)\right] \phi_L(\varphi^{i+1}_{z_2} - \varphi_F) \right\|_{L^2(\Gamma_C)} \\ & + \left\| \psi(u^{i+1}_{z'_2\nu} - g) \left[\phi_L(\varphi^{i+1}_{z_2} - \varphi_F) - \phi_L(\varphi^{i+1}_{z'_2} - \varphi_F)\right] \right\|_{L^2(\Gamma_C)} \\ & + \left\| \left[k_c(u^{i+1}_{z_3\nu} - g) - k_c(u^{i+1}_{z'_3\nu} - g)\right] \phi_L(\theta^{i+1}_{z_3} - \theta_F) \right\|_{L^2(\Gamma_C)} \\ & + \left\| k_c(u^{i+1}_{z'_3\nu} - g) \left[\phi_L(\theta^{i+1}_{z_3} - \theta_F) - \phi_L(\theta^{i+1}_{z'_3} - \theta_F)\right] \right\|_{L^2(\Gamma_C)}. \end{aligned}$$

Using $(H_4)$-$(H_8)$ and the Lipschitz-continuously dependence of the $x_z^{i+1}$ on $z^{i+1}$, we deduce that there exists a constant $c_7 > 0$ such that

$$\begin{aligned} & \left\|\Lambda z - \Lambda z'\right\|_{L^2(\Gamma_C)^3} \\ & \quad \leqslant c\,(\bar\mu + L_\tau)\left\|z_1 - z'_1\right\|_{L^2(\Gamma_C)} + (L_\psi L + M_\psi)\left\|z_2 - z'_2\right\|_{L^2(\Gamma_C)} \\ & + (L_{k_c} L + M_{k_c})\left\|z_3 - z'_3\right\|_{L^2(\Gamma_C)} \\ & \quad \leqslant c_7\,(\bar\mu + L_\tau + L_\psi L + M_\psi + L_{k_c} L + M_{k_c})\left\|z - z'\right\|_{L^2(\Gamma_C)^3}. \end{aligned}$$

We pose $L^* = \dfrac{1}{c_7}$, hence if $\bar\mu + L_\tau + L_\psi L + M_\psi + L_{k_c} L + M_{k_c} \leqslant L^*$, we deduce that $\Lambda$ is a contraction operator. Then, it comes from Banach fixed point theorem that $\Lambda$ has a unique fixed point $z^*$ ($\Lambda z^* = z^*$) and therefore $x^{i+1}_{z^*} = (u^{i+1}_{z^*}, \varphi^{i+1}_{z^*}, \theta^{i+1}_{z^*}) \in X$ is the unique solution of the Problem $(PV^i)$.

## 6. CONVERGENCE ANALYSIS

For all $i = 0, \cdots, n-1$, let $x^{i+1} = (u^{i+1}, \varphi^{i+1}, \theta^{i+1})$ be the unique solution of the problem $(PV^i)$. In order to study the convergence of $\{x^{i+1}\}_{i=0,\cdots,n-1}$ when $n \to \infty$, we introduce the following continuous functions defined on $[t_i, t_{i+1}]$ $(i = 0, \cdots, n-1)$ by

$$u^n(t) = u^i + \frac{t-t_i}{\Delta t}\,\Delta u^i\,, \quad \varphi^n(t) = \varphi^i + \frac{t-t_i}{\Delta t}\,\Delta\varphi^i\,, \quad \theta^n(t) = \theta^i + \frac{t-t_i}{\Delta t}\,\Delta\theta^i \tag{6.1}$$

and the following piecewise constant functions defined on $(t_i, t_{i+1}]$ $(i = 0, \cdots, n-1)$ as follows

$$\bar{u}^n(t) = u^{i+1}\,, \quad \bar{\varphi}^n(t) = \varphi^{i+1}\,, \quad \bar{\theta}^n(t) = \theta^{i+1}\,, \quad \bar{g}^n(t) = g^{i+1} \tag{6.2}$$

where $g : [0, T] \to X$ given by

$$g = (f, q_e, \tilde{\Theta}) = (f, q_e, \Delta t\,\Theta) \in C([0, T], X). \tag{6.3}$$

Using some elementatry manipulations, we have the following results.

**Lemma 6.1.** *There exists $k_1 \geqslant 0$ and $k_2 \geqslant 0$ such that*

$$\|x^{i+1}\|_X \leqslant k_1 \sum_{j=0}^{i+1} \|g^j\|_X + k_2\,. \tag{6.4}$$

**Proof.** Taking (5.10)-(5.12), then from (5.1)-(5.3) that for all $y = (v, \xi, \eta) \in X$, we have

$$\begin{aligned}
&(Ax^{i+1}, y - x^{i+1})_X + (Bx^{i+1}, y - x^{i+1})_X + J(x^{i+1}, y - x^i) - J(x^{i+1}, x^{i+1} - x^i) \\
&\qquad - \Delta t\,\tilde{\chi}(x^{i+1}, y - x^{i+1}) + \Delta t\,\tilde{\omega}(x^{i+1}, y - x^{i+1}) + \tilde{\ell}(x^{i+1}, y - x^{i+1}) \\
&\geqslant\ (g^{i+1}, y - x^{i+1})_X + (\theta^i, \eta - \theta^{i+1})_{L^2(\Omega)} + (\mathcal{M}\,\varepsilon(u^i), \eta - \theta^{i+1})_{L^2(\Omega)} \\
&\qquad - (\mathcal{P}\,\nabla\varphi^i, \eta - \theta^{i+1})_{L^2(\Omega)},
\end{aligned} \tag{6.5}$$

where the functionals $\tilde{\chi}$, $\tilde{\omega}$ and $\tilde{\ell}$ are given by

$$\begin{aligned}
&\tilde{\chi}(x^{i+1}, y - x^{i+1}) = \chi(u^{i+1}; \frac{u^{i+1} - u^i}{\Delta t}; \eta - \theta^{i+1}), \\
&\tilde{\omega}(x^{i+1}, y - x^{i+1}) = \omega((u^{i+1}, \theta^{i+1}), \eta - \theta^{i+1}), \\
&\tilde{\ell}(x^{i+1}, y - x^{i+1}) = \ell((u^{i+1}, \varphi^{i+1}), \xi - \varphi^{i+1}).
\end{aligned}$$

Taking $y=0$ in (6.5), we find

$$\begin{aligned}&(Ax^{i+1},x^{i+1})_X+(Bx^{i+1},x^{i+1})_X\\&\leqslant (g^{i+1},x^{i+1})_X+J(x^{i+1},x^i)-J(x^{i+1},x^{i+1}-x^i)+\Delta t\,\tilde{\chi}(x^{i+1},x^{i+1})+\Delta t\,\tilde{\omega}(x^{i+1},x^{i+1})\\&\quad+\tilde{\ell}(x^{i+1},x^{i+1})+(\theta^i,\theta^{i+1})_{L^2(\Omega)}+(\mathcal{M}\,\varepsilon(u^i),\theta^{i+1})_{L^2(\Omega)}-(\mathcal{P}\,\nabla\varphi^i,\theta^{i+1})_{L^2(\Omega)}.\end{aligned}\tag{6.6}$$

Using (3.4), (3.13), (5.12) and the assumption $(H_6)(c)$, we obtain

$$\begin{aligned}J(x^{i+1},x^i)-J(x^{i+1},x^{i+1}-x^i)&=j(u^{i+1},u^i)-j(u^{i+1},u^{i+1}-u^i)\\&\leqslant c_0^2(L_\nu+L_\tau)\,\|x^{i+1}\|_X^2.\end{aligned}\tag{6.7}$$

Moreover, we have

$$\begin{aligned}\Delta t\,\tilde{\chi}(x^{i+1},x^{i+1})&=\Delta t\,\chi(u^{i+1};\frac{u^{i+1}-u^i}{\Delta t};\theta^{i+1})\\&=\Delta t\int_{\Gamma_C}\mu\,p_\nu(u_\nu^{i+1})\,S_c(.,\frac{1}{\Delta t}\|u_\tau^{i+1}-u_\tau^i\|)\theta^{i+1}\,da\\&\leqslant\Delta t\,c_2c_0\,\bar{\mu}\,L_\nu\,S^*\,\|x^{i+1}\|_X^2.\end{aligned}\tag{6.8}$$

and

$$\begin{aligned}\Delta t\,\tilde{\omega}(x^{i+1},x^{i+1})&=\Delta t\,\omega((u^{i+1},\theta^{i+1}),\theta^{i+1})\\&=\Delta t\int_{\Gamma_C}k_c(u_\nu^{i+1})\,\phi_L(\theta^{i+1}-\theta_F)\,\theta^{i+1}\,da\\&\leqslant\Delta t\,c_2\,L\,M_{k_c}\,\|x^{i+1}\|_X.\end{aligned}\tag{6.9}$$

and

$$\begin{aligned}\tilde{\ell}(x^{i+1},x^{i+1})&=\ell((u^{i+1},\varphi^{i+1}),\varphi^{i+1})\\&=\int_{\Gamma_C}k_c(u_\nu^{i+1}-g)\phi_L(\theta^{i+1}-\theta_F)\varphi^{i+1}\,da\\&\leq M_{k_c}\,L\,(mes(\Gamma_C))^{\frac{1}{2}}\,c_1\,\|x^{i+1}\|_X.\end{aligned}\tag{6.10}$$

We combine (5.15), (5.19) and (6.6)-(6.10) to get

$$\begin{aligned}(m-C)\,\|x^{i+1}\|_X^2\leqslant\;&\|g^{i+1}\|_X\,\|x^{i+1}\|_X+M_{k_c}\,L\,(mes(\Gamma_C))^{\frac{1}{2}}\,c_1\,\|x^{i+1}\|_X\\&+\Delta t\,c_2\,L\,M_{k_c}\,\|x^{i+1}\|_X\\&+c_0^2(L_\nu+L_\tau)\,\|x^{i+1}\|_X^2+\Delta t\,c_2c_0\,\bar{\mu}\,L_\nu\,S^*\,\|x^{i+1}\|_X^2\\&+\sqrt{3}\max(1,M_{\mathcal{P}},M_{\mathcal{M}})\,\|x^i\|_X\,\|x^{i+1}\|_X\end{aligned}$$

Hence

$$\big[[(m-C)-c_0^2(L_\nu+L_\tau)]-\bar{\mu}\Delta t\, c_2\, c_0\, L_\nu\, S^*\big]\,\|x^{i+1}\|_X$$
$$\leqslant \|g^{i+1}\|_X+[\sqrt{3}\max(1,M_{\mathcal{P}},M_{\mathcal{M}})+M_{k_c}\,L\,(mes(\Gamma_C))^{\frac{1}{2}}\,c_1\;+\Delta t\, c_2\, L\, M_{k_c}\,]\,\|x^i\|_X$$

and then, the lemma 6.1 is established.

**Lemma 6.2.** *There exists* $k_3 \geqslant 0$ *such that*

$$\|\Delta x^i\|_X \leq k_3 \sum_{j=0}^{i} \|\Delta g^j\|_X \tag{6.11}$$

**Proof.** Taking (6.5) at time $t_{i+1}$ and $t_i$, we find for all $y \in X$ that

$$\begin{aligned}(Ax^{i+1}, y-x^{i+1})_X &+ (Bx^{i+1}, y-x^{i+1})_X + J(x^{i+1}, y-x^i) - J(x^{i+1}, x^{i+1}-x^i)\\ &- \Delta t\,\tilde{\chi}(x^{i+1}, y-x^{i+1}) + \Delta t\,\tilde{\omega}(x^{i+1}, y-x^{i+1}) + \tilde{\ell}(x^{i+1}, y-x^{i+1})\\ &\geqslant (g^{i+1}, y-x^{i+1})_X + (\theta^i, \eta-\theta^{i+1})_{L^2(\Omega)} + (\mathcal{M}\,\varepsilon(u^i), \eta-\theta^{i+1})_{L^2(\Omega)}\\ &- (\mathcal{P}\,\nabla\varphi^i, \eta-\theta^{i+1})_{L^2(\Omega)},\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}(Ax^i, y-x^i)_X &+ (Bx^i, y-x^i)_X + J(x^i, y-x^{i-1}) - J(x^i, x^i-x^{i-1})\\ &- \Delta t\,\tilde{\chi}(x^i, y-x^i) + \Delta t\,\tilde{\omega}(x^i, y-x^i) + \tilde{\ell}(x^i, y-x^i)\\ &\geqslant (g^i, y-x^i)_X + (\theta^{i-1}, \eta-\theta^i)_{L^2(\Omega)} + (\mathcal{M}\,\varepsilon(u^{i-1}), \eta-\theta^i)_{L^2(\Omega)}\\ &- (\mathcal{P}\,\nabla\varphi^{i-1}, \eta-\theta^i)_{L^2(\Omega)},\end{aligned} \tag{6.13}$$

We take $y = x^i$ in (6.12) and $y = x^{i+1}$ in (6.13) to get

$$\begin{aligned}&(Ax^{i+1}-Ax^i, x^{i+1}-x^i)_X + (Bx^{i+1}-Bx^i, x^{i+1}-x^i)_X\\ &\quad\leqslant (g^{i+1}-g^i, x^{i+1}-x^i)_X + J(x^i, \Delta x^i+\Delta x^{i-1}) - J(x^i, \Delta x^{i-1}) - J(x^{i+1}, \Delta x^i)\\ &\qquad + \Delta t\,\tilde{\chi}(x^{i+1}, \Delta x^i) - \Delta t\,\tilde{\chi}(x^i, \Delta x^i) + \Delta t\,\tilde{\omega}(x^{i+1}, \Delta x^i) - \Delta t\,\tilde{\omega}(x^i, \Delta x^i)\\ &\qquad + \tilde{\ell}(x^{i+1}, \Delta x^i) - \tilde{\ell}(x^i, \Delta x^i) + (\Delta\theta^{i-1}, \Delta\theta^i)_{L^2(\Omega)}\\ &\qquad + (\mathcal{M}\,\varepsilon(u^i) - \mathcal{M}\,\varepsilon(u^{i-1}), \Delta\theta^i)_{L^2(\Omega)} - (\mathcal{P}\,\nabla\varphi^i - \mathcal{P}\,\nabla\varphi^{i-1}, \Delta\theta^i)_{L^2(\Omega)}.\end{aligned} \tag{6.14}$$

Moreover, using the propriety of $J$ and $\big||u_\tau^{i+1}-u_\tau^{i-1}|-|u_\tau^i-u_\tau^{i-1}|\big| \leqslant |u_\tau^{i+1}-u_\tau^i|$ to deduce

$$\begin{aligned}J(x^i,\Delta x^i+\Delta x^{i-1})-J(x^i,\Delta x^{i-1}) &= j(u^i,\Delta u^i+\Delta u^{i-1})-j(u^i,\Delta u^{i-1})\\ &\leqslant j(u^i,\Delta u^i)\\ &\leqslant J(x^i,\Delta x^i)\end{aligned}$$

and

$$\begin{aligned}J(x^i,\Delta x^i)-J(x^{i+1},\Delta x^i) &= j((u^i,\Delta u^i)-j(u^{i+1},\Delta u^i)\\ &= \int_{\Gamma_C}\big[p_\nu(u_\nu^i-g)-p_\nu(u_\nu^{i+1}-g)\big]\,\Delta u_\nu^i\,da\\ &\quad+\int_{\Gamma_C}\big[p_\tau(u_\nu^i-g)-p_\tau(u_\nu^{i+1}-g)\big]\,\|\Delta u_\tau^i\|\,da\\ &\leqslant c_0^2\,(L_\nu+L_\tau)\,\|\Delta x^i\|_X^2.\end{aligned}\tag{6.15}$$

On another hand, we have

$$\begin{aligned}&\tilde\chi(x^{i+1},\Delta x^i)-\tilde\chi(x^i,\Delta x^i)=\chi(u^{i+1};\frac{u^{i+1}-u^i}{\Delta t};\Delta\theta^i)-\chi(u^i;\frac{u^i-u^{i-1}}{\Delta t};\Delta\theta^i)\\ &=\int_{\Gamma_C}\big[p_\tau(u^{i+1}-g)-p_\tau(u^i-g)\big]\,S_c(.,\frac{1}{\Delta t}\|u_\tau^{i+1}-u_\tau^i\|)\,\Delta\theta^i\,da\\ &\quad+\int_{\Gamma_C}\mu\,p_\nu(u^i-g)\,\big[S_c(.,\frac{1}{\Delta t}\|u_\tau^{i+1}-u_\tau^i\|)-S_c(.,\frac{1}{\Delta t}\|u_\tau^i-u_\tau^{i-1}\|)\,\Delta\theta^i\,da\\ &\leqslant c_2\,S^*\,L_\tau\,c_0\,\|\Delta x^i\|_X^2\\ &\quad+\int_{\Gamma_C}\mu\,p_\nu(u^i-g)\,\big[S_c(.,\frac{1}{\Delta t}\|\Delta u_\tau^i\|)-S_c(.,\frac{1}{\Delta t}\|\Delta u_\tau^{i-1}\|)\big]\,\Delta\theta^i\,da\\ &\leqslant c_2\,S^*\,L_\tau\,c_0\,\|\Delta x^i\|_X^2+\bar\mu\,L_{p_\nu}\,\frac{L_{s_c}}{\Delta t}\int_{\Gamma_C}\|u_\tau^{i+1}-u_\tau^{i-1}\|\,\Delta\theta^i\,da\\ &\leqslant c_2\,S^*\,L_\tau\,c_0\,\|\Delta x^i\|_X^2+c_0\,c_2\,\bar\mu\,L_{p_\nu}\,\frac{L_{s_c}}{\Delta t}\,\|\Delta x^{i+1}\|_X^2+c_0\,c_2\,\bar\mu\,L_{p_\nu}\,\frac{L_{s_c}}{\Delta t}\,\|\Delta x^i\|_X\,\|\Delta x^{i-1}\|_X.\end{aligned}$$

Hence

$$\begin{aligned}\Delta t\,\tilde\chi(x^{i+1},\Delta x^i)-\Delta t\,\tilde\chi(x^i,\Delta x^i) &\leqslant \Delta t\,c_2\,S^*\,L_\tau\,c_0\,\|\Delta x^i\|_X^2\\ &\quad+c_0\,c_2\,\bar\mu\,L_{p_\nu}\,L_{s_c}\,\|\Delta x^{i+1}\|_X^2+c_0\,c_2\,\bar\mu\,L_{p_\nu}\,L_{s_c}\,\|\Delta x^i\|_X\,\|\Delta x^{i-1}\|_X.\end{aligned}\tag{6.16}$$

$$\begin{aligned}\tilde\omega(x^{i+1},\Delta x^i)-\tilde\omega(x^i,\Delta x^i) &= \omega((u^{i+1},\theta^{i+1}),\Delta\theta^i)-\omega((u^i,\theta^i),\Delta\theta^i)\\ &\leqslant (L_{k_c}\,L\,c_0\,c_2+M_{k_c}\,c_2^2)\,\|\Delta x^{i+1}\|_X^2,\end{aligned}\tag{6.17}$$

and

$$\begin{aligned}\tilde{\ell}(x^{i+1},\Delta x^i)-\tilde{\ell}(x^i,\Delta x^i)&=\ell((u^{i+1},\varphi^{i+1}),\Delta\varphi^i)-\ell((u^i,\varphi^i),\Delta\varphi^i)\\&\leqslant (L_\psi\, L\, c_0\, c_1+M_{k_c}\, c_1^2)\,\|\Delta x^{i+1}\|_X^2.\end{aligned}\tag{6.18}$$

and by using (6.12)-(6.22), we obtain that there exists a positive constant $L^*$ such that

$$\begin{aligned}(m-C)\|\Delta x^i\|_X^2 \leqslant{}& \|\Delta g^i\|_X\,\|\Delta x^i\|_X + L^*(\bar{\mu}+L_\tau+LL_{k_c}+LL_\psi+M_{k_c}+M_\psi)\|\Delta x^{i+1}\|_X^2\\&+c_0^2\,(L_\nu+L_\tau)\|\Delta x^{i+1}\|_X^2+c_0\,c_2\,\bar{\mu}\,L_{p_\nu}\,L_{s_c}\,\|\Delta x^i\|_X\,\|\Delta x^{i-1}\|_X\\&+\sqrt{3}\,c_2\,\max(1,M_{\mathcal{P}},M_{\mathcal{M}})\,\|\Delta x^i\|_X\,\|\Delta x^{i-1}\|_X.\end{aligned}$$

Thus

$$\begin{aligned}\big((m-C)-c_0^2\,(L_\nu+L_\tau)-&L^*(\bar{\mu}+L_\tau+LL_{k_c}+LL_\psi+M_{k_c}+M_\psi)\big)\,\|\Delta x^i\|_X^2\\&\leqslant \|\Delta g^i\|_X\,\|\Delta x^i\|_X+c_0\,c_2\,\bar{\mu}\,L_{p_\nu}\,L_{s_c}\,\|\Delta x^i\|_X\,\|\Delta x^{i-1}\|_X\\&\quad+\sqrt{3}\,c_2\,\max(1,M_{\mathcal{P}},M_{\mathcal{M}})\,\|\Delta x^i\|_X\,\|\Delta x^{i-1}\|_X.\end{aligned}$$

Finally, for $\bar{\mu}+L_\tau+LL_{k_c}+LL_\psi+M_{k_c}+M_\psi$ sufficiently small, the lemma 6.2 is proved.

□

**Proposition 6.1.** *From the sequences* $(u^n)$, $(\varphi^n)$ *and* $(\theta^n)$, *we can extract subsequences, still denoted* $(u^n)$, $(\varphi^n)$ *and* $(\theta^n)$ *such that*

$$\begin{aligned}&u^n\to u && \textit{weak}\,*\ \textit{in} && L^\infty(0,T;V),\\&\varphi^n\to\varphi && \textit{weak}\,*\ \textit{in} && L^\infty(0,T;W),\\&\theta^n\to\theta && \textit{weak}\,*\ \textit{in} && L^\infty(0,T;Q).\end{aligned}\tag{6.19}$$

*and*

$$\begin{aligned}&\dot{u}^n\to \dot{u} && \textit{weak}\,*\ \textit{in} && L^\infty(0,T;V),\\&\dot{\varphi}^n\to\dot{\varphi} && \textit{weak}\,*\ \textit{in} && L^\infty(0,T;W),\\&\dot{\theta}^n\to\dot{\theta} && \textit{weak}\,*\ \textit{in} && L^\infty(0,T;Q).\end{aligned}\tag{6.20}$$

**Proof.** using (6.2) and the assumption $(H_8)$, we find that $g\in W^{1,\infty}(0,T;X)$. Then, we get

$$\|g^i\|_X\leqslant\|g\|_{L^\infty(0,T;X)}\ ,\quad \Big\|\frac{\Delta g^i}{\Delta t}\Big\|_X\leqslant\|\dot{g}\|_{L^\infty(0,T;X)}\,.\tag{6.21}$$

Let $x^n=x^i+\dfrac{t-t_i}{\Delta t}\Delta x^i$, by lemmas 6.1 and 6.2 we find that there exists a canstant $d>0$ such that

$$\|x^n\|_{W^{1,\infty}(0,T;X)}\leqslant d\,.\tag{6.22}$$

Applying (5.8), we deduce that

$$\begin{aligned} &\|u^n\|_{W^{1,\infty}(0,T;V)} \leqslant d \\ &\|\varphi^n\|_{W^{1,\infty}(0,T;W)} \leqslant d \\ &\|\theta^n\|_{W^{1,\infty}(0,T;Q)} \leqslant d \end{aligned} \tag{6.23}$$

and then, the sequences $(u^n)$, $(\varphi^n)$ and $(\theta^n)$ are uniformly bounded, respectively in $W^{1,\infty}(0,T;V)$, $W^{1,\infty}(0,T;W)$ and $W^{1,\infty}(0,T;Q)$. Hence, we can extract from it subsequences still denoted $(u^n)$, $(\varphi^n)$ and $(\theta^n)$ such that (6.19) and (6.20) and thus the proposition 6.1 is proved. □

**Proposition 6.2.** *There exists subsequences of $(\bar{u}^n)$, $(\bar{\varphi}^n)$ and $(\bar{\theta}^n)$, still denoted $(\bar{u}^n)$, $(\bar{\varphi}^n)$ and $(\bar{\theta}^n)$ such that for almost every $t \in [0,T]$ we have*

$$\begin{aligned} &\bar{u}^n(t) \to u(t) \quad \textit{weak in } V \\ &\bar{\varphi}^n(t) \to \varphi(t) \quad \textit{weak in } W \\ &\bar{\theta}^n(t) \to \theta(t) \quad \textit{weak in } Q. \end{aligned} \tag{6.24}$$

**Proof.** It comes from (6.21) that $(\bar{u}^n)$ is uniformly bounded in $L^\infty(0,T;V)$ and then $(\bar{u}^n)$ converges weak $*$ for subsequence in $L^\infty(0,T;V)$. Moreover, we have

$$\|u^n(t) - \bar{u}^n(t)\|_V \leqslant \frac{T}{n}\,\|\dot{u}^n(t)\|_V\,, \ \ \forall\, t \in \ [0,T]\,.$$

This inequality combined with (6.23) leads to

$$\|u^n(t) - \bar{u}^n(t)\|_{L^\infty(0,T;V)} \leqslant d\,\frac{T}{n}\,. \tag{6.25}$$

We thus deduce from (6.19) that

$$(\bar{u}^n) \to u \ \text{ weak} * \text{for subsequence in } \ L^\infty(0,T;V). \tag{6.26}$$

On another hand, (6.19), (6.20) and $W^{1,\infty}(0,T;V) \subset C(0,T;V)$ imply that

$$u^n(t) \to u(t) \ \text{ weak in } \ V \ \text{ for a.e } \ t \in \ [0.T]. \tag{6.27}$$

We conclude from (6.25)-(6.27) that for almost every $t \in [0,T]$, we have $\bar{u}^n(t) \to u(t)$ weak in $V$ for a subsequence. Finally, by the same manner, the proof is finished.

□

**Remark.** *Let $\bar{f}^n(t) = f^{i+1}$, $\bar{q_e}^n(t) = q_e^{i+1}$ and $\bar{\Theta}^n(t) = \Theta^{i+1}$ for all $t \in [t_i, t_{i+1}]$. Under the assumption $(H_8)$, we have $f \in W^{1,\infty}(0,T;V)$, $q_e \in W^{1,\infty}(0,T;W)$ and $\Theta \in W^{1,\infty}(0,T;Q)$. Then*

$$\begin{aligned} &\bar{f}^n \to f \quad \textit{strongly in } \ L^2(0,T;V) \\ &\bar{q_e}^n \to q_e \quad \textit{strongly in } \ L^2(0,T;W) \\ &\bar{\Theta}^n \to \Theta \quad \textit{strongly in } \ L^2(0,T;Q)\,. \end{aligned} \tag{6.28}$$

**Proposition 6.3.** *The triple* $(u,\varphi,\theta)\in L^2(0,T;V\times W\times Q)$ *is solution of the problem* $(PV)$.

**Proof.** If we substitute $v$ by $u^i+v\,\Delta t$ in (5.1) and divide the resulting inequality by $\Delta t$ and if we divide (5.1) by $\Delta t$, the problem $(PV^i)$ will be written for all $v\in L^2(0,T;V)$, $\xi\in L^2(0,T;W)$ and $\eta\in L^2(0,T;Q)$ as follows

$$
\begin{aligned}
&(\mathfrak{F}\,\varepsilon(\bar u^n(t)),\varepsilon(v(t))-\varepsilon(\frac{d}{dt}u^n(t)))_{\mathcal{H}}+(\mathcal{E}^*\,\nabla\bar\varphi^n(t),\varepsilon(v(t))-\varepsilon(\frac{d}{dt}u^n(t)))_H\\
&\quad-(\mathcal{M}\,\bar\theta^n(t),\varepsilon(v(t))-\varepsilon(\frac{d}{dt}u^n(t)))_H+j(\bar u^n(t),v(t))-j(\bar u^n(t),\frac{d}{dt}u^n(t))\\
&\quad\geqslant(\bar f^n(t),v(t)-\frac{d}{dt}u^n(t))_V
\end{aligned}
$$

$$
\begin{aligned}
&(\beta\,\nabla\bar\varphi^n(t),\nabla\xi(t))_H-(\mathcal{E}\varepsilon(\bar u^n(t)),\nabla\xi(t))_H-(\mathcal{P}\,\bar\theta^n(t),\nabla\xi(t))_H+l((\bar u^n(t),\bar\varphi^n(t));\xi(t))\\
&\quad=(\bar{q_e}^n(t),\xi(t))_W
\end{aligned}
$$

$$
\begin{aligned}
&(\frac{d}{dt}\theta^n(t),\eta(t))_{L^2(\Omega)}+(\mathcal{K}\,\nabla\bar\theta^n(t),\nabla\eta(t))_H+(\mathcal{M}\,\varepsilon(\frac{d}{dt}u^n(t)),\eta(t))_{L^2(\Omega)}\\
&\quad-(\mathcal{P}\,\nabla\frac{d}{dt}\varphi^n(t),\eta(t))_{L^2(\Omega)}-\chi(\bar u^n(t);\frac{d}{dt}u^n(t);\eta(t))+\omega((\bar u^n(t),\bar\theta^n(t)),\eta(t))\\
&\quad=(\bar\Theta^n(t),\eta(t))_{L^2(\Omega)}
\end{aligned}
$$

Integrating both sides of the previous inequalities on $[0,T]$, we find that

$$
\begin{aligned}
&\int_0^T(\mathfrak{F}\,\varepsilon(\bar u^n(t)),\varepsilon(v(t))-\varepsilon(\frac{d}{dt}u^n(t)))_{\mathcal{H}}\,dt+\int_0^T(\mathcal{E}^*\,\nabla\bar\varphi^n(t),\varepsilon(v(t))-\varepsilon(\frac{d}{dt}u^n(t)))_H\,dt\\
&\quad-\int_0^T(\mathcal{M}\,\bar\theta^n(t),\varepsilon(v(t))-\varepsilon(\frac{d}{dt}u^n(t)))_H\,dt+\int_0^T j(\bar u^n(t),v(t))-\int_0^T j(\bar u^n(t),\frac{d}{dt}u^n(t))\,dt\\
&\quad\geqslant\int_0^T(\bar f^n(t),v(t)-\frac{d}{dt}u^n(t))_V\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T(\beta\,\nabla\bar\varphi^n(t),\nabla\xi(t))_H\,dt-\int_0^T(\mathcal{E}\varepsilon(\bar u^n(t)),\nabla\xi(t))_H\,dt-\int_0^T(\mathcal{P}\,\bar\theta^n(t),\nabla\xi(t))_H\,dt\\
&\quad+\int_0^T l((\bar u^n(t),\bar\varphi^n(t));\xi(t))\,dt=\int_0^T(\bar{q_e}^n(t),\xi(t))_W\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T(\frac{d}{dt}\theta^n(t),\eta(t))_{L^2(\Omega)}\,dt+\int_0^T(\mathcal{K}\,\nabla\bar\theta^n(t),\nabla\eta(t))_H\,dt+\int_0^T(\mathcal{M}\,\varepsilon(\frac{d}{dt}u^n(t)),\eta(t))_{L^2(\Omega)}\,dt\\
&\quad-\int_0^T(\mathcal{P}\,\nabla\frac{d}{dt}\varphi^n(t),\eta(t))_{L^2(\Omega)}\,dt-\int_0^T\chi(\bar u^n(t);\frac{d}{dt}u^n(t);\eta(t))\,dt\\
&\quad+\int_0^T\omega((\bar u^n(t),\bar\theta^n(t)),\eta(t))\,dt=\int_0^T(\bar\Theta^n(t),\eta(t))_{L^2(\Omega)}\,dt
\end{aligned}
$$

By adding the previous inequalities, we obtain

$$
\begin{aligned}
&\int_0^T (\mathfrak{F}\,\varepsilon(\bar{u}^n(t)), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t)))_{\mathcal{H}}\,dt + \int_0^T (\beta\,\nabla\bar{\varphi}^n(t), \nabla\xi(t))_H\,dt \\
&+ \int_0^T (\mathcal{K}\,\nabla\bar{\theta}^n(t), \nabla\eta(t))_H\,dt + \int_0^T \omega((\bar{u}^n(t), \bar{\theta}^n(t)), \eta(t))\,dt \\
&+ \int_0^T (\frac{d}{dt}\theta^n(t), \eta(t))_{L^2(\Omega)}\,dt + \int_0^T (\mathcal{E}^*\,\nabla\bar{\varphi}^n(t), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t)))_H\,dt \\
&- \int_0^T (\mathcal{E}\varepsilon(\bar{u}^n(t)), \nabla\xi(t))_H\,dt - \int_0^T (\mathcal{M}\,\bar{\theta}^n(t), \varepsilon(v(t)) - \varepsilon(\frac{d}{dt}u^n(t)))_H\,dt \qquad (6.29)\\
&+ \int_0^T (\mathcal{M}\,\varepsilon(\frac{d}{dt}u^n(t)), \eta(t))_{L^2(\Omega)}\,dt - \int_0^T (\mathcal{P}\,\bar{\theta}^n(t), \nabla\xi(t))_H\,dt \\
&- \int_0^T (\mathcal{P}\,\nabla\frac{d}{dt}\varphi^n(t), \eta(t))_{L^2(\Omega)}\,dt + \int_0^T j(\bar{u}^n(t), v(t)) - \int_0^T j(\bar{u}^n(t), \frac{d}{dt}u^n(t))\,dt \\
&- \int_0^T \chi(\bar{u}^n(t); \frac{d}{dt}u^n(t); \eta(t))\,dt + \int_0^T l((\bar{u}^n(t), \bar{\varphi}^n(t)); \xi(t))\,dt \\
&\geqslant \int_0^T (\bar{f}^n(t), v(t) - \frac{d}{dt}u^n(t))_V + \int_0^T (\bar{q_e}^n(t), \xi)_W\,dt + \int_0^T (\bar{\Theta}^n(t), \eta(t))_{L^2(\Omega)}\,dt.
\end{aligned}
$$

To finish the proof of the proposition 6.3, we use the following Lemmas

**Lemma 6.3.** *For any $v \in L^2(0,T;V)$, $\xi \in L^2(0,T;W)$ and $\eta \in L^2(0,T;Q)$, we have*

$$\lim_{n\to\infty} \int_0^T (\mathfrak{F}\,\varepsilon(\bar{u}^n(t)), \varepsilon(v(t)))_{\mathcal{H}}\,dt = \int_0^T (\mathfrak{F}\,\varepsilon(u(t)), \varepsilon(v(t)))_{\mathcal{H}}\,dt \qquad (6.30)$$

$$\lim_{n\to\infty} \int_0^T (\mathcal{E}^*\,\nabla\bar{\varphi}^n(t), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t)))_H\,dt = \int_0^T (\mathcal{E}^*\,\nabla\varphi(t), \varepsilon(v(t)) - \varepsilon(\dot{u}(t)))_H\,dt \qquad (6.31)$$

$$\lim_{n\to\infty} \int_0^T (\mathcal{M}\,\bar{\theta}^n(t), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t)))_H\,dt = \int_0^T (\mathcal{M}\,\theta(t), \varepsilon(v(t)) - \varepsilon(\dot{u}(t)))_H\,dt \qquad (6.32)$$

$$\lim_{n\to\infty} \int_0^T (\beta\,\nabla\bar{\varphi}^n(t), \nabla\xi(t))_H\,dt = \int_0^T (\beta\,\nabla\varphi(t), \nabla\xi(t))_H\,dt \qquad (6.33)$$

$$\lim_{n\to\infty} \int_0^T (\mathcal{E}\varepsilon(\bar{u}^n(t)), \nabla\xi(t))_H\,dt = \int_0^T (\mathcal{E}\varepsilon(u(t)), \nabla\xi(t))_H\,dt \qquad (6.34)$$

*and*

$$\lim_{n\to\infty}\int_0^T (\mathcal{P}\,\bar{\theta}^n(t),\nabla\xi(t))_H\,dt = \int_0^T (\mathcal{P}\,\theta(t),\nabla\xi(t))_H\,dt \tag{6.35}$$

$$\lim_{n\to\infty}\int_0^T (\frac{d}{dt}\theta^n(t),\eta(t))_{L^2(\Omega)}\,dt = \int_0^T (\dot{\theta}(t),\eta(t))_{L^2(\Omega)}\,dt \tag{6.36}$$

$$\lim_{n\to\infty}\int_0^T (\mathcal{K}\,\nabla\bar{\theta}^n(t),\nabla\eta(t))_H\,dt = \int_0^T (\mathcal{K}\,\nabla\theta(t),\nabla\eta(t))_H\,dt \tag{6.37}$$

$$\lim_{n\to\infty}\int_0^T (\mathcal{M}\,\varepsilon(\frac{d}{dt}u^n(t)),\eta(t))_{L^2(\Omega)}\,dt = \int_0^T (\mathcal{M}\,\varepsilon(\dot{u}(t)),\eta(t))_{L^2(\Omega)}\,dt \tag{6.38}$$

$$\lim_{n\to\infty}\int_0^T (\mathcal{P}\,\nabla\frac{d}{dt}\varphi^n(t),\eta(t))_{L^2(\Omega)}\,dt = \int_0^T (\mathcal{P}\,\nabla\dot{\varphi}(t),\eta(t))_{L^2(\Omega)}\,dt \tag{6.39}$$

$$\lim_{n\to\infty}\int_0^T \omega(\bar{u}^n(t),\bar{\theta}^n(t)),\eta(t))\,dt = \int_0^T \omega((u(t),\theta(t)),\eta(t))\,dt \tag{6.40}$$

$$\lim_{n\to\infty}\int_0^T j((\bar{u}^n(t),v(t))\,dt = \int_0^T j(u(t),v(t))\,dt, \tag{6.41}$$

$$\lim_{n\to\infty}\int_0^T l((\bar{u}^n(t),\bar{\varphi}^n(t));\xi(t))\,dt = \int_0^T l((u(t),\varphi(t));\xi(t))\,dt \tag{6.42}$$

$$\lim_{n\to\infty}\int_0^T (\bar{f}^n(t),v(t)-\frac{d}{dt}u^n(t))_V\,dt = \int_0^T (f(t),v(t)-\dot{u}(t))_V\,dt \tag{6.43}$$

$$\lim_{n\to\infty}\int_0^T (\bar{q_e}^n(t),\xi(t))_W\,dt = \int_0^T (q_e(t),\xi(t))_W\,dt \tag{6.44}$$

$$\lim_{n\to\infty}\int_0^T (\bar{\Theta}^n(t),\eta(t))_{L^2(\Omega)}\,dt = \int_0^T (\Theta(t),\eta(t))_{L^2(\Omega)}\,dt \tag{6.45}$$

**Proof.** It follows from (6.19) and (6.25) that,

$$\begin{aligned} \bar{u}^n &\rightharpoonup u \quad \text{weak in } L^2(0,T,V)\\ \bar{\varphi}^n &\rightharpoonup \varphi \quad \text{weak in } L^2(0,T,W)\\ \bar{\theta}^n &\rightharpoonup \theta \quad \text{weak in } L^2(0,T,Q) \end{aligned} \tag{6.46}$$

and the lemma 6.3 results from (6.46). □

**Lemma 6.4.** *For any* $v \in L^2(0,T;V)$ *and* $\eta \in L^2(0,T;Q)$*, we have*

$$\liminf_{n\to\infty}\int_0^T (\mathfrak{F}\,\varepsilon(\bar{u}^n(t)),\varepsilon(\frac{d}{dt}u^n(t)))_{\mathcal{H}}\,dt \geqslant \int_0^T (\mathfrak{F}\,\varepsilon(u(t)),\varepsilon(\dot{u}(t)))_{\mathcal{H}}\,dt \tag{6.47}$$

$$\liminf_{n\to\infty}\int_0^T j(\bar{u}^n(t),\frac{d}{dt}u^n(t)) \geqslant \int_0^T j(u(t),\dot{u}(t))\,dt \tag{6.48}$$

$$\liminf_{n\to\infty}\int_0^T \chi(\bar{u}^n(t);\frac{d}{dt}u^n(t);\eta(t))\,dt \geqslant \int_0^T \chi(u(t);\dot{u}(t);\eta(t))\,dt. \tag{6.49}$$

**Proof.** For the proof of (6.47), we refer to [7] and for (6.48), we write

$$j(\bar{u}^n(t), \frac{d}{dt}u^n(t)) = j(\bar{u}^n(t), \frac{d}{dt}u^n(t)) - j(u(t), \frac{d}{dt}u^n(t)) + j(u(t), \frac{d}{dt}u^n(t))$$

and

$$\int_0^T j(\bar{u}^n(t), \frac{d}{dt}u^n(t)) - j(u(t), \frac{d}{dt}u^n(t))\, dt$$
$$\leqslant \bar{\mu}\, L_1 \left\| R\big(\sigma_\nu(\bar{u}^n) - \sigma_\nu(u)\big)\right\|_{L^2(0,T;L^2(\Gamma_C))} \|\dot{u}_\tau^n\|_{L^2(0,T;L^2(\Gamma_C)^d)} .$$

Keeping in mind that the mapping $R$ is compact, we have $\lim\limits_{n\to\infty} R\big(\sigma_\nu(\bar{u}^n) - \sigma_\nu(u)\big) = 0$. Thus

$$\liminf_{n\to\infty} \int_0^T j(\bar{u}^n(t), \frac{d}{dt}u^n(t)) - j(u(t), \frac{d}{dt}u^n(t))\, dt \leqslant 0$$

and then, we conclude that

$$\liminf_{n\to\infty} \int_0^T j(u(t), \frac{d}{dt}u^n(t))\, dt \geqslant \int_0^T j(u(t), \dot{u}(t))\, dt$$

In the same way, we can prove (6.49) and that finish the proof of lemma 6.4.

□

Now, we come back to the proof of proposition 6.3. By passing to the limit in (6.29), we obtain

$$\begin{aligned}
&\int_0^T (\mathfrak{F}\,\varepsilon(u(t)), \varepsilon(v(t)) - \varepsilon(\dot{u}(t)))_{\mathcal{H}}\, dt + \int_0^T (\beta\,\nabla\varphi(t), \nabla\xi(t))_H\, dt \\
&+ \int_0^T (\mathcal{K}\,\nabla\theta(t), \nabla\eta(t))_H\, dt + \int_0^T \omega((u(t),\theta(t)),\eta(t))\, dt + \int_0^T (\dot{\theta}(t), \eta(t))_{L^2(\Omega)}\, dt \\
&+ \int_0^T (\mathcal{E}^*\,\nabla\varphi(t), \varepsilon(v(t)) - \varepsilon(\dot{u}(t)))_{\mathcal{H}}\, dt - \int_0^T (\mathcal{E}\varepsilon(u(t)), \nabla\xi(t))_H\, dt \\
&- \int_0^T (\mathcal{M}\,\theta(t), \varepsilon(v(t)) - \varepsilon(\dot{u}(t)))_{\mathcal{H}}\, dt + \int_0^T (\mathcal{M}\,\varepsilon(\dot{u}(t)), \eta(t))_{L^2(\Omega)}\, dt \qquad (6.50)\\
&- \int_0^T (\mathcal{P}\,\theta(t), \nabla\xi(t))_H\, dt - \int_0^T (\mathcal{P}\,\nabla\dot{\varphi}(t), \eta(t))_{L^2(\Omega)}\, dt \\
&+ \int_0^T j(u(t), v(t))\, dt - \int_0^T j(u(t), \dot{u}(t))\, dt \\
&- \int_0^T \chi(u(t); \dot{u}(t); \eta(t))\, dt + \int_0^T l((u(t),\varphi(t)); \xi(t))\, dt \\
&\geqslant \int_0^T (f(t), v(t) - \dot{u}(t))_V\, dt + \int_0^T (q_e(t), \xi(t))_W\, dt + \int_0^T (\Theta(t), \eta(t))_{L^2(\Omega)}\, dt.
\end{aligned}$$

By the same method as in the proof of lemma 5.1, we find that

$$\begin{aligned}
&\int_0^T (\mathfrak{F}\,\varepsilon(u(t)),\varepsilon(v(t))-\varepsilon(\dot u(t)))_{\mathcal H}\,dt+\int_0^T (\mathcal E^*\,\nabla\varphi(t),\varepsilon(v(t))-\varepsilon(\dot u(t)))_{\mathcal H}\,dt\\
&\quad-\int_0^T (\mathcal M\,\theta(t),\varepsilon(v(t))-\varepsilon(\dot u(t)))_{\mathcal H}\,dt+\int_0^T j(u(t),v(t))\,dt\\
&\quad-\int_0^T j(u(t),\dot u(t))\,dt\geqslant\int_0^T (f(t),v(t)-\dot u(t))_V\,dt,
\end{aligned}$$

$$\begin{aligned}
&\int_0^T (\beta\,\nabla\varphi(t),\nabla\xi(t))_H\,dt-\int_0^T (\mathcal E\varepsilon(u(t)),\nabla\xi(t))_H\,dt-\int_0^T (\mathcal P\,\theta(t),\nabla\xi(t))_H\,dt\\
&\quad+\int_0^T l((u(t),\varphi(t));\xi(t))\,dt=\int_0^T (q_e(t),\xi(t))_W\,dt,
\end{aligned}$$

$$\begin{aligned}
&\int_0^T (\dot\theta(t),\eta(t))_{L^2(\Omega)}\,dt+\int_0^T (\mathcal K\,\nabla\theta(t),\nabla\eta(t))_H\,dt\\
&\quad+\int_0^T (\mathcal M\,\varepsilon(\dot u(t)),\eta(t))_{L^2(\Omega)}\,dt-\int_0^T (\mathcal P\,\nabla\dot\varphi(t),\eta(t))_{L^2(\Omega)}\,dt\\
&\quad-\int_0^T \chi(u(t);\dot u(t);\eta(t))\,dt+\int_0^T \omega((u(t),\theta(t)),\eta(t))\,dt\\
&\quad=\int_0^T (\Theta(t),\eta(t))_{L^2(\Omega)}\,dt.
\end{aligned}$$

Setting in this inequalities that

$$v(t)=\begin{cases} v & \text{for } t\in(s,s+\lambda)\\ \dot u(t) & \text{elsewhere.}\end{cases}\qquad\text{and}\qquad \xi(t)=\xi\ ,\quad \eta(t)=\eta\quad\text{for } t\in(s,s+\lambda).$$

Then, we obtain

$$\begin{aligned}
&\frac{1}{\lambda}\int_s^{s+\lambda} (\mathfrak{F}\,\varepsilon(u(t)),\varepsilon(v)-\varepsilon(\dot u(t)))_{\mathcal H}\,dt+\frac{1}{\lambda}\int_s^{s+\lambda} (\mathcal E^*\,\nabla\varphi(t),\varepsilon(v)-\varepsilon(\dot u(t)))_{\mathcal H}\,dt\\
&\quad-\frac{1}{\lambda}\int_s^{s+\lambda} (\mathcal M\,\theta(t),\varepsilon(v)-\varepsilon(\dot u(t)))_{\mathcal H}\,dt+\frac{1}{\lambda}\int_s^{s+\lambda} j(u(t),v)\,dt\\
&\quad-\frac{1}{\lambda}\int_s^{s+\lambda} j(u(t),\dot u(t))\,dt\\
&\quad\geqslant\frac{1}{\lambda}\int_s^{s+\lambda} (f(t),v-\dot u(t))_V\,dt,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\lambda}\int_s^{s+\lambda} (\beta\,\nabla\varphi(t),\nabla\xi)_H\,dt-\frac{1}{\lambda}\int_s^{s+\lambda} (\mathcal E\varepsilon(u(t)),\nabla\xi)_H\,dt-\frac{1}{\lambda}\int_s^{s+\lambda} (\mathcal P\,\theta(t),\nabla\xi)_H\,dt\\
&\quad+\int_s^{s+\lambda} l((u(t),\varphi(t));\xi(t))\,dt=\frac{1}{\lambda}\int_s^{s+\lambda} (q_e(t),\xi)_W\,dt
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{1}{\lambda}\int_{s}^{s+\lambda}(\dot{\theta}(t),\eta)_{L^2(\Omega)}\,dt+\frac{1}{\lambda}\int_{s}^{s+\lambda}(\mathcal{K}\,\nabla\theta(t),\nabla\eta)_{H}\,dt\\
&\quad+\frac{1}{\lambda}\int_{s}^{s+\lambda}(\mathcal{M}\,\varepsilon(\dot{u}(t)),\eta)_{L^2(\Omega)}\,dt-\frac{1}{\lambda}\int_{s}^{s+\lambda}(\mathcal{P}\,\nabla\dot{\varphi}(t),\eta)_{L^2(\Omega)}\,dt\\
&\quad-\frac{1}{\lambda}\int_{s}^{s+\lambda}\chi(u(t);\dot{u}(t);\eta(t))\,dt+\frac{1}{\lambda}\int_{s}^{s+\lambda}\omega((u(t),\theta(t)),\eta(t)\,dt\\
&\quad=\frac{1}{\lambda}\int_{s}^{s+\lambda}(\Theta(t),\eta)_{L^2(\Omega)}\,dt.
\end{aligned}
$$

Using lebesgue's theorem, we pass to the limit with respect to $s$ in the above inequalities, we deduce that $(u,\varphi,\theta)$ satisfies the Problem $(PV)$ and thus, the proposition 6.3 is proved.

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*Authors' addresses*:

*El-Hassan BENKHIRA*, University Moulay Ismaïl, Laboratory MACS, ESTM, BP 3103, Toulal-Meknes, Morocco. e-mail: benkhirahassan@yahoo.fr.

*Abdelhadi HACHLAF*, University Cadi Ayyad, Faculty of Semlalia, B.P. 2390 , 40000, Marrakech, Morocco. e-mail: abdelhadi.hachlaf@hotmail.com.

*Rachid FAKHAR*, University Sultan Moulay Slimane, Laboratory LS3M, 25000 Khouribga, Morocco. e-mail: rachidfakhar@yahoo.fr.

*Lahcen OUMOUACHA*, University Sultan Moulay Slimane, Laboratory LS3M, 25000 Khouribga, Morocco. e-mail: lahcen.oumouacha@gmail.com.