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Existence and Uniqueness Results for Hilfer Langevin Fractional Pantograph Differential Equations and Inclusions

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Abstract

This paper discuss the existence and uniqueness of solution for Hilfer Langevin fractional pantograph differential equation and inclusion, which are a special class of delay differential equations. The novelty of this work is that it is more general than the works based on the derivative of Caputo and Riemann-Liouville, because when $\beta = 0$ we get the Riemann-Liouville fractional derivative and when $\beta = 1$ we get the Caputo fractional derivative. In the first, we give some definitions, theorems, lemmas that are used through this manuscript. secondly, we give our existence results, based on Krasnoselskii's fixed point, and Banach's contraction principle. After that we investigate the inclusion version, and to obtain the existence result we use the Leray–Schauder alternative. Finaly, we give an illustrative example to support our results.
Keywords: Hilfer fractional derivative, Pantograph fractional differential inclusions, Caputo fractional derivative, Langevin equation, Fractional Langevin inclusion.

1. INTRODUCTION

In recent decades, fractional differential equations have been the subject of important studies, because is a useful tool for investigating a variety of complicated problems in physics, chemistry, biology, economics, and control theory. In comparison to classical order differential equations, fractional order differential equations accurately represent

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many real-world processes. More details on fractional differential equations theory and applications can be found here [4, 9–14, 16, 18, 19, 21, 23].

In the literature, the most used derivatives of fractional order are the Caputo and the RiemannLiouville derivatives, in this paper, we study the generalization of these derivatives, called the Hilfer fractional derivative, introduced by R. Hilfer in [10], and also we investigate another important class of fractional differential equations are the pantograph equations, these equations that are a special class of delay equations [3, 15, 25, 26].

Langevin equation has been considered a part of fractional calculus, [1, 2, 8, 20, 22, 27, 28], an equation of the form $m\frac{d^2x}{dt^2} = \lambda\frac{dx}{dt} + \eta(t)$ is called Langevin equation, introduced by Paul Langevin in 1908. The Langevin equation is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [19]. For some new developments on the fractional Langevin equation, see, for example, [5, 6, 17].

In this paper, we consider a new class of Hilfer fractional pantograph differential equations of the following nonlocal boundary value problem :

$$\begin{cases} {}^HD^{\alpha_1,\beta_1}({}^HD^{\alpha_2,\beta_2} + \mu)x(t) = f(t, x(t), x(\theta t)), \quad a \leq t \leq b \\ x(a) = 0 \quad , \qquad x(b) = \sum_{i=1}^{n} \omega_i (I^{\sigma_i}(x))(\eta_i), \end{cases} \tag{1}$$

Where ${}^HD^{\alpha_j,\beta_j}, j = 1, 2$ is the Hilfer fractional derivative of order α_j, $0 < \alpha_j < 1$ and parameter β_j, $0 \leq \beta_j \leq 1$, $j = 1, 2$, $1 < \alpha_1 + \alpha_2 \leq 2$, $\mu \in \mathbb{R}$, $a \geq 0$, I^{σ_i} is the Riemann-Liouville fractional integral of order $\sigma_i > 0$, $\omega_i \in \mathbb{R}$, $i = 1, ..., n$, $a < \eta_1 ..... < \eta_n < b$, $0 < \theta < 1$ and $f : [a, b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function.

As a second problem, we study the multivalued version of (1) by considering the inclusion problem :

$$\begin{cases} {}^HD^{\alpha_1,\beta_1}({}^HD^{\alpha_2,\beta_2} + \mu)x(t) \in F(t, x(t), x(\theta t)), \quad a \leq t \leq b \\ x(a) = 0 \quad , \qquad x(b) = \sum_{i=1}^{n} \omega_i (I^{\sigma_i}(x))(\eta_i), \end{cases} \tag{2}$$

Where $F : [a, b] \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

This paper is organized as follows, in section 2, we give some definitions and lemmas of fractional calculus and multivalued analysis which are needed throughout this paper. In section 3, we give the existence and uniqueness results for the first problem, by using the fixed point theorems. In section 4, we establish the existence results for the inclusion version, by using Leray–Schauder alternative. In section 5, we give an example to illustrate our results.

2. PRELIMINARIES

In this section we call some basic definitions and notations of fractional calculus and multivalued analysis which are needed throughout this paper.

2.1. Fractional Calculus

Definition 2.1. [11] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a continuous function $f : [a, \infty) \longrightarrow \mathbb{R}$ can be defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{3}$$

Provided that the right-hand side exists on (a, ∞).

Definition 2.2. [11] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function f is defined by

$$^{RL}D^{\alpha} f(t) := D^n I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \tag{4}$$

Where $n-1 < \alpha < n, n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. [11] The Caputo fractional derivative of order $\alpha > 0$ of a continuous function f is defined by

$$^{C}D^{\alpha} f(t) := I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \Big(\frac{d}{dt}\Big)^n f(s) ds, \tag{5}$$

Where $n-1 < \alpha < n, n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.4. (Hilfer fractional derivative [10]) The Hilfer fractional derivative of order α and parameter β of a function (also known as the generalized Riemann-Liouville fractional derivative) is defined by

$$^{H}D^{\alpha,\beta} f(t) = I^{\beta(n-\alpha)} D^n I^{(1-\beta)(n-\alpha)} f(t), \tag{6}$$

where $n-1 < \alpha < n$, $0 \le \beta \le 1$, $t > a$, $D = \Big(\frac{d}{dt}\Big)$.

Remark : When $\beta = 0$, the Hilfer fractional derivative corresponds to the Riemann-Liouville fractional derivative:

$$^{H}D^{\alpha,0}f(t) = D^{n}I^{(n-\alpha)}f(t), \tag{7}$$

While $\beta = 1$, The Hilfer fractional derivative corresponds to the Caputo fractional derivative:

$$^{H}D^{\alpha,1}f(t) = I^{(n-\alpha)}D^{n}f(t), \tag{8}$$

The following lemma plays a fundamental role in establishing the existence results for the given problem.

Lemma 2.5. *[10] Let $f \in L(a,b)$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $0 \leq \beta \leq 1$, and $I^{(1-\beta)(n-\alpha)}f \in AC^{k}[a,b]$ then*

$$\left(I^{\alpha H}D^{\alpha,\beta}f\right)(t) = f(t) - \sum_{k=1}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} . \lim_{t\to +a} \frac{d^{k}}{dt^{k}}\left(I^{(1-\beta)(n-\alpha)}f\right)(t) \tag{9}$$

The following lemma deals with a linear variant of the boundary value problem (1).

Lemma 2.6. *Let $a \geq 0$, $0 < \alpha_i < 1$, $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, $i = 1,2$, $1 < \alpha_1 + \alpha_2 \leq 2$ and $h \in C([a,b],\mathbb{R})$. Then the function x is a solution of the boundary value problem:*

$$\begin{cases} ^{H}D^{\alpha_1,\beta_1}(^{H}D^{\alpha_2,\beta_2} + \mu)x(t) = h(t), \quad a \leq t \leq b, \\ x(a) = 0 \quad , \qquad x(b) = \sum_{i=1}^{n} \omega_i (I^{\sigma_i}(x))(\eta_i)ds, \quad a < \eta_i < b, i = 1, ..., n, \end{cases} \tag{10}$$

if and only if

$$\begin{aligned} x(t) = I^{\alpha_1+\alpha_2}h(t) - \mu I^{\alpha_2}x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[I^{\alpha_1+\alpha_2}h(b) \\ - \mu I^{\alpha_2}x(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i}h(\eta_i) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i}x(\eta_i) \Bigg], \end{aligned} \tag{11}$$

where

$$\Lambda = \sum_{i=1}^{n} \omega_i \frac{(\eta_i - a)^{\gamma_1+\alpha_2+\sigma_i-1}}{\Gamma(\gamma_1+\alpha_2+\sigma_i)} - \frac{(b-a)^{\gamma_1+\alpha_2-1}}{\Gamma(\gamma_1+\alpha_2)} \neq 0 \tag{12}$$

Proof. Operating Riemann-Liouville fractional integral of order α_1 to both sides of (10) we get by using Lemma (2.5)

$$^{H}D^{\alpha_2,\beta_2}x(t) + \mu x(t) = I^{\alpha_1}h(t) + \frac{d_0}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1}, \tag{13}$$

where d_0 constant and $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1$. Operating Riemann-Liouville fractional integral of order α_2 to both sides of (13) we get by using Lemma (2.5)

$$x(t) = I^{\alpha_1+\alpha_2}h(t) - \mu I^{\alpha_2}x(t) + \frac{d_0}{\Gamma(\gamma_1+\alpha_2)}(t-a)^{\gamma_1+\alpha_2-1} + \frac{d_1}{\Gamma(\gamma_2)}(t-a)^{\gamma_2-1}. \quad (14)$$

From using the boundary condition $x(a) = 0$ in (14) we obtain $d_1 = 0$. Then, we get

$$x(t) = I^{\alpha_1+\alpha_2}h(t) - \mu I^{\alpha_2}x(t) + \frac{d_0}{\Gamma(\gamma_1+\alpha_2)}(t-a)^{\gamma_1+\alpha_2-1}. \quad (15)$$

From using the boundary condition $x(b) = \sum_{i=1}^{n} \omega_i(I^{\sigma_i}(x))(\eta_i)ds$, in (15) we find

$$d_0 = \frac{1}{\Lambda}\left[I^{\alpha_1+\alpha_2}h(b) - \mu I^{\alpha_2}x(b) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i}h(\eta_i) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x(\eta_i)\right]. \quad (16)$$

Substituting the value of (d_0) in (13) we obtain the solution (11). The converse follows by direct computation. □

2.2. Multivalued Analysis

For a normed space $(X, \|.\|)$, we define :
$\mathcal{P}(X) = \{Y \subset X : Y \neq \varnothing\}$, $\mathcal{P}_{c,cp}(X) = \{Y \subset X : Y \ is \ convex \ and \ compact\}$
For the basic concepts of multivalued analysis, we refer to ([9, 10])

Definition 2.7. A multivalued map $F : [a, b] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if :

(i) $t \longrightarrow F(t, x)$ is measurable for each $x \in \mathbb{R}$.

(ii) $x \longrightarrow F(t, x)$ is upper semicontinuous for almost all $t \in [a, b]$.
Furthermore, a Carathéodory function F is called $\mathbb{L}^1$-Carathéodory if :

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in \mathbb{L}^1([a, b]; \mathbb{R})$ such that

$$\|F(t, x)\| = sup\{|v| : v \in F(t, x)\} \leq \varphi_\rho(t)$$

for all $x \in \mathbb{R}$ with $\|x\| \leq \rho$ and for a.e. $t \in [a, b]$.

Fixed point theorems play a major role in establishing the existence theory for the problem (1) and the problem (2). We collect here some well-known fixed point theorems used in this paper.

Theorem 2.8. *(Krasnoselskii's fixed point theorem [7]) Let $\mathcal{M}$ be a closed, bounded, convex, and non-empty subset of a Banach space. Let $\mathcal{A}, \mathcal{B}$ be the operators such that*

(i) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ *whenever* $x, y \in \mathcal{M}$,

(ii) $\mathcal{A}$ *is compact and continuous,*

(iii) $\mathcal{B}$ *is contraction mapping.*

Then there exists $z \in \mathcal{M}$ *such that* $z = \mathcal{A}z + \mathcal{B}z$.

Theorem 2.9. *(Leray–Schauder alternative [7])*

Let X be a Banach space, C a closed, convex subset of X, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \longrightarrow C$ is a continuous, compact ($F(\overline{U})$ s a relatively compact subset of C) map. Then either

(i) F *has a fixed point in* $\overline{U}$, *or*

(ii) there exists a $x \in \partial U$ *(the boundary of U in C) and* $\varepsilon \in (0, 1)$ *with* $x = \varepsilon F(x)$.

3. EXISTENCE AND UNIQUENESS RESULTS FOR PROBLEM (1)

In this section, we present the existence and uniqueness results to the problem (1).
To simplify the computations, we use the following notations:

$$\Phi_1 = \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\left[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\eta_i-a)^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\right], \tag{17}$$

and

$$\Phi_2 = |\mu|\left\{\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\eta_i-a)^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\right]\right\}. \tag{18}$$

In view of Lemma 2.6 we define the operator $\mathcal{K} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$(\mathcal{K}x)(t) = I^{\alpha_1+\alpha_2}f(t, x(t), x(\theta t)) - \mu I^{\alpha_2}x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2}f(b, x(b), x(\theta b)) - \mu I^{\alpha_2}x(b) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i}f(\eta_i, x(\eta_i), x(\theta\eta_i)) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x(\eta_i)\Bigg], \tag{19}$$

where $\mathcal{C} = C([a, b], \mathbb{R})$ denotes the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ with the norm $\|x\| := sup\{|x(t)|; t \in [a, b]\}$. It is obvious that the boundary value problem (1) has a solution if and only if the operator $\mathcal{K}$ has fixed point.

3.1. Existence result via Krasnoselskii's fixed point theorem

Our first result is an existence result, based on well-known Krasnoselskii's fixed point theorem.

Theorem 3.1. *Assume that:*

(H1). $f : [a,b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is a continuous function such that* $|f(t,x,y)| \leq \psi(t)$, $\forall (t,x,y) \in [a,b] \times \mathbb{R}^2$, *with* $\psi \in C([a,b];\mathbb{R}^+)$

(H2). $\Phi_2 < 1$, *where* Φ_2 *is given by* (18).

Then there exists at least one solution for the boundary value problem (1) *on* $[a,b]$.

Proof. Let $sup_{t\in[a,b]}|\psi(t)| = \|\psi\|$ and $\mathcal{B}_r = \{x \in \mathcal{C}; \|x\| \leq r\}$, where $r \geq \dfrac{\|\psi\|\Phi_1}{1-\Phi_2}$, we will show that the operator $\mathcal{K}$ defined by (19), satisfies the assumptions of Krasnoselskii's fixed point theorem, for that we split the operator $\mathcal{K}$ into the sum of two operators $\mathcal{K}_1$ and $\mathcal{K}_2$ defined, on the closed ball, by

$$\begin{aligned}(\mathcal{K}_1 x)(t) =& I^{\alpha_1+\alpha_2} f(t,x(t),x(\theta t)) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2} f(b,x(b),x(\theta b)) \\ & - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i} f(\eta_i, x(\eta_i), x(\theta\eta_i))\Bigg], \end{aligned} \tag{20}$$

and

$$(\mathcal{K}_2 x)(t) = -\mu I^{\alpha_2} x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[-\mu I^{\alpha_2} x(b) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i} x(\eta_i)\Bigg]. \tag{21}$$

For any $x, y \in \mathcal{B}_r$ we have

$$\begin{aligned}\big|(\mathcal{K}_1 x)(t) + (\mathcal{K}_2 y)(t)\big| \leq & I^{\alpha_1+\alpha_2}|f(t,x(t),x(\theta t))| + |\mu| I^{\alpha_2}|y(t)| + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\ & \times \Bigg[I^{\alpha_1+\alpha_2}|f(b,x(b),x(\theta b))| + \sum_{i=1}^{n}|\omega_i| I^{\alpha_1+\alpha_2+\sigma_i}|f(\eta_i,x(\eta_i),x(\theta\eta_i))| \\ & + |\mu| I^{\alpha_2}|y(b)| + |\mu| \sum_{i=1}^{n}|\omega_i| I^{\alpha_2+\sigma_i}|y(\eta_i)|\Bigg], \\ & \leq \|\psi\| \Bigg\{ \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \end{aligned}$$

$$
\begin{aligned}
&\times\left[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\sum_{i=1}^{n}|\omega_i|\frac{(\eta_i-a)^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\right]\Bigg\}\\
&+\|y\||\mu|\Bigg\{\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\\
&\times\left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\sum_{i=1}^{n}|\omega_i|\frac{(\eta_i-a)^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\right]\Bigg\},\\
&\le \|\psi\|\Phi_1+r\Phi_2,\\
&\le r,
\end{aligned}
$$

then $\|\mathcal{K}_1x+\mathcal{K}_2y\|\le r$, which implies that $\mathcal{K}_1x+\mathcal{K}_2y\in\mathcal{B}_r$.
Since f is continuous, then the operator $\mathcal{K}_1$ is continuous and it is uniformly bounded on $\mathcal{B}_r$ as :

$$
\|\mathcal{K}_1x\|\le\Phi_1\|\psi\| \tag{22}
$$

Next, we prove that the operator $\mathcal{K}_1$ is compact, for that setting $sup_{(t,x,y)\in[a,b]\times\mathcal{B}_r}|f(t,x,y)|=\overline{f}<\infty$, and let $t_1,t_2\in[a,b]$, $t_1<t_2$, we obtain

$$
\begin{aligned}
\Big|(\mathcal{K}_1x)(t_2)-(\mathcal{K}_1x)(t_1)\Big| &= \Big|I^{\alpha_1+\alpha_2}f(t_2,x(t_2),x(\theta t_2))+\frac{(t_2-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\\
&\times\left[I^{\alpha_1+\alpha_2}f(b,x(b),x(\theta b))-\sum_{i=1}^{n}\omega_iI^{\alpha_1+\alpha_2+\sigma_i}f(\eta_i,x(\eta_i),x(\theta\eta_i))\right]\\
&-I^{\alpha_1+\alpha_2}f(t_1,x(t_1),x(\theta t_1))-\frac{(t_1-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2}f(b,x(b),x(\theta b))
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{n}\omega_iI^{\alpha_1+\alpha_2+\sigma_i}f(\eta_i,x(\eta_i),x(\theta\eta_i))\Bigg]\Big|\\
&\quad\le\frac{\overline{f}}{\Gamma(\alpha_1+\alpha_2)}\Big|\int_a^{t_1}\Big((t_2-s)^{\alpha_1+\alpha_2-1}-(t_1-s)^{\alpha_1+\alpha_2-1}ds\Big)\\
&\quad+\int_{t_1}^{t_2}(t_2-s)^{\alpha_1+\alpha_2-1}ds\Big|+\frac{(t_2-a)^{\gamma_1+\alpha_2-1}-(t_1-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\\
&\quad\times\left[\overline{f}\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\overline{f}\sum_{i=1}^{n}|\omega_i|\frac{(\eta_i-a)^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\right],
\end{aligned}
$$

the right hand side tends to zero as $t_2-t_1\longrightarrow 0$; indepently of $x\in\mathcal{B}_r$. Then $\mathcal{K}_1$ is equicontinuous and hence $\mathcal{K}_1$ is relatively compact on $\mathcal{B}_r$. By the Arzelà–Ascoli theorem, it follows that $\mathcal{K}_1$ is compact on $\mathcal{B}_r$.

In the final step, we will show that $\mathcal{K}_2$ is a contraction mapping, for that let $x, y \in \mathcal{C}$, and for $t \in [a,b]$ we have

$$\begin{aligned}
\big|(\mathcal{K}_2x)(t) + (\mathcal{K}_2y)(t)\big| &\leq |\mu| I^{\alpha_2}|x(t) - y(t)| + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\
&\times \left[|\mu| I^{\alpha_2}|x(b) - y(b)| + |\mu| \sum_{i=1}^{n} |\omega_i| I^{\alpha_2+\nu_i}|x(\eta_i) - y(\eta_i)| \right], \\
&\leq \|x-y\||\mu| \left\{ \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \right. \\
&\times \left. \left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\eta_i - a)^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)} \right] \right\}, \\
&\leq \Phi_2 \|x-y\|.
\end{aligned}$$

This shows that $\|\mathcal{K}_2x + \mathcal{K}_2y\| \leq \Phi_2\|x-y\|$, then by using $(H2)$, we deduce that $\mathcal{K}_2$ is a contraction mapping. It follows by Krasnoselskii's fixed point theorem, that the problem (1) has at least one solution on $[a,b]$. □

3.2. Uniqueness result via Banach's fixed point theorem

To deal with the uniqueness of solution for our problem (1), we use Banach's fixed point theorem.

Theorem 3.2. *Assume that* $|f(t,x,y) - f(t,z,w)| \leq L\big(|x-z| + |y-w|\big)$ *;* $L > 0$*, for each* $t \in [a,b]$ *and* $x, y, z, w \in \mathbb{R}$.
If $2L\Phi_1 + \Phi_2 < 1$*, where* Φ_1, Φ_2 *are respectively given by* (17) *and* (18)*, then the problem* (1) *has a unique solution on* $[a,b]$.

Proof. Consider the operator $\mathcal{K}$ defined in (19). The problem (1) is then transformed into a fixed point problem $x = \mathcal{K}x$. By using Banach contraction principle we will show that $\mathcal{K}$ has a unique fixed point.
We set $sup_{t\in[a,b]} = |f(t,0,0)| = M < \infty$, and choose $\rho > 0$ such that

$$\rho \geq \frac{M\Phi_1}{1 - 2L\Phi_1 - \Phi_2}, \tag{23}$$

$\mathcal{B}_\rho = \{x \in \mathcal{C}([a,b],\mathbb{R}); \|x\| \leq \rho)\}$, where Φ_1, Φ_2 are respectively given by (17) and (18).
Step 1: We show that $\mathcal{K}\mathcal{B}_\rho \subset \mathcal{B}_\rho$.

For any $x \in \mathcal{B}_\rho$ we have

$$\begin{aligned}|f(t,x(t),x(\theta t))| &\leq |f(t,x(t),x(\theta t)) - f(t,0,0)| + |f(t,0,0)| \\ &\leq L\big(|x(t)| + |x(\theta t)|\big) + M \\ &\leq 2L\|x\| + M,\end{aligned}$$

then we have

$$\begin{aligned}\big|(\mathcal{K}x)(t)\big| &\leq I^{\alpha_1+\alpha_2}|f(t,x(t),x(\theta t))| + |\mu| I^{\alpha_2}|x(t)| + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\ &\times \Bigg[I^{\alpha_1+\alpha_2}|f(b,x(b),x(\theta b))| + \sum_{i=1}^{n} |\omega_i| I^{\alpha_1+\alpha_2+\sigma_i}|f(\eta_i,x(\eta_i),x(\theta\eta_i))| \\ &+ |\mu| I^{\alpha_2}|x(b)| + |\mu| \sum_{i=1}^{n} |\omega_i| I^{\alpha_2+\sigma_i}|x(\eta_i)| \Bigg], \\ &\leq \Big(2L\|x\| + M\Big)\Bigg\{ \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\ &\times \Bigg[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\eta_i-a)^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)} \Bigg] \Bigg\} \\ &+ \|x\||\mu| \Bigg\{ \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\ &\times \Bigg[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\eta_i-a)^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)} \Bigg] \Bigg\}, \\ &\leq \Big(2L\|x\| + M\Big)\Phi_1 + \|x\|\Phi_2, \\ &\leq \Big(2L\rho + M\Big)\Phi_1 + \rho\Phi_2, \\ &\leq \rho,\end{aligned}$$

which implies that $\mathcal{K}\mathcal{B}_\rho \subset \mathcal{B}_\rho$.

Step 2: We show that the operator $\mathcal{K}$ is a contraction.

For any $x, y \in \mathcal{C}$, and for $t \in [a,b]$, we have

$$\begin{aligned}\big|(\mathcal{K}x)(t) - (\mathcal{K}y)(t)\big| \leq &\Bigg\{ \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\ &\times \Bigg[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\eta_i-a)^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)} \Bigg] \Bigg\} 2L\|x-y\|\end{aligned}$$

$$
\begin{aligned}
&+|\mu|\Bigg\{\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\\
&\times\left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\sum_{i=1}^{n}|\omega_i|\frac{(\eta_i-a)^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\right]\Bigg\}\|x-y\|\\
&\le\Big(2L\Phi_1+\Phi_2\Big)\|x-y\|,
\end{aligned}
$$

which implies, $\|(\mathcal{K}x)(t)-(\mathcal{K}y)(t)\|\le\Big(2L\Phi_1+\Phi_2\Big)\|x-y\|$. As $2L\Phi_1+\Phi_2<1$, then $\mathcal{K}$ is a contraction. Therefore, by Banach's fixed-point theorem, then the operator $\mathcal{K}$ has a unique fixed point which is indeed the unique solution of our problem (1). □

4. EXISTENCE RESULTS FOR THE MULTI-VALUED VERSION (2)

In this part, we will deal with the existence result for the inclusion version defined as problem (2)

Definition 4.1. A continuous function x is said to be a solution of problem (2) if $x(a)=0; x(b)=\sum_{i=1}^{n}\omega_i(I^{\sigma_i}(x))(\eta_i)$ and there exists a function $v\in\mathbb{L}^1([a,b],\mathbb{R})$ with $v\in F(t,x(t),x(\theta t))$ a.e, on $[a,b]$ such that

$$
\begin{aligned}
x(t)=I^{\alpha_1+\alpha_2}v(t)-\mu I^{\alpha_2}x(t)+\frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[&I^{\alpha_1+\alpha_2}v(b)\\
-\mu I^{\alpha_2}x(b)-\sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i}v(\eta_i)+\mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x(\eta_i)&\Bigg],
\end{aligned}
\tag{24}
$$

for each $x\in\mathcal{C}([a,b],\mathbb{R}$, define the set of selections of F by

$$
\mathcal{S}_{F,x}:=\big\{v\in\mathbb{L}^1([a,b],\mathbb{R}):v\in F(t,x(t),x(\theta t))\quad on\quad [a,b]\big\}
\tag{25}
$$

Lemma 4.2. *([12]) Let X be a Banach space, and $F:[a,b]\times\mathbb{R}^2\longrightarrow\mathcal{P}_{c,cp}$ be a $\mathbb{L}^1$-Carathéodory multivalued map. And let Υ be a linear continuous mapping from $L^1([a,b],X)$ to $\mathcal{C}([a,b],X)$. Then the operator :*

$$
\Upsilon\circ\mathcal{S}_{F}:\mathcal{C}([a,b],X)\longrightarrow\mathcal{P}_{c,cp}\big(\mathcal{C}([a,b],X)\big);\quad x\longrightarrow\big(\Upsilon\circ\mathcal{S}_{F}\big)(x)=\Upsilon\big(\mathcal{S}_{F,x}\big),
$$

is a closed graph operator in $\mathcal{C}([a,b],X)\times\mathcal{C}([a,b],X)$.

Our existence result, is based on the nonlinear alternative of the Leray-Schauder for multivalued maps [7]

Theorem 4.3. *Suppose that $(H2)$, and the following assumptions hold :*

(H3). $F : [a,b] \times \mathbb{R}^2 \longrightarrow \mathcal{P}_{c,cp}(\mathbb{R})$ *is $\mathbb{L}^1$-Carathéodory and has nonempty compact and convex values, and for each fixed $x \in \mathcal{C}([a,b],\mathbb{R})$ the set :*

$$\mathcal{S}_{F,x} = \{v \in \mathbb{L}^1([a,b],X) : v(t) \in F(t,x(t),x(\theta t)); \quad t \in [a,b]\}$$

is convex and nonempty.

(H4). $|F(t,x,y)| := sup\{|v| : v \in F(t,x,y)\} \leq p(t)\Psi(|x|+|y|)\}$ *for all $t \in [a,b]$ and all $x,y \in \mathcal{C}([a,b],X)$, where $p \in \mathbb{L}^1([a,b],\mathbb{R}^+)$ and $\Psi : \mathbb{R}^+ \longrightarrow [0,+\infty)$ is continuous and nondecreasing function.*

(H5). *there exists a constant $M > 0$ such that :*

$$\frac{(1-\Phi_2)M}{\|p\|\Psi(2M)\Phi_1} > 1,$$

where Φ_1, Φ_2 are respectively given by (17) *and* (18).

Then, there exists at least one solution for problem (2) *on* $[a,b]$.

Proof. Let us introduce the operator $\mathcal{K} : \mathcal{C}([a,b],\mathbb{R}) \longrightarrow \mathcal{P}_{c,cp}([a,b],\mathbb{R})$, in order to transform the problem (2) into a fixed point problem:
$\mathcal{K}(x) :=$

$$\left\{ k \in \mathcal{C}([a,b],\mathbb{R}) : k(t) = \begin{cases} I^{\alpha_1+\alpha_2}v(t) - \mu I^{\alpha_2}x(t) + \dfrac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\ \times \Bigg[I^{\alpha_1+\alpha_2}v(b) - \mu I^{\alpha_2}x(b) - \displaystyle\sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i}v(\eta_i) \\ +\mu \displaystyle\sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i}x(\eta_i) \Bigg]; \quad t \in [a,b], \quad v \in \mathcal{S}_{F,x} \end{cases} \right\}$$

We will show that the operator $\mathcal{K}$ satisfies all conditions of the Leray-Schauder nonlinear alternative [7], for the poof we give it in steps:
Step 1 : $\mathcal{K}(x)$ is convex for each $x \in \mathcal{C}([a,b],\mathbb{R})$.
Indeed, if k_1, k_2 belong to $\mathcal{K}(x)$, then there exist $v_1, v_2 \in \mathcal{S}_{F,x}$ such that for each $t \in [a,b]$ we have:

$$\begin{aligned} k_j(t) = I^{\alpha_1+\alpha_2}v_j(t) - \mu I^{\alpha_2}x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[I^{\alpha_1+\alpha_2}v_j(b) \\ - \mu I^{\alpha_2}x(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i}v_j(\eta_i) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i}x(\eta_i) \Bigg], \end{aligned} \tag{26}$$

For $j = 1, 2$. Let $0 \leq \lambda \leq 1$ then, for each $t \in [a, b]$, we have

$$\lambda k_1(t) + (1-\lambda)k_2(t) = I^{\alpha_1+\alpha_2}\Big[\lambda v_1(s) + (1-\lambda)v_2(s)\Big] - \mu I^{\alpha_2}x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}$$
$$\times \Bigg[I^{\alpha_1+\alpha_2}\Big[\lambda v_1(s) + (1-\lambda)v_2(s)\Big] - \mu I^{\alpha_2}x(b)$$
$$- \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i}\Big[\lambda v_1(\eta_i) + (1-\lambda)v_2(\eta_i)\Big] + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x(\eta_i)\Bigg],$$

Thus $\lambda v_1 + (1-\lambda)v_2 \in \mathcal{K}(x)$ $\big($ because $\mathcal{S}_{F,x}$ is convex $\big)$, then $\mathcal{K}(x)$ is convex for each $x \in \mathcal{C}([a, b], \mathbb{R})$

Step 2 : $\mathcal{K}(x)$ maps bounded set into bounded set in $\mathcal{C}([a, b], \mathbb{R})$.
Indeed, it is enough to show that there exists a positive constant l such that for each $k \in \mathcal{K}(x)$; $x \in \mathcal{B}_\rho = \{x \in \mathcal{C}([a, b], \mathbb{R} : \|x\| \leq \rho\}$ we have $\|k\| \leq l$.
If $k \in \mathcal{K}(x)$ then there exist $v \in \mathcal{S}_{F,x}$, such that :

$$k(t) = I^{\alpha_1+\alpha_2}v(t) - \mu I^{\alpha_2}x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2}v(b)$$
$$- \mu I^{\alpha_2}x(b) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i}v(\eta_i) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x(\eta_i)\Bigg], \tag{27}$$

then for every $t \in [a, b]$ we have

$$\big|k(x)(t)\big| \leq \sup_{t\in[a,b]}\Bigg\{I^{\alpha_1+\alpha_2}|v(t)| + |\mu|I^{\alpha_2}|x(t)| + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}$$
$$\times\Bigg[I^{\alpha_1+\alpha_2}|v(b)| + \sum_{i=1}^{n}|\omega_i|I^{\alpha_1+\alpha_2+\sigma_i}|v(\eta_i)|$$
$$+ |\mu|I^{\alpha_2}|x(b)| + |\mu|\sum_{i=1}^{n}|\omega_i|I^{\alpha_2+\sigma_i}|x(\eta_i)|\Bigg]\Bigg\}$$
$$\leq \|p\|\Psi(2\|x\|)\Bigg\{\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}$$
$$\times\Bigg[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\eta_i-a)^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\Bigg]\Bigg\}$$
$$+ \|x\||\mu|\Bigg\{\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}$$
$$\times\Bigg[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\eta_i-a)^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\Bigg]\Bigg\}$$
$$\leq \|p\|\Psi(2\|x\|)\Phi_1 + \|x\|\Phi_2 \leq \|p\|\Psi(\rho)\Phi_1 + \rho\Phi_2,$$

Then

$$\|k\| \le \|p\|\Psi(\rho)\Phi_1 + \rho\Phi_2 := l,$$

where Φ_1, Φ_2 are respectively given by (17) and (18).

Step 3 : $\mathcal{K}$ maps bounded set into equicontinuous sets of $\mathcal{C}([a,b],\mathbb{R})$.

Let $t_1, t_2 \in [a,b]; t_1 < t_2$, and $x \in \mathcal{B}_\rho$ where $\mathcal{B}_\rho$, as above, is a bounded set of $\mathcal{C}([a,b],\mathbb{R})$; for each $x \in \mathcal{B}_\rho$ and $k \in \mathcal{K}(x)$, there exist $v \in \mathcal{S}_{F,x}$ then we obtain :

$$\begin{aligned}
\Big|k(t_2)-k(t_1)\Big| \le{}& \frac{1}{\Gamma(\alpha_1+\alpha_2)}\Big|\int_a^{t_1}\Big((t_2-s)^{\alpha_1+\alpha_2-1}-(t_1-s)^{\alpha_1+\alpha_2-1}\Big)v(s)ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha_1+\alpha_2-1}v(s)ds\Big|\\
&+\frac{|\mu|}{\Gamma(\alpha_2)}\Big|\int_a^{t_1}\Big((t_2-s)^{\alpha_2-1}-(t_1-s)^{\alpha_2-1}\Big)x(s)ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}x(s)ds\Big| + \frac{(t_2-a)^{\gamma_1+\alpha_2-1}-(t_1-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\\
&\times\Bigg[\|v(s)\|\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\|v(s)\|\sum_{i=1}^n|\omega_i|\frac{(\eta_i-a)^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\\
&+\|x(b)\||\mu|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\|x(\eta_i)\||\mu|\sum_{i=1}^n|\omega_i|\frac{(\eta_i-a)^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\Bigg],
\end{aligned}$$

$$\begin{aligned}
\le{}&\frac{\|p\|\Psi(2\rho)}{\Gamma(\alpha_1+\alpha_2)}\Big|\int_a^{t_1}\Big((t_2-s)^{\alpha_1+\alpha_2-1}-(t_1-s)^{\alpha_1+\alpha_2-1}\Big)ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha_1+\alpha_2-1}ds\Big|\\
&+\frac{\rho|\mu|}{\Gamma(\alpha_2)}\Big|\int_a^{t_1}\Big((t_2-s)^{\alpha_2-1}-(t_1-s)^{\alpha_2-1}\Big)ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}ds\Big| + \frac{(t_2-a)^{\gamma_1+\alpha_2-1}-(t_1-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)}\\
&\times\Bigg[\|p\|\Psi(2\rho)\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\|p\|\Psi(2\rho)\sum_{i=1}^n|\omega_i|\frac{(\eta-a)^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\\
&+\rho|\mu|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\rho|\mu|\sum_{i=1}^n|\omega_i|\frac{(\eta_i-a)^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\Bigg],
\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero, implies that $\mathcal{A}(x)$ is equicontinuous. Therefore it follows by Arzela-Ascoli theorem that

$\mathcal{K} : \mathcal{C}([a,b],\mathbb{R}) \longrightarrow \mathcal{P}_{c,cp}(\mathcal{C}([a,b],\mathbb{R}))$ is relatively compact then $\mathcal{K}$ is completely continuous.

Now, we show that the operator $\mathcal{K}$ is upper semicontinuous. To prove this, is enough to show that $\mathcal{K}$ has a closed graph.

Step 4 : $\mathcal{K}$ has a closed graph.

Let $x_n \longrightarrow x_*$, $k_n \in \mathcal{K}(x_n)$ and $k_n \longrightarrow k_*$, we shall prove that $k_* \in \mathcal{A}(x_*)$.

$k_n \in \|(x_n)$ then there exists $v_n \in \mathcal{S}_{F,x_n}$ such that for each $t \in [a,b]$,

$$k_n(t) = I^{\alpha_1+\alpha_2}v_n(t) - \mu I^{\alpha_2}x_n(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2}v_n(b) - \mu I^{\alpha_2}x_n(b) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i}v_n(\eta_i) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x_n(\eta_i)\Bigg], \tag{28}$$

We should prove that $v_* \in \mathcal{S}_{F,x_*}$ such that for each $t \in [a,b]$:

$$k_*(t) = I^{\alpha_1+\alpha_2}v_*(t) - \mu I^{\alpha_2}x_*(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2}v_*(b) - \mu I^{\alpha_2}x_*(b) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\nu_i}v_*(\eta_i) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x_*(\eta_i)\Bigg], \tag{29}$$

we have that:

$$\Bigg\|\Bigg(k_n(t) + \lambda I^{\alpha_2}x_n(t) - \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[-\mu I^{\alpha_2}x_n(b) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x_n(\eta_i)\Bigg]\Bigg) - \Bigg(k_*(t) + \mu I^{\alpha_2}x_*(t) - \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[-\lambda I^{\alpha_2}x_*(b) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x_*(\eta_i)\Bigg]\Bigg)\Bigg\| \longrightarrow 0,$$

as $n \longrightarrow \infty$ Consider the linear operator :

$$\Upsilon : \mathbb{L}^1([a,b],\mathbb{R}) \longrightarrow \mathcal{C}([a,b],\mathbb{R})$$

$$v \longrightarrow \Upsilon(v)(t).$$

With

$$\Upsilon(v)(t) = I^{\alpha_1+\alpha_2}v(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[I^{\alpha_1+\alpha_2}v(b) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i}v(\eta_i)\Bigg],$$

From lemma 4.2, $\Upsilon \circ \mathcal{S}_F$ is a closed graph operator then we have that :

$$\Bigg(k_n(t)+\mu I^{\alpha_2}x_n(t)-\frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\Bigg[-\mu I^{\alpha_2}x_n(b)+\mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x_n(\eta_i)\Bigg]\Bigg) \in \Upsilon\Big(\mathcal{S}_{F,x_n}\Big). \tag{30}$$

Since $x_n \longrightarrow x_*$, and $k_n \longrightarrow k_*$ then :

$$\left(k_*(t) + \mu I^{\alpha_2} x_*(t) - \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\left[-\mu I^{\alpha_2} x_*(b) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i} x_*(\eta_i)\right]\right) = \Upsilon(v_*) \in \Upsilon\Big(\mathcal{S}_{F,x_*}\Big). \tag{31}$$

It follows that $v_* \in \mathcal{S}_{F,x_*}$ such that

$$\begin{aligned} k_*(t) = I^{\alpha_1+\alpha_2} v_*(t) - \mu I^{\alpha_2} x_*(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \Bigg[I^{\alpha_1+\alpha_2} v_*(b) \\ - \mu I^{\alpha_2} x_*(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i} v_*(\eta_i) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i} x_*(\eta_i)\Bigg], \end{aligned} \tag{32}$$

Step 5: $\mathcal{K}$ has a fixed point in $\mathcal{B}_\rho$.
We show that (ii) from theorem 2.9 is not possible. Then if $x \in \varepsilon\mathcal{K}(x)$ for $\varepsilon \in]0,1[$ there exist $v \in \mathcal{S}_{F,x}$ such that $x(t) = \varepsilon k(t)$ implies $|x(t)| \leq |k(t)|$ then

$$\|x\| \leq \|p\|\Psi(2\|x\|)\Phi_1 + \|x\|\Phi_2,$$

then

$$(1-\Phi_2)\|x\| \leq \|p\|\Psi(2\|x\|)\Phi_1, \tag{33}$$

if (ii) from theorem 2.9 hold then there exist $\varepsilon \in]0,1[$ and $x \in \partial\mathcal{B}_M$ with $x = \varepsilon k(x)$ wich means that x is solution to (2) with $\|x\| = M$ then we have from (32) that :

$$(1-\Phi_2)M \leq \|p\|\Psi(2M)\Phi_1, \tag{34}$$

then

$$\frac{(1-\Phi_2)M}{\|p\|\Psi(2M)\Phi_1} \leq 1, \tag{35}$$

which contredicts $(H5)$. Consequently $\mathcal{K}$ has fixed point in $[a,b]$.
By the nonlinear alternative of Leray–Schauder we deduce that our problem (2) has at least one solution. □

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