International Journal of Difference Equations (IJDE).
ISSN 0973-6069 Volume 18, Number 1 (2023), pp. 119-143

https://www.ripublication.com/ijde.htm

On Strongly Quasilinear Elliptic Systems with Variable Exponent and Nonlinear Physical Data

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Abstract

In this paper, we study the existence results for a class of quasilinear elliptic system with homogeneous Dirichlet boundary conditions. Under regularity, growth, and coercivity conditions on the operator a, we prove the existence of weak solutions in a suitable variable exponent Sobolev spaces by using Galerkin's approximation and the theory of Young measure.
Keywords: Quasilinear elliptic systems, weak solutions, Young measures, Galerkin's approximation.

1. INTRODUCTION AND MAIN RESULTS

The main goal of this work is to establish the existence of weak solutions for the following quasilinear elliptic system $(QES)_{f,h}$:

$$\begin{cases} -div\ a(x,u,Du) + |u|^{p(x)-2}u + b(x,u,Du) = v(x) + f(x,u) + div\ h(x,u) & in\ \ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1}$$

which is a Dirichlet problem where v belongs to the dual space $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. Let Ω denote a bounded open domain in $\mathbb{R}^m$ and $\mathbb{M}^{m\times n}$ denote the set of real m by n matrices equipped with the usual inner

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product $M : N = \sum_{i,j} M_{ij}N_{ij}$(with the usual summation convention).

Endowed with the Dirichlet boundary condition, the following quasilinear elliptic problem was studied in [17] by Hungerbühler

$$-div\ \sigma(x, u, Du) = f \quad \text{in } \Omega, \tag{2}$$

The author used the tool of Young measures and weak monotonicity over σ to achieve his result. We find a generalization of (2) in [2], where the following quasilinear elliptic system

$$-div\ \sigma(x, u, Du) = v(x) + f(x, u) + div\ h(x, u) \quad \text{in } \Omega \tag{3}$$

was considered. This system corresponds to a diffusion problem with a source v in a moving and dissolving substance, where the motion is described by g and the dissolution by f. The authors proved existence of a weak solution for this system under classical regularity, growth, and coercivity conditions for σ, but with only very mild monotonicity assumptions. For more results we address the reader to the monographs [3, 5].

Inspired by the previous works (especially [2, 4, 11, 16, 18, 21]), we will study the existence result for the problem (1) in the framework of variable exponent Sobolev spaces $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$. This work, can be seen as an extension of [2] (i.e. of (3)). We will use the Young measure as a technical tool combined with the theory of variable exponent Sobolev spaces to obtain the desired result.

The paper is organized as follows: In Section 2, we specify the assumptions on σ, f, b and g needed in the present study and introduce the definition of a weak solution of (1). We present in Section 3 an overview on Young measure, section 4 contains the general convergence result for functions a, while Section 5 is devoted to present the main result and its proof.

2. ASSUMPTIONS ON THE DATA AND THE DEFINITION OF A WEAK SOLUTION

Throughout this paper, we suppose that the following assumptions hold true on a and b for some $p(x) \in (1,\ \infty)$:

(H$_0$) (Continuity)

a : $\Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e. $x \mapsto a(x, u, F)$ is measurable for every $(u, F) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(u, F) \mapsto a(x, u, F)$ is continuous for almost every $x \in \Omega$.

($\mathbf{H}_1$) (Growth and coercivity)
There exist $c_1 \geq 0,\ c_2 > 0,\ \lambda_1 \in L^{p'(x)}(\Omega),\ \lambda_2 \in L^1(\Omega)\ ,\quad 0 < \alpha(x) < p^-,\ \lambda_3 \in L^{(p(x)/\alpha(x))'}(\Omega)$ and $\ 0 < \beta(x) \leq \dfrac{n}{n-p(x)}(p(x)-1)$ such that

$$|a(x,u,F)| \leq \lambda_1(x) + c_1(|u|^{\beta(x)} + |F|^{p(x)-1}),$$

$$a(x,u,F)\ :\ F \geq -\lambda_2(x) - \lambda_3(x)|u|^{\alpha(x)} + c_2|F|^{p(x)}.$$

($\mathbf{H}_2$) (Monotonicity) a satisfies one of the following conditions:

(a) For all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $F \mapsto a(x,u,F)$ is a C^1-function and monotone, i.e.

$$(a(x,u,F) - a(x,u,G))\ :\ (F-G) \geq 0,$$

for all $x \in \Omega,\ u \in \mathbb{R}^m$ and $F,\ G \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $a(x,u,F) = \dfrac{\partial W}{\partial F}(x,u,F)$, and $F \mapsto W(x,u,F)$ is convex and C^1 for all $x \in \Omega$ and all $u \in \mathbb{R}^m$.

(c) For all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $F \mapsto a(x,u,F)$ is strictly monotone

(d) a is strictly $p(x)$-quasimonotone in F, i.e.,

$$\int_{\mathbb{M}^{m\times n}} (a(x,\ u,\ \lambda) - a(x,\ u, \overline{\lambda})); (\lambda - \overline{\lambda}) dv(\lambda) > 0,$$

where $\overline{\lambda} = \langle v_X, \text{ id } \rangle,\ v = \{v_X\}_\chi \in \Omega$ is any family of Young measures generated by a sequence in $L^p(\Omega)$ and not a Dirac measure for a.e. $x \in \Omega$.

(e) For a.e. $x \in \Omega$ and all $s \in \mathbb{R}^m$, the map $F \to a(x,\ s,\ F)$ is strictly quasimonotone, i.e., there exists a constant $\gamma > 0$ such that

$$\int_\Omega (a(x,\ s,\ Ds) - a(x,\ s,\ Dv)) : (Ds - Dv)dx \geq \gamma \int_\Omega |Ds - Dv|^{p(x)} dx.$$

($\mathcal{B}$) $b : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ is a Carathéodory function in the sense of ($\mathbf{H}_0$). Moreover, we assume that one of the following conditions holds:

(i) There exist $c > 0$ and a function $d_3(x) \in L^{p'(x)}(\Omega)$ such that

$$|b(x,s,F)| \leq d_3(x) + c(|s|^{p(x)-1} + |F|^{p(x)-1})$$

for a.e. $x \in \Omega$ and all $(s,F) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$.

(ii) The function b is independent of the third variable or, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^m$, the mapping $F \mapsto b(x,s,F)$ is linear.

And f, h satisfy the following continuity and growth conditions:

($\mathcal{F}_0$) (Continuity) $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function, i.e. $x \mapsto f(x,u)$ is measurable for every $u \in \mathbb{R}^m$ and $u \mapsto f(x,u)$ is continuous for almost every $x \in \Omega$.

($\mathcal{F}_1$) (Growth) There exist $0 < \gamma(x) < p(x) - 1,\ b_1 \in L^{p'(x)}(\Omega)$ and $b_2 \in L^{\frac{n}{p(x)}}(\Omega)$ such that

$$|f(x,u)| \leq b_1(x) + b_2(x)|u|^{\gamma(x)}.$$

(**G**$_0$) (Continuity) $g : \Omega \times \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a Carathéodory function.

(**G**$_1$) (Growth) There exist $0 < \eta < p(x) - 1,\ b_4 \in L^{p'(x)}(\Omega)$ and $b_5 \in L^{\frac{n}{p(x)-1}}(\Omega)$ such that

$$|h(x,u)| \leq b_4(x) + b_5(x)|u|^{q(x)}.$$

Condition $(\mathbf{H}_0)$ takes certain that $a(x,u(x),U(x))$ is measurable on Ω for measurable functions $u : \Omega \to \mathbb{R}^m$ and $U : \Omega \to \mathbb{M}^{m\times n}$; see e.g. [24, Appendix ”Measurable functions” page 1013].

Condition $(\mathbf{H}_1)$ describes the standard growth and coercivity conditions. The essential point is that we do not require strict monotonicity of a typical Leray-Lions operator [20] or monotonicity in the variables (u,F) in $(\mathbf{H}_2)$ as it is usually assumed in previous works. As a result, the classic monotone operator approaches [7,9,20,24,25] proposed by Visik, Minty, Browder, Brézis, Lions, and others do not apply in general for functions satisfying only $(\mathbf{H}_0)$-$(\mathbf{H}_2)$ conditions. For example, the assumption $(\mathbf{H}_2)$ permits us to consider the elliptic problem (QES) with $a(x,u,F) = \dfrac{\partial W}{\partial F}(x,u,F)$ and take a potential $W(x,u,F)$, which is only convex but not strictly convex in F. The issue is that where W is not strictly convex, the gradients of approximation solutions do not have to converge pointwise. The notion here is that W is not absolutely convex, it is locally affine, so passage to the limit should still be achievable locally. Technically, the Young measure formed by the sequence of gradients of approximation solutions can be used to do this. The assumption (d) in $(\mathbf{H}_2)$ is motivated by the study of nonlinear elastostatics by Ball. For non-hyperelastic materials the static equation is not given by a potential map. Subsequently quasimonotone systems have been studied by Zhang and Chabrowski [10] who investigated the existence of solutions for perturbed systems.

However, a slightly different notion of quasimonotonicity is used in the mentioned papers. The regularity problems for such systems were studied by Fuchs [15].

Conditions ($\mathcal{F}_0$) and ($\mathbf{G}_0$) ensure that $f(x, u(x))$ and $h(x, u(x))$ are measurable on Ω for any measurable function $u : \Omega \to \mathbb{R}^m$.
(F_1) and ($\mathbf{G}_1$) state standard growth conditions. In particular, if $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ then $f(\cdot, u(\cdot)) \cdot u(\cdot)$ and $h(\cdot, u(\cdot)) : Du(\cdot)$ belong to $L^1(\Omega)$.

For (1), the notation $(QES)_{f,h}$ should aid the reader and lighten the text. As a result, the subscripts f and g indicate the nature of the right side. When $h = 0$ in particular, the system is denoted as $(QES)_f$. It's worth noting that the $(QES)_{f,h}$ system is more general than the $(QES)_f$ system. Because no condition on the derivatives of h or on the monotonicity of f and h is applied, the term $div\ h$ cannot be absorbed in $div\ a$ or f. We shall prove the existence of a weak solution for the system(1) by adapting the methods employed in [17]:

Definition 1. *A measurable function* $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ *is called a weak solution to problem* (1) *if*

$$\int_\Omega a(x, u(x), Du(x)) : D\varphi(x)dx + \int_\Omega |u(x)|^{p(x)-2}u(x) \cdot \varphi(x)dx + \int_\Omega b(x,\ u(x),\ Du(x)) \cdot \varphi(x)dx$$
$$= \langle v,\ \varphi\rangle + \int_\Omega f(x,\ u(x)).\varphi(x)dx + \int_\Omega h(x,\ u(x)) : D\varphi(x)dx$$

holds for every function $\varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$.

Let us recall the definition of the Young measure and some of its properties that will be needed in the sequel of this paper.

3. PRELIMINARIES

We recall some necessary notations, definitions and properties for our function spaces and an overview about Young measures. For each open bounded subset Ω of $\mathbb{R}^n (n \geq 2)$, we denote $C_+(\bar{\Omega}) = \{p(x) \in C(\bar{\Omega}), p(x) > 1 \text{ for any } x \in \Omega\}$. We define for every $p \in C^+(\bar{\Omega})$,

$$p^- = \inf_{x\in\Omega} p(x) \text{ and } p^+ = \sup_{x\in\Omega} p(x).$$

The Sobolev space $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ consists of all functions u in the Lebesgue space

$$L^{p(x)}(\Omega; \mathbb{R}^m) = \left\{ u : \Omega \to \mathbb{R}^m \text{ measurable} : \int_\Omega |u(x)|^{p(x)}dx < \infty \right\},$$

such that $Du \in L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$. The space $L^{p(x)}(\Omega; \mathbb{R}^m)$ endowed with the norm

$$\|u\|_{p(x)} = \inf\left\{\beta > 0, \int_\Omega \left|\frac{u(x)}{\beta}\right|^{p(x)} dx \le 1\right\},$$

is a Banach space. Moreover, it is reflexive if and only if $1 < p^- \le p^+ < \infty$. Its dual is defined by $L^{p'(x)}(\Omega; \mathbb{R}^m)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega; \mathbb{R}^m)$ and $v \in L^{p'(x)}(\Omega; \mathbb{R}^m)$, the generalized Hölder inequality

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p^+}\right) \|u\|_{p(x)}\|v\|_{p'(x)} \le 2\|u\|_{p(x)}\|v\|_{p'(x)}$$

holds true. The space $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ is endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|Du\|_{p(x)}.$$

Proposition 1. *([16]) . We denote $\rho(u) = \int_\Omega |u|^{p(x)}dx, \forall u \in L^{p(x)}(\Omega; \mathbb{R}^m)$. If $u_k, u \in L^{p(x)}(\Omega; \mathbb{R}^m)$ and $p^+ < \infty$, then:*

i) $\|u\|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$.

ii) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \le \rho(u) \le \|u\|_{p(x)}^{p^+}; \|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \le \rho(u) \le \|u\|_{p(x)}^{p^-}$.

iii) $\|u_k\|_{p(x)} \to 0 \Leftrightarrow \rho(u_k) \to 0; \|u_k\|_{p(x)} \to +\infty \Leftrightarrow \rho(u_k) \to +\infty$.

We denote by $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ the closure of $C_0^\infty(\Omega; \mathbb{R}^m)$ in $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ and $W^{-1,p'(x)}(\Omega; \mathbb{R}^m)$ is its dual space. We denote $p^*(x) = \frac{np(x)}{n-p(x)}$ for $p(x) < n; = \infty$ for $p(x) > n$

Proposition 2. *([16])*

i) *Under the assumption $1 < p^-$, the spaces $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ are separable and reflexive Banach spaces.*

ii) *If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \bar{\Omega}$, then $W^{1,p(x)}(\Omega; \mathbb{R}^m) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega; \mathbb{R}^m)$ is compact and continuous. In particular, we have $W_0^{1,p(x)}(\Omega; \mathbb{R}^m) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega; \mathbb{R}^m)$ is compact and continuous.*

iii) *There exists a constant $c_3 > 0$, such that*

$$\|u\|_{p(x)} \le c_3\|Du\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m),$$

hence $\|Du\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are two equivalent norms on $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$.

4. A REVIEW OF YOUNG MEASURES

The main theorem we will advocate to solve nonlinear PDEs systems is the following result due to Ball:

Theorem 1. *([6]) (Ball).Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $u_j : \Omega \to \mathbb{R}^m$, $j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $u_j \to K$ in measure as$j \to \infty$, $i.e.$ given any open neighborhood U of K in $\mathbb{R}^m$*

$$\lim_{j\to\infty} |\{x \in \Omega : u_j(x) \notin U\}| = 0. \tag{4}$$

Then there exists a subsequence (u_k) of (u_j) and a family (ν_x) , $x \in \Omega$, of positive measures on $\mathbb{R}^m$, depending measurably on x, such that

(i) $\|\nu_x\|_{\text{meas}} \equiv \int_{\mathbb{R}^m} d\nu_x \leq 1$ *for* $a.e.\ x \in \Omega$,

(ii) $\text{supp}\nu_x \subset K$ *for* $a.e.\ x \in \Omega$, *and*

(iii) $f(u_k) \to *\langle \nu_x,\ f\rangle = \int_{\mathbb{R}^m} f(\lambda)d\nu_x(\lambda)$ *in* $L^\infty(\Omega)$ *for each continuous function* $f : \mathbb{R}^m \to \mathbb{R}$ *satisfying* $\lim_{|\lambda|\to\infty} f(\lambda) = 0$.

Suppose further that $\{u_k\}$ satisfies the boundedness condition

$$\forall R > 0 : \lim_{L\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_R : |u_k(x)| \geq L\}| = 0, \tag{5}$$

where $B_R = B_R(0)$. Then

$$\|\nu_x\|_{\text{meas}} = 1 \quad for\ a.e.\ x \in \Omega \tag{6}$$

*(*i.e. *ν_x is a probability measure), and there holds:*

$$For\ any\ measurable\ A \subset \Omega\ and\ any\ continuous\ function\ f : \mathbb{R}^m \to \mathbb{R}$$

$$such\ that\ \{f(u_k)\}\ is\ sequentially\ weakly\ relatively\ compact\ in\ L^1(A) \tag{7}$$

$$we\ have\ f(u_k) \to \langle \nu_x,\ f\rangle\ in\quad L^1(A).$$

Improved versions of this theorem exist: In [19, Theorem 1.2], it is shown that (5) is necessary for (6) and (7) to hold, and that in fact (5), (6) and (7) are equivalent. In this article, we will adopt the following terminology:

Convention 1. *Choosing $K = \mathbb{R}^m$, the assumptions of Ball's Theorem 1 are always fulfilled. Thus a family $(\nu_x)_{x\in\Omega}$ satisfying* $(\mathrm{i}) - (\mathrm{ii})$ *always exists. Moreover, once the subsequence (u_k) of (u_j) is fixed, $(\nu_x)_{x\in\Omega}$ obtained by this way is unique and is a sub-probability family on $\mathbb{R}^m$ by (i):* A *sub-probability family $(\tau_x)_{x\in\Omega}$ on $\mathbb{R}^m$ is a family of measures such that $\|\tau_x\|_{\mathrm{meas}} \leq 1$ for a.e. $x \in \Omega$. Such a family $(\nu_x)_{x\in\Omega}$ is called a Young measure on $\Omega \times \mathbb{R}^m$. Thus, in this sense, each sequence generates a Young measure.*

Theorem 1 has useful applications, in particular in non-linear PDE theory. The following technical statements build the basic tools used in the next sections.

Proposition 3. *([6]) If $|\Omega| < \infty$ and ν_x is the Young measure (see Convention 1) generated by the (whole) sequence u_j, then there holds*

$$u_j \to u \quad \textit{measure if an only if} \quad \nu_x = \delta_{u(x)} \;\; for\ a.e.\ x \in \Omega.$$

For the proof, see [19, Proposition 1.3].

Proposition 4. *([6]) Let $|\Omega| < \infty$. If the sequences $u_j : \Omega \to \mathbb{R}^m$ and $v_j : \Omega \to \mathbb{R}^d$ generate the Young measures $\delta_{u(x)}$ and ν_x respectively, then $(u_j,\ v_j)$ generates the Young measure $\delta_{u(x)} \otimes \nu_x$.*

For the proof, see [19, Proposition 1.4]. This result also holds for sequences $\mu_j,\ \lambda_j$ of Young measures converging in the narrow topology to μ and λ respectively. However it is false in general if both μ and λ are not Dirac measures. The third application is a Fatou-type lemma:

Lemma 1. *([14]) Let $F : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $u_k : \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that Du_k generates the Young measure v_x, with $\|\nu_x\|_{\mathrm{meas}} = 1$ for almost every $x \in \Omega$. Then*

$$\liminf_{k\to\infty} \int_\Omega F(x,\ u_k(x),\ Du_k(x))dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x,\ u,\ \xi)d\nu_x(\xi)dx,$$

provided that the negative part $F^-(x,\ u_k(x),\ Du_k(x))$ is equiintegrable.

5. THE CONVERGENCE IN TERM OF YOUNG MEASURE

This section presents a general convergence result for functions a satisfying similar conditions as stated in Section 1. In fact, an elliptic div-curl inequality is the key ingredient to prove that one can pass to the limit in our quasilinear elliptic system. Since they are, in part, independent of the differential equation, we state them in a general form using only a set of hypotheses:

($\mathcal{A}_\infty$) The sequence (u_k) is uniformly bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ for some $p(x) > 1$ and hence a subsequence converges weakly in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ to an element denoted by u.

(A_2) $a : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function.

(A_3) The sequence $a_k(x) \equiv a(x,\ u_k(x),\ Du_k(x))$ is uniformly bounded in the space $L^{p'(x)}(\Omega;\mathbb{M}^{m\times n})$ and hence equiintegrable. The equiintegrability follows from the Hölder inequality.

(A_4) The sequence $(a_k(x) : Du_k)^-$ is equiintegrable.

(A_5) There exists a sequence (v_k) such that $v_k \to u$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ and $\int_\Omega a_k(x) : (Du_k - Dv_k)dx \to 0$ as $k \to \infty$.

Note that the assumption ($\mathcal{A}_1$) ensures even a strong convergence in some Lebesgue spaces:

Lemma 2. *Let $p > 1$ and (u_k) be a sequence which is uniformly bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. Then there exists a subsequence of (u_k) (for convenience not relabeled) and a function $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that*

$$u_k \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \tag{8}$$

and such that

$$u_k \to u \quad \text{in measure on } \Omega \text{ and in } L^{s(x)}(\Omega;\mathbb{R}^m) \tag{9}$$

for all $s < p^*$.

Proof. Since (u_k) is bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, (8) follows directly from Eberlein-Smuljan Theorem [8]. Moreover, the Rellich-Kondrachov Theorem [1] implies that (u_k) converges to an element $\tilde{u}$ in $L^s(\Omega;\mathbb{R}^m)$ for all $s < p^*$. Notice that in order to have the strong convergence simultaneously for all $s < p^*$, the usual diagonal sequence procedure applies. By unicity of the limit, $\tilde{u} = u$. Finally, the sequence converges in measure [13, Proposition 2.29] since $p(x) > 1$. □

Now, under the conditions $(\mathcal{A}_1) - (\mathbf{A}_5)$, we can prove the following div-curl inequality:

Lemma 3. *(div-curl inequality). Suppose $(\mathcal{A}_1) - (\mathbf{A}_5)$ and assume (after passing to a suitable subsequence if necessary) that (Du_k) generates the Young measure ν_x. Then the following inequality holds:*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,\ u,\ \lambda) : \lambda d\nu_x(\lambda)dx \leq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,\ u,\ \lambda) : Dud\nu_x(\lambda)dx. \tag{10}$$

Proof. Let us consider the sequence

$$I_k \equiv a(x,\ u_k,\ Du_k)\ :\ (Du_k - Du) = a_k\ :\ Du_k - a_k\ :\ Du.$$

By conditions $(\mathbf{A}_3)$ and $(\mathbf{A}_4)$, the negative part I_k^- of I_k is equiintegrable. Hence, we may use the Fatou-Lemma 1 which gives that

$$X \equiv \liminf_{k\to\ \infty} \int_\Omega I_k dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,\ u,\ \lambda) : (\lambda - Du) d\nu_x(\lambda) dx.$$

It remains to prove that $X \leq 0$. For this, we note that by (A5) we have

$$X = \liminf_{k\to\ \infty} (\int_\Omega a_k\ :\ (Du_k - Dv_k)dx + \int_\Omega a_k\ :\ (Dv_k - Du)dx)$$

$$= \lim_{k\to\infty} \int_\Omega a_k : (Dv_k - Du)dx \leq \lim_{k\to\infty} \underbrace{\|a_k\|_{p'(x)}}_{\leq C} \|v_k - u\|_{1,p(x)} = 0,$$

where we used the Hölder inequality and $(\mathbf{A}_3)$. Thus the conclusion follows. □

Remark 1. *The naming "div-curl inequality" can be explained as follows. Suppose for a moment that div* $a(x,\ u_k,\ Du_k) = 0$ *for all* k *and that* $a(x,\ u_k,\ Du_k) : Du_k$ *is equiintegrable. Then, the weak limit of* $a(x,\ u_k,\ Du_k) : Du_k$ *in* $L^1(\Omega)$ *is given by* $\int_{\mathbb{M}^{m\times n}} a(x,\ u,\ \lambda) : \lambda d\nu_x(\lambda)$ *. On the other hand, by the usual div-curl lemma we conclude that* $\int_\Omega a(x,\ u_k,\ Du_k) : Du_k dx$ *converges to* $\int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,\ u,\ \lambda) : Du d\nu_x(\lambda) dx$ *and hence, the lemma would follow with equality.*

The div-curl inequality will be the key ingredient to pass to the limit in the approximating equations. However, we need some additional information on the Young measure ν_x generated by the sequence of the gradients (Du_k) to exploit (10). These properties are the following:

$(\mathbf{Z}_1)$ ν_x is a probability measure for almost every $x \in \Omega$.

$(\mathbf{Z}_2)$ ν_x is a homogeneous $W^{1,p(x)}$ gradient Young measure for almost every $x \in \Omega$ in the sense that for $x \in \Omega$ fixed there exists a sequence $\tilde{u}(z)$ such that the Young measure $(\tilde{\nu}_z)_{z\in\Omega}$ generated by $D\tilde{u}(z)$ is homogeneous and equal to $\nu_x : \tilde{\nu}_z = \nu_x$ for almost every $z \in \Omega$.

$(\mathbf{Z}_3)$ ν_x satisfies $\langle \nu_x, \text{id} \rangle = Du(x)$ for almost every $x \in \Omega$.

The properties $(\mathbf{Z}_1) - (\mathbf{Z}_3)$ follow in particular from the two estimates formulated in the next Lemma:

Lemma 4. *Let Ω be a bounded subset in $\mathbb{R}^n$ and $(u_k)_k$ a sequence in $W_0^{1,1}(\Omega;\mathbb{R}^m)$. Suppose that there exist $r(x) > 0,\ p(x) > 1$ and some constants $C,\ M$ and L such that*

$$\sup_{k\in\mathbb{N}} \int_\Omega |u_k|^{r(x)} dx \leq C$$

and

$$\sup_{k\in\mathbb{N}} \int_{|u_k|\leq R} |Du_k|^{p(x)} dx \leq MR + L \quad \forall\ R > 0.$$

Then the Young measure ν_x generated by (a subsequence of) Du_k has finite $p(x)$-th moment for almost every $x \in \Omega$ and satisfies $(\mathbf{Z}_1) - (\mathbf{Z}_3)$.

For the proof, see [12, Lemma 9]. In particular, $(\mathbf{Z}_1) - (\mathbf{Z}_3)$ hold if the condition $(\mathcal{A}_1)$ is fulfilled (actually if $(\mathcal{A}_1)$ is satisfied, $(\mathbf{Z}_1) - (\mathbf{Z}_3)$ can also be verified directly). In any case, the conditions $(\mathbf{Z}_1) - (\mathbf{Z}_3)$ will be sufficient to pass to the limit as shown by the following convergence result for a:

Proposition 5. *Suppose that $(\mathcal{A}_1) - (\boldsymbol{A}_5)$ hold. Further assume that the Young measure ν_x generated by the gradients Du_k satisfies $(\mathrm{N}_1) - (\mathrm{N}_3)$ and that one of the following conditions holds:*

(a) *The map $F \mapsto a(x, u, F)$ is monotone and continuously differentiable for all $(x, u) \in \Omega \times \mathbb{R}^m$.*

(b) $a(x, u, F) = \dfrac{\partial W}{\partial F}(x, u, F)$ *and $F \mapsto W(x, u, F)$ is a convex C^1-function for all $(x, u) \in \Omega \times \mathbb{R}^m$.*

(c) *The map $F \mapsto a(x, u, F)$ is strictly monotone for all $(x, u) \in \Omega \times \mathbb{R}^m$.*

(d) *The map $F \mapsto a(x, u, F)$ is strictly $p(x)$-quasimonotone.*

Then (after passage to a subsequence) the sequence a_k converges weakly in the space $L^1(\Omega; \mathbb{M}^{m\times n})$ as $k \to \infty$ and the weak limit $\overline{a}$ is given by

$$\overline{a}(x) = a(x, u(x), Du(x)).$$

If (b), (c) or (d) holds, then

$$a(x, u_k(x), Du_k(x)) \to a(x, u(x), Du(x)) \quad in \quad L^1(\Omega; \mathbb{M}^{m\times n}).$$

In cases (c) and (d), it follows in addition that (after extraction of a further subsequence if necessary) $Du_k \to Du$ in measure and almost everywhere in Ω. Before we prove Proposition (5)*, we state a technical lemma which allows to localize the support of the Young measures ν_x.*

Lemma 5. *Suppose that* $(\mathcal{A}_1)-(\boldsymbol{A}_5)$ *hold. Further assume that* ν_x *is the Young measure generated by the gradients* Du_k *and satisfies* $(\mathbf{Z}_1)-(\mathbf{Z}_3)$ *. If the map* $F\mapsto a(x,\ u,\ F)$ *is monotone for all* $(x,\ u)\in\Omega\times\mathbb{R}^m$*, then*

$$spt\nu_x\subset\{\lambda\in\mathbb{M}^{m\times n}:(a(x,\ u,\ \lambda)-a(x,\ u,\ Du)):(\lambda-Du)=0\}. \tag{11}$$

Proof. By $(\mathbf{Z}_1)$ and $(\mathbf{Z}_3)$, we have (with $\overline{\lambda}=Du(x)$)

$$\int_{\mathbb{M}^{m\times n}} a(x,\ u,\overline{\lambda}):(\lambda-\overline{\lambda})d\nu_x(\lambda)$$

$$=\int_{\mathbb{M}^{m\times n}} a(x,\ u,\overline{\lambda}):\lambda d\nu_x(\lambda)-\int_{\mathbb{M}^{m\times n}} a(x,\ u,\overline{\lambda}):\overline{\lambda}d\nu_x(\lambda)$$

$$=a(x,u,\overline{\lambda}):\underbrace{\int_{\mathbb{M}^{m\times n}}\lambda d\nu_x(\lambda)}_{=\overline{\lambda}}-a(x,\overline{\lambda}):\overline{\lambda}\underbrace{\int_{\mathbb{M}^{m\times n}}d\nu_x(\lambda)}_{=1}=0$$

By conditions $(\mathcal{A}_1)-(\mathbf{A}_5)$, we have $\overline{\lambda}=Du(x)$ and we infer from inequality (10) in Lemma (3) that

$$\int_\Omega\int_{\mathbb{M}^{m\times n}}(a(x,u,\lambda)-a(x,\ u,\overline{\lambda})):(\lambda-\overline{\lambda})d\nu_x(\lambda)dx\leq 0. \tag{12}$$

On the other hand, the integrand in (12) is non negative by monotonicity. It follows that the integrand must vanish almost everywhere with respect to the product measure $d\nu_x\otimes dx$. Hence, the conclusion follows. □

Proof of Proposition 5. We start with the easiest case:

Case (c): Since a is monotone by assumption, (11) holds by Lemma 5. By strict monotonicity, it follows from (11) that $\nu_x=\delta_{Du(x)}$ for almost all $x\in\Omega$, and hence $Du_k\to Du$ in measure for $k\to\infty$ by Proposition 3. Since we have already that $u_k\to u$ in measure for $k\to\infty$ by $(\mathcal{A}_1)$ and Lemma 3 , we may infer that (after extraction of a suitable subsequence, if necessary [13, Theorem 2.30]) $u_k\to u$ and $Du_k\to Du$ almost everywhere in Ω for $k\to\infty$. From the continuity condition $(\mathbf{A}_2)$, it follows that $a(x,\ u_k,\ Du_k)\to a(x,\ u,\ Du)$ almost everywhere in Ω. Since, by assumption $(\mathbf{A}_3)$, $a_k(x)$ is equiintegrable, it follows from the Vitali convergence Theorem [13, Page 180] that $a(x,\ u_k,\ Du_k)\to a(x,\ u,\ Du)\ in\ L^1(\Omega;\mathbb{M}^{m\times n})$ for $k\to\infty$, which proves the proposition in this case.

Case (d): Assume that ν_x is not a Dirac mass on a set $x \in M$ of positive Lebesgue measure $|M| > 0$. Then, by the strict p-quasimonotonicity of $a(x,\ u,\ \cdot)$ and $(\mathbf{Z}_2)$, we have for a.e. $x \in M$ (with $\bar{\lambda} = \langle \nu_x,\ \mathrm{id} \rangle = Du(x)$ by $(\mathbf{Z}_3)$).

$$\begin{aligned}\int_{\mathbb{M}^{m\times n}} & a(x,u,\lambda) : \lambda d\nu_x(\lambda) \\ &> \underbrace{\int_{\mathbb{M}^{m\times n}} a(x,u,\bar{\lambda}) : \lambda d\nu_x(\lambda)}_{=a(x,u,\bar{\lambda}):\bar{\lambda}} - \underbrace{\int_{\mathbb{M}^{m\times n}} a(x,u,\bar{\lambda}) : \bar{\lambda} d\nu_x(\lambda)}_{=a(x,u,\bar{\lambda}):\bar{\lambda}.1} \\ &\quad + \int_{\mathbb{M}^{m\times n}} a(x,u,\lambda) : \bar{\lambda} d\nu_x(\lambda) \\ &= \int_{\mathbb{M}^{m\times n}} a(x,u,\lambda) : \bar{\lambda} d\nu_x(\lambda),\end{aligned} \tag{13}$$

where we used $(\mathbf{Z}_1)$. We claim now that we obtain a contradiction. Indeed, by integrating (5) over Ω and using the div-curl inequality (10) in Lemma (3), we get

$$\begin{aligned}\int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,u,\lambda) : \lambda d\nu_x(\lambda) dx &> \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,u,\lambda) : \bar{\lambda} d\nu_x(\lambda) dx \\ &\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,u,\lambda) : \lambda d\nu_x(\lambda) dx\end{aligned}$$

as desired. Hence, we have $\nu_x = \delta_{\overline{\lambda}} = \delta_{Du(x)}$ for almost every $x \in \Omega$. Thus, it follows again by Proposition (3) that $Du_k \to Du$ in measure for $k \to \infty$. The reminder of the proof in this case is exactly as in case (c).

Case (b): We start by showing that for almost all $x \in \Omega$, the support of ν_x is in the set where W agrees with the supporting hyper-plane $L \equiv \{(\lambda,\ W(x,\ u, \bar{\lambda}) + a(x,u,\bar{\lambda})(\lambda - \bar{\lambda}))\}$ in $\bar{\lambda} = Du(x)$, i.e. we want to show that

$$spt\ \nu_x \subset K_x = \{\lambda \in \mathbb{M}^{m\times n} : W(x,u,\lambda) = W(x,u,\bar{\lambda}) + a(x,u,\bar{\lambda}) : (\lambda - \bar{\lambda})\}.$$

Since a admits a potential, a is monotone and then (11) holds by Lemma (5). Thus, if $\lambda \in spt\nu_x$ then by (11)

$$(1-t)(a(x,\ u,\bar{\lambda}) - a(x,u,\lambda)) : (\bar{\lambda} - \lambda) = 0 \ \ forall\ t \in [0,1]. \tag{14}$$

On the other hand, by monotonicity, we have for $t \in [0,1]$ that

$$0 \leq (1-t)(a(x,u,\bar{\lambda} + t(\lambda - \bar{\lambda})) - a(x,u,\lambda)) : (\bar{\lambda} - \lambda). \tag{15}$$

Subtracting (14) from (15), we get

$$0 \leq (1-t)(a(x,u,\bar{\lambda} + t(\lambda - \bar{\lambda})) - a(x,u,\bar{\lambda})) : (\bar{\lambda} - \lambda), \tag{16}$$

for all $t \in [0,1]$. But by monotonicity, in (16) also the reverse inequality holds and we may conclude, that

$$(a(x,u,\bar{\lambda}+t(\lambda-\bar{\lambda}))-a(x,u,\bar{\lambda})):(\lambda-\bar{\lambda})=0, \tag{17}$$

for all $t \in [0,1]$, whenever $\lambda \in spt\ \nu_x$. Now, it follows from (17) that

$$\begin{aligned} W(x,u,\lambda) &= W(x,u,\bar{\lambda})+(W(x,u,\lambda)-W(x,u,\bar{\lambda})) \\ &= W(x,u,\bar{\lambda})+\int_0^1 a(x,u,\bar{\lambda}+t(\lambda-\bar{\lambda})):(\lambda-\bar{\lambda})dt \\ &= W(x,\ u,\bar{\lambda})+a(x,u,\bar{\lambda}):(\lambda-\bar{\lambda}) \end{aligned}$$

By the convexity of W we have $W(x,u,\lambda) \geq W(x,u,\bar{\lambda})+a(x,u,\bar{\lambda}):(\lambda-\bar{\lambda})$ for all $\lambda \in \mathbb{M}^{m\times n}$ and thus L is a supporting hyper-plane for all $\lambda \in K_x$. Since the mapping $\lambda \mapsto W(x,u,\lambda)$ is by assumption continuously differentiable we obtain

$$a(x,u,\lambda)=a(x,u,\bar{\lambda})\ for\ all\ \lambda \in K_x \supset spt\ \nu_x \tag{18}$$

and thus

$$\bar{a}(x) \equiv \int_{\mathbb{M}^{m\times n}} a(x,u,\lambda)d\nu_x(\lambda)=a(x,u,\bar{\lambda}). \tag{19}$$

Now consider the Carathéodory function

$$\psi(x,u,p)=|a(x,u,p)-\bar{a}(x)|.$$

The sequence $\psi_k(x)=\psi(x,u_k(x),Du_k(x))$ is equiintegrable and thus by Ball's Theorem 1

$$\psi_k \rightharpoonup \bar{\psi} \quad weakly \quad in \quad L^1(\Omega)$$

and the weak limit $\bar{\psi}$ is given by

$$\begin{aligned} \bar{\psi}(x) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} |a(x,\eta,\lambda)-\bar{a}(x)|d\delta_{u(x)}(\eta)\otimes d\nu_x(\lambda) \\ &= \int_{\text{spt }\nu_x} |a(x,u(x),\lambda)-\bar{a}(x)|d\nu_x(\lambda)=0 \end{aligned}$$

by (18) and (19). Since $\psi_k \geq 0$ it follows that

$$\psi_k \to 0 \quad strongly \quad in \quad L^1(\Omega).$$

Thus the proof of the case (b) is finished.

Case (a): First we note that since a is monotone, 11 holds by Lemma (5). We claim that in this case for almost all $x \in \Omega$ the following identity holds for all $M \in \mathbb{M}^{m\times n}$ on the support of ν_x:

$$a(x,u,\lambda) : M = a(x,u,\bar{\lambda}) : M + (\nabla_F\, a(x,u,\bar{\lambda})M) : (\bar{\lambda} - \lambda), \qquad (20)$$

where ∇_F is the derivative with respect to the third variable of a and $\bar{\lambda} = Du(x)$. Indeed, by the monotonicity of a we have for all $t \in \mathbb{R}$

$$(a(x,u,\lambda) - a(x,u,\bar{\lambda} + tM)) : (\lambda - \bar{\lambda} - tM) \geq 0,$$

whence, by (11)

$$\begin{aligned} -a(x,u,\lambda) : (tM) &\geq -a(x,u,\bar{\lambda}) : (\lambda - \bar{\lambda}) + a(x,u,\bar{\lambda} + tM) : (\lambda - \bar{\lambda} - tM) \\ &= t((\nabla_F a(x,u,\bar{\lambda})M)(\lambda - \bar{\lambda}) - a(x,\ u,\bar{\lambda}) : M) + o(t). \end{aligned}$$

The claim follows from this inequality since the sign of t is arbitrary. Since the sequence $a_k(x)$ is equiintegrable by (A3), its weak L^1-limit $\bar{a}$ is given by

$$\begin{aligned} \bar{a}(x) &= \int_{\text{spt }\nu_x} a(x,u,\lambda)d\nu_x(\lambda) \\ &= \int_{\text{spt }\nu_x} a(x,u,\bar{\lambda})d\nu_x(\lambda) + (\nabla_F a(x,u,\bar{\lambda})^t \underbrace{\int_{\text{spt }\nu_x} (\bar{\lambda} - \lambda)d\nu_x(\lambda)}_{=\bar{\lambda} - \langle l/_x, \text{id}\rangle = 0} \\ &= a(x,\ u,\bar{\lambda})\ , \end{aligned}$$

where we used (20) in this calculation. This finishes the proof of the case (c) and hence of the proposition.

Remark 2. *In case (b), we remark, that the relation* (19) *already states that* $a(x,u,\bar{\lambda})$ *is the weak* L^1*-limit of* $a(x,u_k,Du_k)$ *, which is enough to pass to the limit in an equation which holds in the distributional sense. However, we wanted to point out that in this case, the convergence is even strong in* $L^1(\Omega;\mathbb{M}^{m\times n})$.

6. EXISTENCE OF A WEAK SOLUTION

The main result we prove in this paper is the following:

Theorem 2. If $p(x) \in (1,n)$ and if a satisfies the conditions $(\mathbf{H}_0) - (\mathbf{H}_2)$, for every $v \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$, every f satisfying $(\mathcal{F}_0) - (\mathcal{F}_1)$, every g satisfying $(G_0) - (G_1)$ and every b satisfying (B). Then there exists at least one weak solution to the Dirichlet problem (1) in the sense of Definition1

Proof. To prove Theorem 2, we will apply a Galerkin scheme. First we recall that by the Poincaré and the Sobolev inequalities, there exists a constant $A \geq 1$ such that

$$\max(\|u\|_{p(x)},\ \|u\|_{p^*(x)}) \leq A\|Du\|_{p(x)}\ \ \forall\, u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \tag{21}$$

Note that we write A, in general without further comment, to point to the use of (21). This relation and the Hölder inequality are central to establish the required estimates to prove the desired results.

Lemma 6. *For arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ and $v \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$, the functional $F(u)$: $W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ given by*

$$\begin{aligned} w \mapsto \int_\Omega a(x,u(x),Du(x)) : Dw(x)dx + \int_\Omega |u(x)|^{p(x)-2}u(x)\cdot w(x)dx \\ + \int_\Omega b(x,u(x),Du(x))\cdot w(x)dx - \langle v,\ w\rangle \\ - \int_\Omega f(x,u(x))\cdot w(x)dx + \int_\Omega h(x,u(x)) : Dw(x)dx \end{aligned}$$

is well defined, linear and bounded.

Proof. On the one hand, the growth condition in (Z_1) allows us to estimate $I \equiv \displaystyle\int_\Omega a(x,u(x),Du(x)) : Dw(x)dx$ for each $w \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$:

$$\begin{aligned} |I| &\leq \int_\Omega |a(x,u,Du)||Dw|dx \\ &\leq \int_\Omega \lambda_1|Dw|dx + c_1\int_\Omega |u|^{\beta(x)}|Dw|dx + c_1\int_\Omega |Du|^{p(x)-1}|Dw|dx \\ &\leq \|Dw\|_{p(x)}\big(\|\lambda_1\|_{p'(x)} + c_1(A^{\frac{p^*(x)}{p'(x)}}\|Du\|_{p(x)}^{\frac{p^*(x)}{p'(x)}} + \|Du\|_{p(x)}^{p(x)-1})\big), \end{aligned}$$

by the Hölder inequality and the bound for $\beta(x)$. Next, the generalized Hölder inequality implies that

$$|\langle v,w\rangle| \leq \|v\|_{-1,p'(x)}\|w\|_{1,p(x)} \leq A\|v\|_{-1,p'(x)}\|Dw\|_{p(x)},$$

and

$$|II| = |\int_\Omega |u|^{p(x)-2}u\cdot w dx| \leq \int_\Omega |u|^{p(x)-1}|w|dx \leq A\|u\|_{p(x)}^{p(x)-1}\|Dw\|_{p(x)}.$$

On the one hand, the growth condition in $(\mathcal{B})$ allows us to estimate $III \equiv \int_\Omega b(x,u(x),Du(x))\cdot w(x)dx$

$$\begin{aligned}|III| &\leq \int_\Omega |b(x,u,Du)||w|dx\\ &\leq \int_\Omega d_3(x)|w|dx + \int_\Omega c|u|^{p(x)-1}|w|dx + \int_\Omega c|Du|^{p(x)-1}|w|dx\\ &\leq ||d_3||_{p'(x)}||w||_{p(x)} + c||u||_{p'(x)}^{p(x)-1}||w||_{p(x)} + c||Du||_{p'(x)}^{p(x)-1}||w||_p\\ &\leq ||Dw||_{p(x)}(A||d_3||_{p'(x)} + cA^{p(x)}||u||^{p(x)-1} + cA^{p(x)}||Du||_{p'(x)}^{p(x)-1}).\end{aligned}$$

On the other hand, if $IV \equiv \int_\Omega f(x,u)\cdot wdx$, it follows from the growth condition $(\mathbf{F}_1)$ (Without loss of generality, we may assume that $\gamma(x) = p(x)-1$). An application of the Hölder inequality to the three functions yields

$$\begin{aligned}|IV| &\leq \int_\Omega |f(x,u)||w|dx\\ &\leq \int_\Omega b_1|w|dx + \int_\Omega b_2|u|^{p(x)-1}|w|dx\\ &\leq \|b_1\|_{p'(x)}\|w\|_{p(x)} + \|b_2\|_{\frac{n}{p(x)}}\|u\|_{p^*(x)}^{p(x)-1}\|w\|_{p^*(x)}\\ &\leq \|Dw\|_{p(x)}(A\|b_1\|_{p'(x)} + A^{p_+}\|b_2\|_{\frac{n}{p(x)}}\|Du\|_{p(x)}^{p(x)-1})\ .\end{aligned}$$

Finally, the growth condition (G1) (Without loss of generality, we may assume that $\eta(x) = p(x)-1$) allows us to estimate $V \equiv \int_\Omega h(x,u) : Dwdx$ for each $w \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$:

$$\begin{aligned}|V| &\leq \int_\Omega |h(x,u)||Dw|dx \leq \int_\Omega b_4|Dw|dx + \int_\Omega b_5|u|^{p(x)-1}|Dw|dx\\ &\leq \|b_4\|_{p'(x)}\|Dw\|_{p(x)} + \|b_5\|_{\frac{n}{p(x)-1}}\|u\|_{p^*(x)}^{p(x)-1}\|Dw\|_{p(x)}\\ &\leq \|Dw\|_{p(x)}(\|b_4\|_{p'(x)} + A^{p_+-1}\|b_5\|_{\frac{n}{p(x)-1}}\|Du\|_{p(x)}^{p(x)-1})\ .\end{aligned}$$

for each $w \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. Since these five expressions are finite by our assumptions, $F(u)$ is well defined. Moreover, $F(u)$ is trivially linear and we have for all $w \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$

$$|\langle F(u),w\rangle| \leq |I| + |\langle v,w\rangle| + |II| + |III| + |IV| + |V| \leq C\|Dw\|_{p(x)},$$

which implies that $F(u)$ is bounded. □

So we can define the operator

$$F : W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \to W^{-1,p'(x)}(\Omega;\mathbb{R}^m), \quad u \mapsto F(u),$$

which satisfies the following property.

Lemma 7. *The restriction of F to a finite dimensional linear subspace V of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is continuous.*

Proof. Let r be the dimension of V and $(\phi_i)_{i=1}^r$ a basis of V. Let $(u_j = a_j^i\phi_i)$ be a sequence in V which converges to $u = a^i\phi_i$ in V (with the standard summation convention). Then on the one hand the sequence (a_j) converges to a in $\mathbb{R}^r$ and so $u_j \to u$ and $Du_j \to Du$ almost everywhere and on the other hand $\|u_j\|_{p(x)}$ and $\|Du_j\|_{p(x)}$ are bounded by a constant C.
Thus, the continuity condition in ($\mathbf{E}_0$), in ($\mathbf{F}_0$) and in $\mathbf{G}_0$) permits to deduce that

$$a(x,u_j,Du_j) : Dw \to a(x,u,Du) : Dw,$$
$$f(x,u_j)\cdot w \to f(x,u).w,$$
$$b(x,u_j,Du_j) : w \to b(x,u,Du) : w,$$

and

$$h(x,u_j) : Dw \to h(x,u) : Dw,$$

almost everywhere as $k \to \infty$. Moreover we infer from the growth conditions ($\mathbf{E}_1$), ($\mathbf{F}_1$) and ($\mathbf{G}_1$) that the sequences $(a(x,u_j,Du_j) : Dw)$, $(b(x,u_j,Du_j)\cdot w)$, $(f(x,u_j)\cdot w)$ and $(h(x,u_j) : Dw)$ are equi-integrable. Indeed, if $\Omega' \subset \Omega$ is a measurable subset and $w \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, then

$$\begin{aligned}\int_{\Omega'} |a(x,u_j,Du_j) : Dw|dx &\leq \int_{\Omega'} (\lambda_1 + c_1(|u_j|^{\beta(x)} + |Du_j|^{p(x)-1}))|Dw|dx \\ &\leq (\int_{\Omega'} |Dw|^{p(x)}dx)^{\frac{1}{p(x)}}(||\lambda_1||_{p'(x)} + c_1(A^{\frac{p*(x)}{p'(x)}}||Du_j||_{p(x)}^{\frac{p*(x)}{p'(x)}} + ||Du_j||_{p(x)}^{p(x)-1})),\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega'} |b(x,u_j,Du_j) : w|dx &\leq (\int_{\Omega'} |w|^{p(x)}dx)^{\frac{1}{p(x)}}(c\int_{\Omega'} (|d_3(x)|^{p'(x)} \\ &\quad +|u_j|^{p(x)} + |Du_j|^{p(x)})dx)^{\frac{1}{p'(x)}} \\ &\leq (\int_{\Omega'} |w|^{p(x)}dx)^{\frac{1}{p(x)}}(c[||d_3||_{p'(x)} + ||u_j||_{p(x)}^{p(x)-1} + ||Du_j||_{p(x)}^{p(x)-1}]) \\ &\leq (\int_{\Omega'} |w|^{p(x)}dx)^{\frac{1}{p(x)}}(c[||d_3||_{p'(x)} + A^{p_+-1}||Du_j||_{p(x)}^{p(x)-1} + ||Du_j||_{p(x)}^{p(x)-1}])\end{aligned}$$

and (Without loss of generality, we may assume that $\gamma(x) = p(x) - 1$),

$$\int_{\Omega'} |f(x, u_j) \cdot w| dx \le \int_{\Omega'} (b_1 + b_2 |u_j|^{p(x)-1}) |w| dx$$
$$\le A(\int_{\Omega'} |Dw|^{p(x)} dx)^{\frac{1}{p(x)}} (||b_1||_{p'} + (A^{p_+ -1} ||b_2||_{\frac{n}{p(x)}}) ||Du_j||_{p(x)}^{p(x)-1})$$

and (Without loss of generality, we may assume that $\eta(x) = p(x) - 1$),

$$\int_{\Omega'} |h(x, u_j) : Dw| dx$$
$$\le (\int_{\Omega'} |Dw|^{p(x)} dx)^{\frac{1}{p(x)}} \Big(||b_4||_{p'(x)} + (A^{p_+ -1} ||b_5||_{\frac{n}{p(x)-1}}) ||Du_j||_{p(x)}^{p(x)-1} \Big),$$

by the Hölder inequality (see the proof of Lemma 6). Applying the Vitali Theorem, it follows that for all $w \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, we have

$$\|F(u_j) - F(u)\|_{-1,p'(x)} = \sup_{\|w\|_{1,p(x)}=1} |\langle F(u_j), w\rangle - \langle F(u), w\rangle| \le C,$$

which implies that $\lim_{j\to\infty} \langle F(u_j)\ , w\rangle = \langle F(u)\ , w\rangle$ as desired. □

Now, the problem (1) is equivalent to find a solution $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ such that

$$\langle F(u), w\rangle = 0 \quad for\ all \quad w \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m). \tag{22}$$

In order to find such a solution we apply a Galerkin scheme. Let $V_1 \subset V_2 \subset \ldots \subset W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\bigcup_{k\in\mathbb{N}} X_k$ is dense in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$. Let us fix some k and assume that X_k has dimension r and that $\phi_1, \ \ldots, \ \phi_r$ is a basis of X_k. Then we define the map

$$G : \mathbb{R}^r \to \mathbb{R}^r, \qquad \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^r \end{pmatrix} \mapsto \begin{pmatrix} \langle F(a^i\phi_i), \phi_1\rangle \\ \langle F(a^i\phi_i), \phi_2\rangle \\ \vdots \\ \langle F(a^i\phi_i), \phi_r\rangle \end{pmatrix}.$$

Proposition 6. *G is continuous and*

$$G(a) \cdot a \to \infty \quad as \quad \|a\|_{\mathbb{R}^r} \to \infty.$$

Proof. Since F restricted to X_k is continuous by Lemma 7, G is continuous. Let be $a \in \mathbb{R}^r$ and $u = a^i\phi_i \in X_k$. Then $G(a) \cdot a = \langle F(u), u\rangle$ and $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,p(x)} \to \infty$. Next, we note the following considerations. First the coercivity condition in $(\mathbf{Z}_1)$ and the Hölder inequality imply that

$$I \equiv \int_\Omega a(x,u,Du) \; : \; Dudx \geq \int_\Omega -\lambda_2(x) - \lambda_3(x)|u|^{\alpha(x)} + c_2|Du|^{p(x)}$$
$$\geq -\|\lambda_2\|_{L^1} - \|\lambda_3\|_{L^1} A^{\alpha_+}\|\lambda_3\|_{(\frac{p(x)}{\alpha(x)})'}\|Du\|_{p(x)}^{\alpha(x)} + c_2\|Du\|_{p(x)}^{p(x)}.$$

Next the generalized Hölder inequality implies that

$$|II| \equiv |\int_\Omega b(x,\; u,\; Du) : Dudx| \leq A\|b_3\|_{p'(x)}\|Du\|_{p(x)} + cA(\|Du\|_{p'(x)}^{p(x)} + \|Du\|_{p(x)}^{p(x)})$$

and

$$|III| \equiv |\langle v,\; u\rangle| \leq \|v\|_{-1,p'(x)}\|u\|_{1,p(x)} \leq A\|v\|_{-1,p'(x)}\|Du\|_{p(x)}.$$

Finally, it follows from the growth conditions $(\mathbf{F}_1)$ and $(\mathbf{G}_1)$ that (see the proof of Lemma 6)

$$IV \equiv \int_\Omega f(x,\; u)\cdot udx \leq A\|b_1\|_{p'(x)}\|Du\|_{p(x)} + A^{\gamma_+ +1}\|b_2\|_{\frac{n}{p(x)}}\|Du\|_{p(x)}^{\gamma(x)+1}$$

and

$$|V| \equiv \left|\int_\Omega h(x,\; u) : Dudx\right| \leq \|b_4\|_{p'(x)}\|Du\|_{p(x)} + A^{\eta_+}\|b_5\|_{\frac{n}{p(x)-1}}\|Du\|_{p(x)}^{\eta(x)+1}.$$

From these estimations it follows that $\langle F(u)\,,u\rangle = I - II - III + IV + V \to \infty$ as $\|u\|_{1,p(x)} \to \infty$,
since $p(x) > \max(1, \alpha(x), \gamma(x)+1, \delta(x)+1, \eta(x)+1)$ and $A, c, c_2 > 0$. □

The properties of G allow us to construct our Galerkin approximations:

Corollary 1. *For all $k \in \mathbb{N}$ there exists $u_k \in X_k$ such that*

$$\langle F(u_k), w\rangle = 0 \quad for\ all \quad w \in X_k. \tag{23}$$

Proof. By Proposition 6 there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(a)\cdot a > 0$ and the usual topological argument [23, Proposition 2.8] gives that $G(x) = 0$ has a solution $x \in B_R(0)$.
Hence, for all k there exists $u_k \in X_k$ such that (23) holds. □

The Galerkin approximations satisfy the following bound:

Proposition 7. *The sequence of the Galerkin approximations constructed in Corollary 1 is uniformly bounded in* $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ *, i.e. there exists a constant* $R > 0$ *such that*

$$\|u_k\|_{1,p(x)} \leq R \quad forall \quad k \in \mathbb{N}. \tag{24}$$

Proof. As in the proof of Proposition 6 we see that $\langle F(u)\,,\,u\rangle \to \infty$ as $\|u\|_{1,p(x)} \to \infty$. Then it follows that there exists $R > 0$ with the property, that $\langle F(u)\,,\,u\rangle > 1$ whenever $\|u\|_{1,p(x)} > R$. Thus, for the sequence of Galerkin approximations $u_k \in X_k$ which satisfy $\langle F(u_k)\,,\,u_k\rangle = 0$ by (23), we have the uniform bound. □

Now, we are able to pass to the limit and so to prove Theorems 2. First, in order to apply Proposition 5, we verify that, under our assumptions, the conditions $(\mathcal{A}_1) - (\mathbf{A}_5)$ and $(\mathbf{Z}_1) - (\mathbf{Z}_3)$ hold for the Galerkin approximations solutions u_k constructed before. $(\mathcal{A}_1)$ holds by Proposition 7. Moreover, it follows then by Lemma 4 that $(\mathbf{Z}_1) - (\mathbf{Z}_3)$ hold.

The condition $(\mathbf{A}_2)$ is equivalent to $(\mathbf{E}_0)$. To obtain $(\mathbf{A}_3)$, we observe that by the growth condition in $(\mathbf{E}_1)$

$$\int_\Omega |a(x,\ u_k,\ Du_k)|^{p'(x)}dx \leq C(\int_\Omega (|\lambda_1(x)|^{p'(x)} + |u_k|^{p^*(x)} + |Du_k|^{p(x)})dx)\ ,$$

which is uniformly bounded in k by (24) since $\|u_k\|_{p^*(x)} \leq A\|Du_k\|_{p(x)}$ by (21). Next, to verify $(\mathbf{A}_4)$, we fix an arbitrary measurable subset $\Omega' \subset \Omega$. Then, on the one hand, the growth condition in (E_1) implies that

$$\int_{\Omega'} |\min(a(x,\ u_k,\ Du_k) : Du_k,\ 0)|dx \leq \int_{\Omega'} |\lambda_2|dx + \int_{\Omega'} |\lambda_3||u_k|^{\alpha(x)}dx$$

by the Hölder inequality and (24). Since a finite set of integrable functions is equi-integrable, the equi-integrability of $(a_k : Du_k)^-$ follows.

Finally, we want to prove $(\mathbf{A}_5)$: According to Mazur's Theorem (see, e.g., [22, Theorem 2,page 120] there exists a sequence X_k in $W_0^{1,p(x)}(\Omega)$ where each X_k is a convex linear combination of $\{u_1,\ \ldots,\ u_k\}$ such that $v_k \to u$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. Means

$$v_k \text{ belongs to the same space as } u_k \text{ and } v_k\ \to u \text{ in } W_0^{1,p(x)}(\Omega;\mathbb{R}^m). \tag{25}$$

This allows us in particular, to use $u_k - v_k$ as a test function in (23). We have

$$\begin{aligned}\int_\Omega a(x,u_k,Du_k) : (Du_k - Dv_k)dx = \langle v, u_k - v_k\rangle - \int_\Omega |u_k|^{p(x)-2}u_k \cdot (u_k - v_k)dx \\ + \int_\Omega f(x,\ u_k)\cdot(u_k - v_k)dx - \int_\Omega b(x,u_k,Du_k).(u_k - v_k)dx \\ - \int_\Omega h(x,\ u_k) : (Du_k - Dv_k)dx.\end{aligned} \tag{26}$$

The first and the second terms on the right in (6) converges to zero since

$$u_k - v_k \to 0 \; in \; W_0^{1,p(x)}(\Omega; \mathbb{R}^m) \tag{27}$$

by the choice of v_k, $(\mathcal{A}_1)$ and Lemma 2. Now, for the third term $III_k \equiv \int_\Omega f(x,\ u_k)\,.$ $(v_k - u_k)dx$ in (6) it follows from the growth condition $(\mathbf{F}_1)$ and the Hölder inequality that

$$|III_k| \leq \|b_1\|_{p'(x)}\|v_k - u_k\|_{p(x)} + \|b_2\|_{\frac{n}{p(x)}}\|u_k\|^{\gamma}_{p^*(x)}\|v_k - u_k\|_{\frac{p^*(x)}{p(x)-\gamma(x)}}.$$

By (21) and (24), $\|u_k\|_{p^*(x)}$ is bounded. Moreover, by the construction of X_k, $(\mathcal{A}_1)$ and Lemma 2 we have

$$\|u_k - v_k\|_{s(x)} \leq \|u_k - u\|_{s(x)} + \|u - v_k\|_{s(x)} \to 0$$

for all $s(x) < p^*(x)$ Since it follows from $\gamma(x) < p(x) - 1$ that $\dfrac{p^*(x)}{p(x) - \gamma(x)} < p^*(x)$, we infer that the second term in (6) vanishes as $k \to \infty$. For the fourth term $IV_k \equiv \int_\Omega b(x, u_k, Du_k).(u_k - v_k)dx$ in (6) it follows from the growth condition (B)(i), (21) and the Hölder inequality that

$$\begin{aligned}|IV_k| &\leq ||d_3||_{p'(x)}||u_k - v_k||_{p(x)} + c||u_k||^{p(x)-1}_{p(x)}||u_k - v_k||_{p(x)} + c||Du_k||^{p(x)-1}_{p(x)}||u_k - v_k||_{p(x)} \\ &\leq (||d_3||_{p'(x)} + cA^{p_+-1}\underbrace{||Du_k||^{p(x)-1}_{p(x)}}_{\leq C} + c\underbrace{||Du_k||^{p(x)-1}_{p(x)}}_{\leq C})||u_k - v_k||_{p(x)}.\end{aligned}$$

Moreover, by the construction of v_k, $(\mathcal{A}_1)$ and Lemma 2 the fifth term in (6) vanishes as $k \to \infty$. Finally, for the last term $V_k \equiv \int_\Omega h(x,\ u_k) : D(v_k - u_k)dx$ in (6) we note that $g(x,\ u_k) \to (hx,\ u)$ strongly in $L^{p'(x)}(\Omega; \mathbb{M}^{m\times n})$ by $(\mathbf{G}_0)$, $(\mathbf{G}_1)$, $(\mathcal{A}_1)$ and (9) in Lemma 2. Indeed we may assume by $(\mathcal{A}_1)$ that $u_k \to u$ almost everywhere (since in the sequel we consider only a subsequence of u_k, not relabeled for convenience, which converges almost everywhere to u. Since by $(\mathbf{G}_1)$ $|h(x,\ u_k)|^{p'(x)}$ is bounded by an integrable function

$$\begin{aligned}|h(x,\ u_k)|^{p'(x)} &\leq C(|b_4|^{p'(x)} + |b_5|^{p'(x)}|u_k|^{p(x)}) \\ &\leq C(|b_4|^{p'(x)} + |b_5|^{p'(x)}(1 + |u|^{p(x)})) \in L^1(\Omega),\end{aligned}$$

the claim follows from $(\mathbf{G}_0)$ and the Dominated Convergence Theorem [13, Theorem 2.24]. We infer then that

$$VI_k \leq ||g(u_k - g(u)||_{p'(x)}\underbrace{||D_v k - Du_k||_{p(x)}}_{\leq C} + |\int_\Omega h(x,u) : D(v_k - u_k)dx| \to 0$$

as $k \to \infty$ by (27). Thus the last term in (6) disappears also as $k \to \infty$ and $(\mathbf{A}_5)$ is fulfilled.
So, $(\mathcal{A}_1) - (\mathbf{A}_5)$ and $(\mathbf{Z}_1) - (\mathbf{Z}_3)$ hold and we may infer from Proposition (5) that $\lim\limits_{n\to\infty} \int_\Omega a(x,\, u_k,\, Du_k) : Dw(x)\, dx = \int_\Omega a(x,\, u,\, Du) : Dw(x) dx \quad \forall w \in \bigcup\limits_{k=1}^{\infty} X_k.$
Moreover, since $u_k \to u$ in measure for $k \to \infty$, we may infer that, after extraction of a suitable subsequence, if necessary, [13, Theorem 2.30] $u_k \to u$ almost everywhere for $k \to \infty$.

Thus, for arbitrary $w \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, it follows from the continuity conditions (B), $(\mathbf{F}_0)$ and $(\mathbf{G}_0)$ that $b(x,\, u_k,\, Du_k) \cdot w(x) \to b(x,\, u,\, Du) \cdot w(x)$, $f(x,\, u_k) \cdot w(x) \to f(x,\, u) \cdot w(x)$ and $g(x,\, u_k) : Dw(x) \to g(x,\, u) : Dw(x)$ almost everywhere.
Since, by the growth conditions (B(i)), $(\mathbf{F}_1)$ and $(\mathbf{G}_1)$ and the uniform bound (24), $b(x,\, u_k,\, Du_k) \cdot w(x)$, $f(x,\, u_k) \cdot w(x)$ and $h(x,\, u_k) : Dw(x)$ are equiintegrable (see the proof of Lemma (7)), it follows that $b(x,\, u_k,\, Du_k) \cdot w(x) \to b(x,\, u,\, Du) \cdot w(x)$, $f(x,\, u_k) \cdot w(x) \to f(x,\, u) \cdot w(x)$ and $h(x,\, u_k) : Dw(x) \to h(x,\, u) : Dw(x)$ in $L^1(\Omega)$ by the Vitali convergence Theorem. This implies that

$$\lim_{k\to\infty} \int_\Omega b(x,\, u_k(x)) \cdot w(x) dx = \int_\Omega b(x,\, u(x)) \cdot w(x) dx \quad \forall\, w \in \bigcup_{k=1}^{\infty} X_k,$$

$$\lim_{k\to\infty} \int_\Omega f(x,\, u_k(x)) \cdot w(x) dx = \int_\Omega f(x,\, u(x)) \cdot w(x) dx \quad \forall\, w \in \bigcup_{k=1}^{\infty} X_k$$

and

$$\lim_{k\to\infty} \int_\Omega h(x,\, u_k(x)) : Dw(x) dx = \int_\Omega h(x,\, u(x)) : Dw(x) dx \quad \forall w \in \bigcup_{k=1}^{\infty} X_k.$$

Since $\bigcup\limits_{k=1}^{\infty} X_k$ is dense in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, then u is a weak solution of (1).

ACKNOWLEDGEMENTS

The authors would like to thank the referees for the useful comments and suggestions that substantially helped improved the quality of the paper

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