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Solution of Non-homogeneous Wave Equation in Extended Colombeau Algebras

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Abstract

In this paper, we are interested to study the non-homogeneous wave equation in Colombeau algebra, we give a result of existence and uniqueness of generalized solution with initial data are distributions. Then we study the association concept with the classical solution.
Keywords: Colombeau algebra, non-homogeneous wave equation, Generalized solution, association.

1. INTRODUCTION

The algebras of Colombeau are constructed by J. F. Colombeau [5] [6], as factor algebras of infinite powers of the space C^∞ modulo a particular class of ideals. Enjoying a list of optimal properties, These algebras contain the space of distributions D' as a subspace with an embedding realized through convolution with a suitable mollifier. Elements of these algebras are classes of nets of smooth functions. This theory was been used for solving the linear and nonlinear partial differential equations with singularities, and in the last few years it was developed and also applied in different domains [5].

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On the other hand these problems have been studied by some authors [2], [1], [11], [12] in different cases, for example M. Oberguggenberger and Y.G. Wang, studied the Delta-waves for semi linear hyperbolic Cauchy problems [8], also Nonlinear stochastic wave equations have been studied by M. Oberguggenberger and F. Russo [9].
The wave equation is a partial differential equation used to describe mechanical waves (water waves, sound waves, and seismic waves) or electromagnetic waves (including light waves). It may be found in domains like as acoustics, electromagnetic, and fluid dynamics. The one-way wave equation may also be used to describe a single wave moving in a specific direction.
In our case, we studied the following non-homogeneous wave equation in Colombeau algebra with initial data are distributions (singular).

$$\begin{cases} \frac{d^2}{dt^2}u(t,x) - c^2\frac{d^2}{dx^2}u(t,x) = F(t,u(t,x)) \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ u(0,x) = a(x) \\ \partial_t u(0,x) = b(x) \end{cases} \tag{1}$$

With $a, b \in D'(\mathbb{R})$.
In the first part of this paper, we introduce Colombeau algebras and provide some properties and tools, in the second part, we study the existence and uniqueness of generalized solutions of non-homogeneous wave equations with singular initial data, and finally, we prove the association of generalized solutions with classical solutions.

2. PRELIMINARIES

In this section we will introduce basic notations and definitions from Colombeau theory. For $q \in \mathbb{N}_0$ with $\mathbb{N}_0 = \mathbf{N} \cup 0$.

$$\mathcal{A}_q = \left\{ \varphi \in \mathcal{D}(\mathbb{R}^n) / \int_{\mathbb{R}^n} \varphi(x)dx = 1 \quad and \int_{\mathbb{R}^n} x^\alpha \varphi(x)dx = 0 \text{ for } 1 \leq \alpha \leq q \right\}$$

$$q = 1, 2, \ldots$$

where $D(\mathbb{R}^n)$ is the space of all C^α functions $\varphi : \mathbb{R}^n \to C$ with compact support.
The elements of the set $\mathcal{A}_q$ are called test functions.
It is simple too see that $A1 \supset A2 \supset A3 \ldots$. Also, $A_i \neq \varnothing$; for all $i \in \mathbb{N}$,

$$\varphi_\epsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right) \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^n)$$

We denote by:
$\mathcal{E}(\mathbb{R}^n) = \{u : \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C}/$ with $u(\varphi, x)$ is $\mathcal{C}^\infty$ to the second variable $x\}$
$u(x, \varphi_\varepsilon) = u_\varepsilon(x) \quad \forall \varphi \in \mathcal{A}_1$

$\mathcal{E}_M(\mathbb{R}^n) = \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n) / \forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N}$ such that $\sup_{x\in K} ||D^\alpha u_\varepsilon(x)|| = \mathcal{O}(\varepsilon^{-N})$ as $\varepsilon \to 0\}$
$\mathcal{N}(\mathbb{R}^n) = \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n) / \forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \forall p \in \mathbb{N}$ such that $\sup_{x\in K} ||D^\alpha u_\varepsilon(x)|| = \mathcal{O}(\varepsilon^p)$ as $\varepsilon \to 0\}$

The Colombeau algebra $\mathcal{G}(\mathbb{R}^n)$ contains the distributions space as subspace by the map : Where $*$ denotes the convolution product of two distributions and is given by:

$$u * \varphi = \langle u(y), \varphi(y-x) \rangle$$

We denote:

$$\begin{aligned}\mathcal{E}_M^e(\mathbb{R}) = \{&(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}) / \forall K \subset\subset \mathbb{R}, \forall \alpha \in \mathbb{R}_+ \cup \{0\}, \exists N \in \mathbb{N} \quad \text{such that} \\ &\sup_{x\in K} |D^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\} \\ \mathcal{N}^e(\mathbb{R}) = \{&(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}) / \forall K \subset\subset \mathbb{R}, \forall \alpha \in \mathbb{R}_+ \cup \{0\}, \forall p \in \mathbb{N} \quad \text{such that} \\ &\sup_{x\in K} |D^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{p}) \quad \text{as} \quad \varepsilon \to 0\}\end{aligned}$$

The extended Colombeau algebra of generalized functions is the factor set:

$$\mathcal{G}^e(\mathbb{R}) = \mathcal{E}_M^e(\mathbb{R}) / \mathcal{N}^e(\mathbb{R})$$

A fractional integral is defined by:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad \alpha > 0$$

The fractional derivative of order $\alpha > 0$ in the Caputo sense is defined by:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau) d\tau}{(t-\tau)^{\alpha+1-m}} \quad , \quad m-1 < \alpha < m$$

Let (f_ε) be a representative of $F \in \mathcal{G}$, then:

$$\begin{aligned} D^\alpha f_\varepsilon(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau \quad 0 < \alpha < 1 \\ \sup_{t\in[0,T]} |D^\alpha f_\varepsilon(t)| &\le \frac{1}{\Gamma(1-\alpha)} \sup_{t\in[0,T]} \left| \int_0^t \frac{f'(\tau) d\tau}{(t-\tau)^\alpha} \right| \\ &\le \frac{1}{\Gamma(1-\alpha)} \|f'\|_{L^\infty([0,T])} \sup_{t\in[0,T]} \int_0^t \frac{d\tau}{(t-\tau)^\alpha} d\tau \\ &\le \frac{1}{\Gamma(1-\alpha)} \varepsilon^{-N} \frac{T^{1-\alpha}}{1-\alpha} \\ &\le C_{\alpha,T} \varepsilon^{-N} \end{aligned}$$

In general [11], for $m-1<\alpha<m$

$$\begin{aligned}
\sup_{t\in[0,T]}|D^{\alpha}f_{\varepsilon}(t)| &\leq \frac{1}{\Gamma(m-\alpha)}\sup_{t\in[0,T]}\int_0^t \frac{\left|f^{(m)}(\tau)\right|}{(t-\tau)^{\alpha+1-m}}d\tau \\
&\leq \frac{1}{\Gamma(m-\alpha)}\left\|f^{(m)}\right\|_{L^{\infty}([0,T])}\sup_{t\in[0,T]}\int_0^t \frac{1}{(t-\tau)^{\alpha+1-m}}d\tau \\
&\leq \frac{1}{\Gamma(m-\alpha)}\varepsilon^{-N}\frac{T^{m-\alpha}}{m-\alpha} \\
&\leq C_{\alpha,T}\varepsilon^{-N}
\end{aligned}$$

The constant $C_{\alpha,T}$ depends on two parameters α and T.

Definition 1. *Let $G_1, G_2 \in \mathcal{G}(\mathbb{R}^n)$ and $G_{1,\varepsilon}, G_{2,z}$ their representatives respectively. We say that $G_1, G_2 \in \mathcal{G}(\mathbb{R}^n)$ are associated and we write $G_1 \approx G_2$, if for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}(G_{1,\varepsilon}-G_{2,\varepsilon})\,\varphi(x)dx = 0$$

3. MAIN RESULTS

This section is devoted to solving the non-homogeneous wave equation in in extended Colombeau algebra $\mathcal{G}(\mathbb{R}^+\times\mathbb{R})$. Recall first that if u is a classical solution of the following problem we consider the following problem:

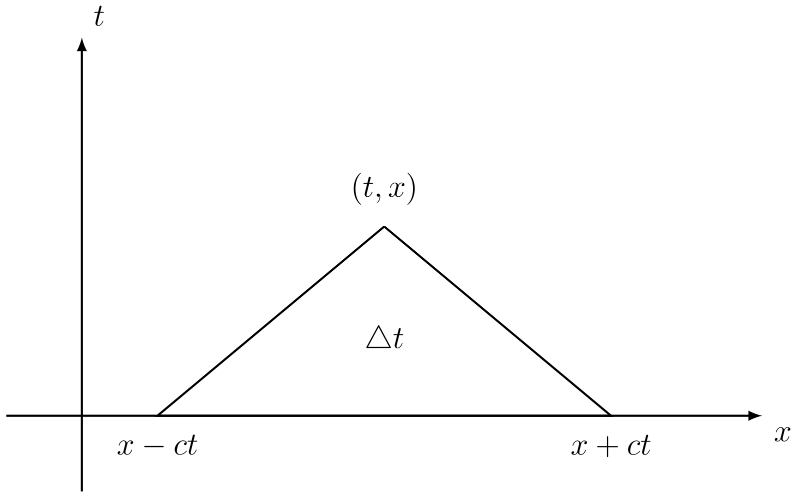
$$\begin{cases}
\dfrac{d^2}{dt^2}u(t,x)-c^2\dfrac{d^2}{dx^2}u(t,x)=F(t,u(t,x)) \quad x\in\mathbb{R}, \quad t\geqslant 0 \\
u(0,x)=a(x) \\
\partial_t u(0,x)=b(x)
\end{cases} \tag{2}$$

With $a, b \in D'(\mathbb{R})$.
Then the integral solution is:

$$u=\frac{1}{2}(a(x-ct)+a(x+ct))+\frac{1}{2c}\int_{x-ct}^{x+ct}b(y)dy+\frac{1}{2c}\int_0^t\int_{x-c(t+s)}^{x+c(t-s)}F(s,u(s,y))dyds. \tag{3}$$

We define the domains:

$$\Delta_s=\{(s,y)\in\mathbb{R}\times[0,\infty)/0\leq s\leq t, y\in I_s\}.$$
$$I_s=\{z\in\mathbb{R}/x-cs\leq z\leq x+cs\}.$$



Using (2), the following estimates are easily deduced $(0 \leq t \leq T)$

$$\|u\|_{L^\infty(\Delta_T)} \leq \|a\|_{L^\infty(I_0)} + T\|b\|_{L^\infty(I_0)} + T\int_0^T \|F(s, u(s, .))\|_{L^\infty(\Delta_s)} ds \qquad (4)$$

$$\|u(t, .)\|_{L^\infty(I_t)} \leq \|a\|_{L^\infty(I_0)} + T\|b\|_{L^\infty(I_0)} + T\int_0^T \|F(s, u(s, .))\|_{L^\infty(I_s)} ds \qquad (5)$$

Definition 2. *An element $F \in \mathcal{G}[\mathbb{R}^n]$ is L^∞ logarithmic type if it has a representative $(F_\varepsilon)_\varepsilon \in \mathcal{E}_M[\mathbb{R}^+ \times \mathbb{R}]$ such that*

$$\|F_\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} = \mathcal{O}(\log(\varepsilon)) \quad \textit{as} \quad \varepsilon \to 0$$

Theorem 1. *The regularization of equation (2), defined as :*

$$\begin{cases} \dfrac{d^2}{dt^2}u_\varepsilon(t, x) - c^2\dfrac{d^2}{dx^2}u_\varepsilon(t, x) = F_\varepsilon(t, u(t, x)) \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ u_\varepsilon(0, x) = a_\varepsilon(x) \\ \partial_t u_\varepsilon(0, x) = b_\varepsilon(x) \end{cases} \qquad (6)$$

where a_ε and b_ε are regularized of a and b, respectively, ∇F_ε is L^∞logarithmic type. Then the problem (6) has a unique solution in $\mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$.

Proof. **Existence :** To prove the existence of the solution, we transform the problem in the Colombeau algebra, then we obtain:

$$\begin{cases} \frac{d^2}{dt^2}u_\varepsilon(t, x) - c^2\frac{d^2}{dx^2}u_\varepsilon(t, x) = F_\varepsilon(t, u_\varepsilon(t, x)) \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ u_\varepsilon(0, x) = a_\varepsilon(x) \\ \partial_t u_\varepsilon(0, x) = b_\varepsilon(x) \end{cases} \qquad (7)$$

With $a_\varepsilon, b_\varepsilon$ and F_ε are the Representative of a, b, F respectively.

The integral solution is :
$u_\varepsilon(t,x) = \frac{1}{2}\left(a_\varepsilon(x-ct)+a_\varepsilon(x+ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct} b_\varepsilon(y)dy+\frac{1}{2c}\int_0^t\int_{x-c(t+s)}^{x+c(t-s)} F_\varepsilon\left(s,u_\varepsilon(s,y)\right)dyds$
We apply the estimate (4) successively to all the derivatives,

$$\|u\|_{L^\infty(\Delta_T)} \leq \|a\|_{L^\infty(I_0)} + T\|b\|_{L^\infty(I_0)} + T\int_0^T \|F(s,u(s,.))\|_{L^\infty(\Delta_s)}ds$$

The first approximation of F_ε :

$$F_\varepsilon\left(s,u_\varepsilon(s,.)\right) = F_\varepsilon(s,0) + ||\nabla F_\varepsilon||u_\varepsilon(s,.) + N_\varepsilon(s,.)$$

with $N_\varepsilon \in \mathcal{N}\left(\mathbb{R}^+\times\mathbb{R}\right)$ Then,

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty(\Delta_T)} \leq{}& \|a_\varepsilon\|_{L^\infty(I_0)} + T\left\|b_\varepsilon\right\|_{L^\infty(I_0)}\\ &+T\int_0^T\left[||F_\varepsilon(.,0)||_{L^\infty([0,T])} + ||\nabla F_\varepsilon||||u_\varepsilon(s,.)||_L||_{L^\infty(\Delta_T)}\right]ds\end{aligned}$$

Therefore,

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty(\Delta_T)} \leq{}& \|a_\varepsilon\|_{L^\infty(I_0)} + T\left\|b_\varepsilon\right\|_{L^\infty(I_0)} + T^2\left\|F_\varepsilon(.,0)\right\|_{L^\infty([0,T])} + T^2\left\|N_\varepsilon\right\|_{L^\infty(\Delta_T)}\\ &+T\int_0^T ||\nabla F_\varepsilon||||u_\varepsilon||_{L^\infty(\Delta_s)}ds\end{aligned}$$

By the Granwall's inequality, we have

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty(\Delta_T)} \leq{}& \Big(\|a_\varepsilon\|_{L^\infty(I_0)} + T\left\|b_\varepsilon\right\|_{L^\infty(I_0)} + T^2\left\|F_\varepsilon(.,0)\right\|_{L^\infty([0,T])}\\ &+T^2||N_\varepsilon||_{L^\infty(\Delta_T)}\Big)\times\exp\left(T^2||\nabla F_\varepsilon||\right)\end{aligned}$$

As $a\in\mathcal{G}(\mathbb{R}), b\in\mathcal{G}(\mathbb{R})$ and ∇F is $L^\infty - \log type$ there exist $M\in\mathbb{N}$ such that

$$\|u_\varepsilon\|_{L^\infty(\Delta_T)} = \mathcal{O}\left(\varepsilon^{-M}\right) \quad \text{as} \quad \varepsilon\to 0$$

Uniqueness:
To prove uniqueness, we consider representatives $u_\varepsilon, v_\varepsilon \in \mathcal{E}\left(\mathbb{R}^+\times\mathbb{R}\right)$ of two solutions u and v. Their difference satisfies

$$\begin{cases}\frac{d^2}{dt^2}\left(u_\varepsilon(t,x)-v_\varepsilon(t,x)\right)-c^2\frac{d^2}{dt^2}\left((u_\varepsilon(t,x)-v_\varepsilon(t,x)\right)=F_\varepsilon\left(t,u_\varepsilon(t,x)\right)\\ \qquad -F_\varepsilon\left(t,v_\varepsilon(t,x)\right)+n_\epsilon(t,x)\quad x\in\mathbb{R}, t\geqslant 0\\ u_\varepsilon(0,x)-v_\varepsilon(0,x)=n_{1,\varepsilon}(x)\\ \partial_t u_\varepsilon(0,x)-\partial_t v_\varepsilon(0,x)=n_{2,\varepsilon}(x)\end{cases}$$

The integral solution is:

$$u_\varepsilon(x,t)-v_\varepsilon(t,x)=\frac{1}{2}\left(n_{1,\varepsilon}(x-ct)+n_{1,\varepsilon}(x+ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}n_{2,\varepsilon}(y)dy$$
$$+\frac{1}{2c}\int_0^t\int_{x-c(t+s)}^{x+c(t-s)}F_\varepsilon\left(t,u_\varepsilon(t,x)\right)-F_\varepsilon\left(t,v_\varepsilon(t,x)\right)+n_\varepsilon(s,x)dyds$$

$\|u_\varepsilon-v_\varepsilon\|_{L^\infty(\Delta_T)} \leq \|n_{1,\varepsilon}\|_{L^\infty(I_0)} + T\|n_{2,\varepsilon}\|_{L^\infty(I_0)} + T\int_0^T\|F_\varepsilon\left(s,u_\varepsilon(s,.)\right)-F_\varepsilon\left(s,v_\varepsilon(s,.)\right)-n_\varepsilon(s,.)\|_{L^\infty(\Delta_s)}ds$

The first approximation of F_ε :

$$F_\varepsilon(s,u_\varepsilon(s,.))=F_\varepsilon(s,0)+\|\nabla F_\varepsilon\|u_\varepsilon(s,.)+N_{1,\varepsilon}(s,.)$$

$$F_\varepsilon\left(s,v_\varepsilon(s,.)\right)=F_\varepsilon(s,0)+\|\nabla F_\varepsilon\|v_\varepsilon(s,.)+N_{2,\varepsilon}(s,.)$$

with $N_1,N_2\in\mathcal{N}\left(\mathbb{R}\times\mathbb{R}^+\right)$
Then there exist $N_3\in\mathcal{N}\left(\mathbb{R}\times\mathbb{R}^+\right)$such that,
$\|u_\varepsilon-v_\varepsilon\|_{L^\infty(\Delta_T)}\leq\|n_{1,\varepsilon}\|_{L^\infty(I_0)}+T\|n_{2,\varepsilon}\|_{L^\infty(I_0)}+T\int_0^T\|\nabla F_\varepsilon\|\left(u_\varepsilon(s,.)-v_\varepsilon(s,.)\right)+N_{3,\varepsilon}(s,.)\|_{L^\infty(\Delta_s)}ds$
So,

$$\|u_\varepsilon-v_\varepsilon\|_{L^\infty(\Delta_T)}\leq\|n_{1,\varepsilon}\|_{L^\infty(I_0)}+T\|n_{2,\varepsilon}\|_{L^\infty(I_0)}$$
$$+T\int_0^T\|\|\nabla F_\varepsilon\|\left(u_\varepsilon(s,.)-v_\varepsilon(s,.)\right)\|_{L^\infty(\Delta_s)}$$
$$+T\int_0^T\|N_{3,\varepsilon}(s,.)\|_{L^\infty(\Delta_s)}ds$$

Then,

$$\|u_\varepsilon-v_\varepsilon\|_{L^\infty(\Delta_T)}\leq\|n_{1,\varepsilon}\|_{L^\infty(I_0)}+T\|n_{2,\varepsilon}\|_{L^\infty(I_0)}+T^2\|N_{3,\varepsilon}\|_{L^\infty(\Delta_T)}$$
$$+T\int_0^T\|\|\nabla F_\varepsilon\|\left(u_\varepsilon(s,.)-v_\varepsilon(s,.)\right)\|_{L^\infty(\Delta_s)}$$

we apply Granwall's inequality on the function $s\mapsto\|u_\varepsilon-v_\varepsilon\|_{L^\infty(\Delta_s)}$ we obtain :
$\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Delta_T)} \leq \left(\|n_{1,\varepsilon}\|_{L^\infty(I_0)}+T\|n_{2,\varepsilon}\|_{L^\infty(I_0)}+T^2\|N_{3,\varepsilon}\|_{L^\infty(\Delta_T)}\right)\times\exp\left(T\int_0^T\|\nabla F_\varepsilon\|ds\right)$
As $N_1,N_2\in\mathcal{N}(\mathbb{R}), N_3\in\mathcal{N}\left(\mathbb{R}^+\times\mathbb{R}\right)$ and ∇F is $L^\infty-$ logtype

$$\|u_\varepsilon-v_\varepsilon\|_{L^\infty(\Delta_T)}=\mathcal{O}\left(\varepsilon^q\right)\quad\text{as}\quad\varepsilon\to0\quad\forall q$$

Then the problem have a unique solution in $\mathcal{G}\left(\mathbb{R}^+\times\mathbb{R}\right)$. □

Theorem 2. *The regularization of equation (2), defined as :*

$$\begin{cases} \frac{d^2}{dt^2}u_\varepsilon(t,x) - c^2\frac{d^2}{dx^2}u_\varepsilon(t,x) = F_\varepsilon(t,u(t,x)) \quad x\in\mathbb{R}, \quad t\geqslant 0 \\ u_\varepsilon(0,x) = a_\varepsilon(x) \\ \partial_t u_\varepsilon(0,x) = b_\varepsilon(x) \end{cases} \tag{8}$$

where a_ε and b_ε are regularized of a and b, respectively, ∇F_ε is L^∞logarithmic type. Then the problem (8) has a unique solution in extended colombeau algebras $\mathcal{G}^e\left(\mathbb{R}^+\times\mathbb{R}^n\right)$.

Proof. We shall prove only the fractional part since the entire part is already proved in theorem(1).
Consider the caputo fractional derivative $D^\alpha; 0<\alpha<1$, without loss of generality. The same holds for $m-1<\alpha<m; m\in\mathbb{N}$.
Take the fractional derivative to the spatial variable to equation (3):

$$D^\alpha u_\varepsilon = \frac{1}{2}\left(D^\alpha a_\varepsilon(x-ct) + D^\alpha a_\varepsilon(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} D^\alpha b_\varepsilon(y)dy + \tag{9}$$

$\frac{1}{2c}\int_0^t \int_{x-c(t+s)}^{x+c(t-s)} D^\alpha F_\varepsilon\left(s,u_\varepsilon(s,y)\right)dyds$

We define the domains:

$$\Delta_s = \left\{(s,y)\in\mathbb{R}\times[0,\infty)/0\leq s\leq t, y\in I_s\right\}. \tag{10}$$

$$I_s = \{z\in\mathbb{R}/x-cs\leq z\leq x+cs\}. \tag{11}$$

Using (9), the following estimates are easily deduced $(0\leq t\leq T)$

$$\|D^\alpha u\|_{L^\infty(\Delta_T)} \leq \|D^\alpha a\|_{L^\infty(I_0)} + T\|D^\alpha b\|_{L^\infty(I_0)} + T\int_0^T \|D^\alpha F(s,u(s,.))\|_{L^\infty(\Delta_s)}ds \tag{12}$$

$$\|D^\alpha u(t,.)\|_{L^\infty(I_t)} \leq \|D^\alpha a\|_{L^\infty(I_0)} + T\|D^\alpha b\|_{L^\infty(I_0)} + T\int_0^T \|D^\alpha F(s,u(s,.))\|_{L^\infty(I_s)}ds \tag{13}$$

By the Granwall's inequality, we have:

$$\|D^\alpha u_\varepsilon\|_{L^\infty(\Delta_T)} \leq \Big(\|D^\alpha a_\varepsilon\|_{L^\infty(I_0)} + T\,\|D^\alpha b_\varepsilon\|_{L^\infty(I_0)} + +T^2||N_\varepsilon||_{L^\infty(\Delta_T)}\Big)) \tag{14}$$

$$\times\exp\left(T^2||\nabla F_\varepsilon||\right)$$

As $a_\varepsilon \in \mathcal{G}(\mathbb{R}), b_\varepsilon \in \mathcal{G}(\mathbb{R})$ and ∇F is $L^\infty - \log type$ there exist $M \in \mathbb{N}$ such that

$$\|D^\alpha u_\varepsilon\|_{L^\infty(\Delta_T)} = \mathcal{O}\left(\varepsilon^{-M}\right) \quad \text{as} \quad \varepsilon \to 0 \tag{15}$$

Then

$$u \in \mathcal{G}^e(\Omega)$$

□

4. ASSOCIATION WITH CLASSICAL SOLUTION

Let v the classical solution to

$$\begin{cases} \dfrac{d^2}{dt^2}v(t,x) - c^2\dfrac{d^2}{dx^2}v(t,x) = 0 \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ v(0,x) = a(x) \\ \partial_t v(0,x) = b(x) \end{cases}$$

And w the classical solution to

$$\begin{cases} \dfrac{d^2}{dt^2}w(t,x) - c^2\dfrac{d^2}{dx^2}w(t,x) = F(w+v) \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ w(0,x) = 0 \\ \partial_t w(0,x) = 0 \end{cases}$$

Proposition 1. *The generalized solution of (2) is associated with $v + w$.*

Proof. Let v_ϵ the solution to

$$\begin{cases} \dfrac{d^2}{dt^2}v_\varepsilon(t,x) - c^2\dfrac{d^2}{dx^2}v_\varepsilon(t,x) = 0 \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ v_\varepsilon(0,x) = a_\varepsilon(x) \\ \partial_t v_\varepsilon(0,x) = b_\varepsilon(x) \end{cases}$$

w_ϵ the solution to

$$\begin{cases} \dfrac{d^2}{dt^2}w_\epsilon(t,x) - c^2\dfrac{d^2}{dx^2}w_\epsilon(t,x) = F_\epsilon\left(w_\epsilon + v_\epsilon\right) \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ w_\epsilon(0,x) = 0 \\ \partial_t w_\epsilon(0,x) = 0 \end{cases}$$

And u_ϵ the solution to

$$\begin{cases} \dfrac{d^2}{dt^2}u_\varepsilon(t,x) - c^2\dfrac{d^2}{dx^2}u_\varepsilon(t,x) = F_\varepsilon\left(u_\varepsilon(t,x)\right) \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ u_\varepsilon(0,x) = a_\varepsilon(x) \\ \partial_t u_\varepsilon(0,x) = b_\varepsilon(x) \end{cases}$$

Then, we have

$$\begin{cases} \left(\frac{d^2}{dt^2} - c^2\frac{d^2}{dx^2}\right)(u_\epsilon(t,x) - v_\epsilon(t,x) - w_\varepsilon(t,x)) = F_\epsilon\left(t, u_\epsilon(t,x)\right) \\ -F_\epsilon\left(t, w_\epsilon(t,x) + v_\epsilon(t,x)\right) + n_\epsilon(t,x) \quad x \in \mathbb{R}, \quad t \geqslant 0 \\ \left(u_\epsilon - v_\epsilon - w_\epsilon\right)(0,x) = n_{1,\epsilon}(x) \\ \partial_t\left(u_\epsilon - v_\epsilon - w_\epsilon\right)(0,x) = n_{2,\epsilon}(x) \end{cases}$$

With $n_1, n_2 \in \mathcal{N}(\mathbb{R})$

The integral solution is:

$$\begin{aligned} u_\varepsilon(x,t) - v_\varepsilon(t,x) w_\varepsilon(t,x) =& \frac{1}{2}\left(n_{1,\varepsilon}(x-ct) + n_{1,\varepsilon}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} n_{2,\varepsilon}(y)dy \\ &+ \frac{1}{2c}\int_0^t \int_{x-c(t+s)}^{x+c(t-s)} F_\varepsilon\left(t, u_\varepsilon(t,x)\right) - F_\varepsilon(t, w_\varepsilon(t,x) \\ &+ v_\varepsilon(t,x)) + n_\varepsilon(t,x) dy ds \end{aligned}$$

Using (4) we have,

$$\begin{aligned} &\left\|u_\varepsilon - v_\varepsilon - w_\varepsilon\right\|_{L^\infty(\Delta_T)\leq} \left\|n_{1,\varepsilon}\right\|_{L^\infty(I_0)} + T\left\|n_{2,\varepsilon}\right\|_{L^\infty(I_0)} \\ &+ T\int_0^T \left\|F_\varepsilon\left(s, u_\varepsilon(s,.)\right) - F_\varepsilon\left(s, w_\varepsilon(s,.) + v_\varepsilon(s,.)\right) - n_\varepsilon(s,.)\right\|_{L^\infty(\Delta_s)} ds \end{aligned}$$

The first approximation of F_ε :

$$\begin{aligned} F_\varepsilon\left(s, u_\varepsilon(s,.)\right) &= F_\varepsilon(s,0) + \|\nabla F_\varepsilon\| u_\varepsilon(s,.) + N_{1,\varepsilon}(s,.) \\ F_\varepsilon\left(s, v_\varepsilon(s,.) + w_\varepsilon(s,.)\right) &= F_\varepsilon(s,0) + \|\nabla F_\varepsilon\|\left(v_\varepsilon(s,.) + w_\varepsilon(s,.)\right) + N_{2,\varepsilon}(s,.) \end{aligned}$$

with $N_1, N_2 \in \mathcal{N}\left(\mathbb{R}\times\mathbb{R}^+\right)$
Then there exist $N_3 \in \mathcal{N}\left(\mathbb{R}\times\mathbb{R}^+\right)$such that,

$$\begin{aligned} &\left\|u_\varepsilon - v_\varepsilon - w_\varepsilon\right\|_{L^\infty(\Delta_T)} \leq \left\|n_{1,\varepsilon}\right\|_{L^\infty(I_0)} + T\left\|n_{2,\varepsilon}\right\|_{L^\infty(I_0)} \\ &+ T\int_0^T \|\nabla F_\varepsilon\|\left(u_\varepsilon(s,.) - v_\varepsilon(s,.) - w_\varepsilon(s,.)\right) + N_{3,\varepsilon}(s,.)\|_{L^\infty(\Delta_s)} ds \end{aligned}$$

So,

$$\begin{aligned} \left\|u_\varepsilon - v_\varepsilon - w_\varepsilon\right\|_{L^\infty(\Delta_T)} \leq& \|n_{1,\varepsilon}\|_{L^\infty(I_0)} + T\|n_{2,\varepsilon}\|_{L^\infty(I_0)} \\ &+ T\int_0^T \|\|\nabla F_\varepsilon\|\left(u_\varepsilon(s,.) - v_\varepsilon(s,.) - w_\varepsilon(s,.)\right)\|_{L^\infty(\Delta_s)} \\ &+ T\int_0^T \|N_{3,\varepsilon}(s,.)\|_{L^\infty(\Delta_s)} ds \end{aligned}$$

Then,

$$\|u_\varepsilon - v_\varepsilon - w_\varepsilon\|_{L^\infty(\Delta_T)} \leq \|n_{1,\varepsilon}\|_{L^\infty(I_0)} + T\,\|n_{2,\varepsilon}\|_{L^\infty(I_0)} + T^2\,\|N_{3,\varepsilon}\|_{L^\infty(\Delta_T)} + T\int_0^T \|\|\nabla F_\varepsilon\|\,(u_\varepsilon(s,.) - v_\varepsilon(s,.) - w_\varepsilon(s,.))\,\|_{L^\infty(\Delta_s)}$$

We apply Granwall's inequality
$\|u_\varepsilon - v_\varepsilon - w_\varepsilon\|_{L^\infty(\Delta_T)} \leq \left(\|n_{1,\varepsilon}\|_{L^\infty(I_0)} + T\|n_{2,\varepsilon}\|_{L^\infty(I_0)} + T^2\|N_{3,\varepsilon}\|_{L^\infty(\Delta_T)}\right) \times \exp\left(T\int_0^T \|\nabla F_\varepsilon\| ds\right)$
As $n_1, n_2 \in \mathcal{N}(\mathbb{R}), N_3 \in \mathcal{N}\left(\mathbb{R}^+ \times \mathbb{R}\right)$ and ∇F is $L^\infty-$ logtype

$$\|u_\varepsilon - v_\varepsilon - w_\varepsilon\|_{L^\infty(\Delta_T)} = \mathcal{O}\left(\varepsilon^q\right) \quad \text{as} \quad \varepsilon \to 0 \quad \forall q$$

Then, u is associated to $w + v$ □

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