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Symmetries, Conservation Laws and Invariant Solutions on One-Dimensional Gas Dynamics in Lagrangian Coordinates

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Abstract

The one-dimensional gas dynamics equations in Lagrangian coordinates are studied in this paper. The advantage of studying in Lagrangian coordinates makes it possible to reduce these flows to a variational Euler-Lagrange equation with a suitable Lagrangian. Group classification of the variational Euler-Lagrange equation with respect to a pressure function in Lagrangian coordinates is performed. Applying Noether's theorem with the found Lagrangian and the admitted generators, conservation laws are constructed and some exact invariant solutions are also presented.

Keywords: Gas Dynamics equations**,** Lie point symmetries, Euler-Lagrange equation, Noether's Theorem, Invariant solutions

1. INTRODUCTION

There are two ways to describe the model phenomena in continuum mechanics. One is called Eulerian description, where focuses on specific locations in the space through which the flows as time passes. It studies individual spatial positions, regardless of what particles reach those positions at a given instant of time. The other is Lagrangian description, where all particles are identified by the positions which they occupy at some initial time. It describes as individual particles are "marked" and their positions, velocities, etc. are described as a function of time. Typically, Lagrangian coordinates are not applied in fluid motion because the experiment data are simply presented in Eulerian coordinates. In the Lagrangian description can be twisted, destroyed and it leads to a cumbersome analysis. However, in some special contexts the Lagrangian description is indeed useful in solving certain problems. In this paper gives one of such demonstrations, we studied on the one-dimensional of gas dynamics equations in Lagrangian coordinates.

In the present paper we consider a class of dispersive models [1]:

$$\dot{\rho} + \rho\, div(u) = 0, \qquad \rho\dot{u} + \nabla p = 0, \qquad \dot{S} = 0,$$
$$p = \rho\frac{\delta W}{\delta\rho} - W = \rho\left(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial\dot{\rho}}\right) - div\left(\frac{\partial W}{\partial\dot{\rho}}u\right)\right) - W, \tag{1}$$

here t is time, ∇ is the gradient operator with respect to the space variables, ρ is the fluid density, u is the velocity field, p is the pressure, S is the entropy and $W(\rho,\dot{\rho},S)$ is a given potential, the "dot" denotes the material time derivative: $\dot{f} = \frac{df}{dt} = f_t + u\nabla f$, and $\frac{\delta W}{\delta\rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u. The class of dispersive models (1) is an example of medium whose behavior depends not only on thermodynamic variables, but also on their derivatives with respect to space and time.

The model (1) were derived in [1] using the Lagrangian

$$L(t,u,\rho,S,\dot{\rho}) = \rho u^2/2 - W(\rho,\dot{\rho},S). \tag{2}$$

One of such models include the Green-Naghdi equations [2] correspond to the potential $W(\rho,\dot{\rho}) = \gamma_1\rho^2 - \gamma\rho\dot{\rho}$.

Moreover the Green-Naghdi model with this potential function in Lagrangian coordinates were studied in [3]. In particular condition $\gamma = 0$ is determined as a potential for the classical hyperbolic shallow water equations and for the gas dynamics equations can be considered in a particular case of the model (1) with the potential function $W = W(\rho, S)$.

1.1 Eulerian and Lagrangian coordinates

The Lagrangian coordinates (t, x_0) and the Eulerian coordinates (t, x) are related by the condition $x = \varphi(t, x_0)$, where the function $\varphi(t, x_0)$ is the solution of the Cauchy problem

$$\varphi_t(t, x_0) = u(t, \varphi(t, x_0)), \; \varphi(t_0, x_0) = x_0.$$

In Lagrangian coordinates, the general solution of the mass conservation law equation is $\rho(t, \varphi(t, x_0)) = \frac{\rho_0(x_0)}{\varphi_{x_0}(t, x_0)}$,

where $\rho_0(x_0)$ is an arbitrary function of integration. Without loss of generally one can assume that $\rho_0 = 1$. Using the change

$$\xi = \alpha(x_0), \tag{3}$$

where $\alpha'(x_0) = \rho_0(x_0)$, one obtains that $\tilde{\rho}(t,\xi) = \frac{1}{\tilde{\varphi}_\xi(t,\xi)}$. Here the functions $\tilde{\varphi}(t,\xi)$ and $\varphi(t, x_0)$ are related by the formula[1]:

[1]The sign ~ is further omitted.

$\tilde{\varphi}(t, \alpha(x_0)) = \varphi(t, x_0)$.

In Lagrangian coordinates the change (3) defines an equivalence transformation: this change simplifies the equations studied. As the variable ξ is related with $\rho_0(x_0)$, it is called the mass Lagrangian coordinate [4].

1.2 Symmetries and conservation laws

One of the tools for studying symmetries is the group analysis method, which is a basic method for constructing exact solutions of partial differential equations [5, 6, 7, 8]. Moreover the knowledge of an admitted Lie group allows us to derive conservation laws. Conservation laws provide information on the basic properties of solutions of differential equations, and they are needed in the analyses of stability and global behavior of solutions. Noether's theorem is a fundamental theorem that provides a connection between the symmetries of a physical system with a Lagrangian and the conservation laws for the associated Euler-Lagrange equations [9]. The one-dimensional gas dynamics equations in general case possess the well-known conservation laws: mass, linear and angular momentum and energy conservation laws. The gas dynamic equations still attract the attention of researchers to derive conservation laws by applying a variety of approaches in combination with Noether's theorem [10, 11, 12, 13].

1.3 Objectives of the present paper

This present paper considered on the one-dimensional gas dynamics equations [1], the Eulerian coordinates of these equations are presented as:

$$\rho(u_t + uu_x) + p_x = 0, \quad \rho_t + u\rho_x + \rho u_x = 0, \quad S_t + uS_x = 0, \tag{4}$$

where $p(\rho, S) = \rho W_\rho - W$ and $W = W(\rho, S)$. The equation for the function $x = \varphi(t, \xi)$ relating the Eulerian coordinate x and the Lagrangian coordinate ξ, we then obtained the Lagrangian in Lagrangian coordinates:

$$L(t, \xi, \varphi, \rho_0, \varphi_t, \varphi_\xi, S_0(\xi)) = \rho_0 \varphi_t^2/2 - \varphi_\xi W\left(\rho_0 \varphi_\xi^{-1}, S_0(\xi)\right). \tag{5}$$

The Euler-Lagrange equation corresponding to this Lagrangian can be derived by applying the variational principle:

$$\frac{\delta L}{\delta \varphi} = 0, \tag{6}$$

where the variational derivatives has the form:

$$\frac{\delta}{\delta\varphi} = \frac{\partial}{\partial\varphi} - D_t \frac{\partial}{\partial\varphi_t} - D_\xi \frac{\partial}{\partial\varphi_\xi} + D_t^2 \frac{\partial}{\partial\varphi_{tt}} + D_t D_\xi \frac{\partial}{\partial\varphi_{t\xi}} + D_\xi^2 \frac{\partial}{\partial\varphi_{\xi\xi}} + \dots$$

D_t and D_ξ are the total derivative with respect to the mass Lagrangian coordinates. The Euler-Lagrange equation (6) with the Lagrangian (5) is presented as

$$W_{\rho\xi}\varphi_\xi^2 - W_{\rho\rho}\varphi_{\xi\xi} - W_\xi \varphi_\xi^3 + \varphi_{tt}\varphi_\xi^3 = 0, \tag{7}$$

since $p(\rho, S) = \rho W_\rho - W$, $W = W(\rho, \xi)$, and $\rho = \varphi_\xi^{-1}$, one finds $W_{\rho\rho} = p_\rho/\rho$ and $W_{\rho\xi} = \left(p_\xi + W_\xi\right)/\rho$. Substituting these expression into (7) and taking into account that $p = P(\xi, \varphi_\xi)$, one obtains equation

$$\varphi_{tt} + P_{\xi} + P_{\varphi_{\xi}}\varphi_{\xi\xi} = 0 \text{ or } \varphi_{tt} + D_{\xi}P = 0, \tag{8}$$

where P is a pressure function.

The model we study in this paper is the gas dynamics equations with a Lgrangian in Lagrangian coordinates having the form of Euler-Lagrange equation which is reduced to one dimensional wave equation (8) for arbitrary function of pressure function $P(\xi, \varphi_{\xi})$.

This paper is organized as follows. In Section 2 the background for applying symmetries to derive conservation laws is presented. A group classification of the Euler-Lagrange equation (8) with an arbitrary pressure function $P(\xi, \varphi_{\xi})$ and invariant solutions of Eqs. (8) are constructed in Section 3. By applying the Noether's theorem, the conservation laws are derived and presented in Section 4.

2. NOETHER'S THEOREM

The background related to the application of symmetries, Noether's Theorem[2], is presented here. Let the expression

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha}\frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha}\frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha}\frac{\partial}{\partial u_{i_1 i_2}^{\alpha}}...,$$

be the Lie-Backlund operator and $F = F(x, u, p)$. Here $x = (x^1, x^2, ..., x^n)$ are the independent variables, $u = (u^1, u^2, ..., u^m)$ are dependent variables, and p are the derivatives of the variables $u_{i_1,...,i_s}^{\alpha}$ up to finite order. Noether's theorem is based on 2 identities.

The first identity is called the Noether identity [14];

$$XF + FD_i(\xi^i) = W^{\alpha}\frac{\delta F}{\delta u^{\alpha}} + D_i(N^i F), \tag{9}$$

where

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} + \sum_{s\geq 1}(-1)^s D_{i_1}...D_{i_s}\frac{\partial}{\partial u_{i_1,...,i_s}^{\alpha}} \quad (\alpha = 1, 2, ..., m) \tag{10}$$

are variational derivatives. The functions $W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}$, $(\alpha = 1, 2, ..., m)$ are called the characteristic functions and the operator N^i is defined as

$$N^i F = \xi^i F + W^{\alpha}\frac{\delta F}{\delta u_i^{\alpha}} + \sum_{s=1} D_{i_1}...D_{i_s}(W^{\alpha})\frac{\delta F}{\delta u_{i i_1 i_2,...,i_s}^{\alpha}}, \ (i = 1, 2, ..., n).$$

Here the variational derivatives $\frac{\delta F}{\delta u_{i_1 i_2,...,i_s}^{\alpha}}$ are obtained from (10) by replacing u^{α} with the corresponding derivative $u_{i_1 i_2,...,i_s}^{\alpha}$. If the generator X such that

$$XF + FD_i(\xi^i) = D_i(B^i), \tag{11}$$

[2]Details on symmetries and conservation laws is referred to [14].

then this identity provides the conservation laws

$$D_i\left(N^i F - B^i\right) = 0$$

for the Euler-Lagrange equation

$$\frac{\delta F}{\delta u^\alpha} = 0\,, \quad (\alpha = 1, 2, ..., m). \tag{12}$$

The second identity is [14, 15]:

$$\frac{\delta}{\delta u^j}\left(XF + FD_i\xi^i - D_iB^i\right) = X\left(\frac{\delta F}{\delta u^j}\right) + \frac{\delta F}{\delta u^\alpha}\left(\frac{\partial \eta^\alpha}{\partial u^j} - \frac{\partial \xi^i}{\partial u^j}u_i^\alpha + \delta_{\alpha j}D_i\xi^i\right).$$

This identity gives the generator X satisfying (11) is an admitted generator of the Euler-Lagrange equations (12). The conservation laws are considered in the divergence form

$$D_tT^t + D_xT^x = 0\,.$$

For the choose of $B^i = 0, (i = 1, 2, ..., n)$, then the symmetry X is called a variational, otherwise it is called a divergent symmetry.

3. GROUP ANALYSIS OF THE EULER-LAGRANGE EQUATION

The purpose of this study is to construct conservation laws by the application of Noether's theorem to Eqs. (8), one needs to study its group properties. A symmetry X of Eqs. (8) is sought in the form

$$X = \xi^t\partial_t + \xi^\xi\partial_\xi + \eta^\varphi\partial_\varphi\,,$$

where the unknown coefficients $\xi^t(t,\xi,\varphi)$, $\xi^\xi(t,\xi,\varphi)$, and $\eta^\varphi(t,\xi,\varphi)$. Applying the prolonged generator to Eqs. (8), substituting the main derivatives and splitting with respect to parametric derivatives, one obtains the determining equation. Solving the determining equations with respect to the arbitrary pressure function $P = P(\xi,\varphi_\xi)$, where $P_\xi \neq 0, P_{\varphi_\xi} < 0$ and $P_{\varphi_\xi\varphi_\xi} \neq 0$, one finds the basis of the kernel of the admitted Lie algebras consists of the generators

$$X_1 = \partial_t\,, \quad X_2 = t\partial_\varphi\,, \quad X_3 = \partial_\varphi. \tag{13}$$

These generators are admitted for all functions $P = P(\xi,\varphi_\xi)$. Calculations found the extensions of the kernel which are summarized in **Table 1**.

TABLE 1. Group classification of Eqs. (8).

	$P(\xi,\varphi_\xi)$	**Extensions**	**Remarks**
M1	$\phi(\xi^{-\alpha}\varphi_\xi)\xi^{\alpha+2\gamma}+h(\xi)$ $h''(\xi)=(\alpha+2\gamma-1)h'(\xi)$	$(\gamma-1)t\partial_t-\xi\partial_\xi-(\alpha+1)\varphi\partial_\varphi$	$\gamma\neq 0,1$ $\alpha\neq -1$
M2	$\phi(e^{\alpha\xi}\varphi_\xi)e^{(2\beta-\alpha)\xi}+h(\xi)$ $h''(\xi)=(2\beta-\alpha)h'(\xi)$	$\beta t\partial_t-\partial_\xi+\alpha\varphi\partial_\varphi$	$\alpha\neq 0$
M3	$\phi(\varphi_\xi)+\beta\xi+\gamma\xi^2$	$\partial_\xi-\gamma t^2\partial_\varphi$	$\gamma,\beta\neq 0$
M4	$\phi(\varphi_\xi)$	$t\partial_t+\xi\partial_\xi+\varphi\partial_\varphi$, ∂_ξ	
M5	$e^{\beta\xi}\varphi_\xi^\gamma$	$(\gamma-1)t\partial_\varphi$ $(\gamma-1)\partial_\xi-\beta\varphi\partial_\varphi$	$\gamma\neq 1$ $\beta\neq 0$
M6	$b(\xi)\varphi_\xi^\gamma+k_1\xi^2$	$(2l-1)t\partial_t+2l\gamma\xi\partial_\xi+2(2l-1+l\gamma)\varphi\partial_\varphi$	$\gamma\neq 0,1$ $l,\beta,k_1\neq 0$
M7	$e^{\beta\xi}\varphi_\xi^\gamma+k_1\xi^2$	$-\beta t\partial_t+2\gamma\partial_\xi-2\beta\varphi\partial_\varphi$	$\gamma\neq 0,1$ $\beta,k_1\neq 0$
M8	$\beta\varphi_\xi^\gamma+k_1\xi^2$	$\partial_\xi-k_1t^2\partial_\varphi$ $t\partial_t+\gamma\xi\partial_\xi+(\gamma+2)\varphi\partial_\varphi$	$\gamma\neq 0,1,-2$ $\beta,k_1\neq 0$
M9	$\beta\varphi_\xi^\gamma$	$(\gamma-1)t\partial_t-2\varphi\partial_\varphi$, ∂_ξ $(\gamma-1)\xi\partial_\xi+(\gamma+1)\varphi\partial_\varphi$	$\gamma\neq 0,1$ $\beta\neq 0$
M10	$b(\xi)\varphi_\xi^{-3}$	$t^2\partial_t+t\varphi\partial_\varphi$ $2t\partial_t+\varphi\partial_\varphi$	

One of the advantages of the Group analysis is the possibility to find the solutions of the original differential equation by solving reduced equations. The reduced equations are obtained by introducing suitable new variables, determined as invariant functions with respect to the infinitesimal generators. Constructing of invariant solutions consists of some steps: choosing a subgroup of the admitted group, finding a representation of solution, substituting the representation into the studied equation and the analysis of compatibility is required for deriving the reduced equations. A survey of this method can be found in [5, 7].

The generators in **Table 1** are admitted by Eqs. (8) which corresponding to the gas dynamics equations with respect to any type of pressure functions $P=P(\xi,\varphi_\xi)$. Analysis of invariant solutions is presented in detail for some examples:

***Consider M_1*:**

The generator $t\partial_t+2\xi\partial_\xi+2\varphi\partial_\varphi$ and $P=\phi(\xi^{-\alpha}\varphi_\xi)\xi^{\alpha+2\gamma}+h(\xi)$.

Choosing conditions $\alpha=0$, $\gamma=\dfrac{1}{2}$, $\phi(\xi)=\xi$, and $h(\xi)=c_1+c_2\xi$ with $c_1=c_2=0$.

Then we can consider the pressure function as $P=\xi\varphi_\xi$ and Eqs. (8) becomes

$$\varphi_{tt} + \varphi_{\xi} + \xi\varphi_{\xi\xi} = 0 . \quad (14)$$

From the generator $t\partial_t + 2\xi\partial_{\xi} + 2\varphi\partial_{\varphi}$, solving the Cauchy problems

$$\frac{d\hat{t}}{da} = \hat{t} , \quad \hat{t}\big|_{a=0} = t , \ \frac{d\hat{\xi}}{da} = 2\hat{\xi} , \quad \hat{\xi}\big|_{a=0} = \xi , \ \frac{d\hat{\varphi}}{da} = 2\hat{\varphi} , \quad \hat{\varphi}\big|_{a=0} = \varphi ,$$

then a representation of the invariant solution on the parameter a with respect to the generator $t\partial_t + 2\xi\partial_{\xi} + 2\varphi\partial_{\varphi}$ has the following form

$$\varphi = f\left(\hat{t}, \hat{\xi}\right)e^{-2a}, \quad \hat{t} = te^{a} , \quad \hat{\xi} = \xi e^{2a} .$$

After substituting this representation of the invariant solution into Eqs. (14), the function $f\left(\hat{t}, \hat{\xi}\right)$ has to satisfy the equation

$$f_{\hat{t}\hat{t}} + f_{\hat{\xi}} + \hat{\xi} f_{\hat{\xi}\hat{\xi}} = 0 .$$

Solving the partial differential equation, one we found the general solution of the last equation as

$$f\left(\hat{t}, \hat{\xi}\right) = F_1\left(\hat{t}\right)F_2\left(\hat{\xi}\right)$$

where $F_1''\left(\hat{t}\right) = c_1 F_1\left(\hat{t}\right)$, $F_2''\left(\hat{\xi}\right) = \frac{1}{\hat{\xi}}\left(c_1 F_2\left(\hat{\xi}\right) - F_2'\left(\hat{\xi}\right)\right)$.

Therefore the invariant solution of Eqs. (8) which satisfied to $P = \xi\varphi_{\xi}$ is

$$\varphi = F_1\left(\hat{t}\right)F_2\left(\hat{\xi}\right)e^{-2a},$$

where $F_1''\left(\hat{t}\right) = c_1 F_1\left(\hat{t}\right)$, $F_2''\left(\hat{\xi}\right) = \frac{1}{\hat{\xi}}\left(c_1 F_2\left(\hat{\xi}\right) - F_2'\left(\hat{\xi}\right)\right)$, $\hat{t} = te^{a}$, $\hat{\xi} = \xi e^{2a}$ and c_1 is a constant.

***Consider M*2:**

The generator $t\partial_t - 2\partial_{\xi} + 2\varphi\partial_{\varphi}$ and $P = \phi\left(e^{\alpha\xi}\varphi_{\xi}\right)e^{(2\beta-\alpha)\xi} + h(\xi)$.

Choosing conditions $\alpha = 1$, $\beta = \frac{1}{2}$, $\phi(\xi) = \xi$, and $h(\xi) = c_1 + c_2\xi$ with $c_1 = c_2 = 0$.

Then we can consider the pressure function as $P = e^{\xi}\varphi_{\xi}$ and Eqs. (8) becomes

$$\varphi_{tt} + e^{\xi}\varphi_{\xi} + e^{\xi}\varphi_{\xi\xi} = 0 . \quad (15)$$

From the generator $t\partial_t - 2\partial_{\xi} + 2\varphi\partial_{\varphi}$, solving the Cauchy problems

$$\frac{d\hat{t}}{da} = \hat{t} , \quad \hat{t}\big|_{a=0} = t , \ \frac{d\hat{\xi}}{da} = -2 , \quad \hat{\xi}\big|_{a=0} = \xi , \ \frac{d\hat{\varphi}}{da} = 2\hat{\varphi} , \quad \hat{\varphi}\big|_{a=0} = \varphi ,$$

then a representation of the invariant solution on the parameter a with respect to the generator $t\partial_t - 2\partial_{\xi} + 2\varphi\partial_{\varphi}$ has the following form

$$\varphi = f\left(\hat{t}, \hat{\xi}\right)e^{-2a}, \quad \hat{t} = te^{a} , \quad \hat{\xi} = \xi - 2a .$$

After substituting this representation of the invariant solution into Eqs. (15), the function $f\left(\hat{t}, \hat{\xi}\right)$ has to satisfy the equation

$$f_{\hat{t}\hat{t}} + e^{\hat{\xi}} f_{\hat{\xi}} + e^{\hat{\xi}} f_{\hat{\xi}\hat{\xi}} = 0 .$$

Solving the partial differential equation, then the general solution of the last equation is
$f\left(\hat{t}, \hat{\xi}\right) = F_1\left(\hat{t}\right) F_2\left(\hat{\xi}\right)$
where $F_1''\left(\hat{t}\right) = c_1 F_1\left(\hat{t}\right)$, $F_2''\left(\hat{\xi}\right) = -c_1 e^{-\hat{\xi}} F_2\left(\hat{\xi}\right) - F_2'\left(\hat{\xi}\right)$.
Therefore the invariant solution of Eqs. (8) which satisfied to $P = e^{\xi}\varphi_{\xi}$ is

$$\varphi = F_1\left(\hat{t}\right) F_2\left(\hat{\xi}\right) e^{-2a},$$

where $F_1''\left(\hat{t}\right) = c_1 F_1\left(\hat{t}\right)$, $F_2''\left(\hat{\xi}\right) = -c_1 e^{-\hat{\xi}} F_2\left(\hat{\xi}\right) - F_2'\left(\hat{\xi}\right)$, $\hat{t} = te^{a}$, $\hat{\xi} = \xi - 2a$ and c_1 is a constant.

***Consider M*3:**
The generator $\partial_{\xi} - \gamma t^2 \partial_{\varphi}$ and $P = \phi\left(\varphi_{\xi}\right) + \beta\xi + \gamma\xi^2$ $\left(\alpha , \beta \neq 0\right)$,
then Eqs. (8) becomes

$$\varphi_{tt} + \left(\beta + 2\gamma\xi\right) + \varphi_{\xi\xi} = 0 . \tag{16}$$

From the generator $\partial_{\xi} - \gamma t^2 \partial_{\varphi}$, solving the Cauchy problems

$$\frac{d\hat{t}}{da} = 0 , \quad \hat{t}\Big|_{a=0} = t , \quad \frac{d\hat{\xi}}{da} = 1 , \quad \hat{\xi}\Big|_{a=0} = \xi , \quad \frac{d\hat{\varphi}}{da} = -\gamma t^2 , \quad \hat{\varphi}\Big|_{a=0} = \varphi ,$$

then a representation of the invariant solution on the parameter a with respect to the generator $\partial_{\xi} - \gamma t^2 \partial_{\varphi}$ has the following form

$$\varphi = f\left(\hat{t}, \hat{\xi}\right) + \gamma t^2 a, \quad \hat{t} = t , \quad \hat{\xi} = \xi + a .$$

After substituting this representation of the invariant solution into Eqs. (16), the function $f\left(\hat{t}, \hat{\xi}\right)$ has to satisfy the equation

$$f_{\hat{t}\hat{t}} + \beta\hat{\xi} + f_{\hat{\xi}\hat{\xi}} = 0 .$$

Solving the partial differential equation, one we found the general solution

$$f\left(\hat{t}, \hat{\xi}\right) = F_1\left(\hat{\xi} - c_1\hat{t}\right) + F_2\left(\hat{\xi} + c_1\hat{t}\right) - \frac{c_2\hat{t}^2\hat{\xi}}{2} ,$$

and c_1 , c_2 are constant.
Therefore the invariant solution of Eqs. (8) which satisfied to $P = \phi\left(\varphi_{\xi}\right) + \beta\xi + \gamma\xi^2$ is

$$\varphi = F_1\left(\hat{\xi} - c_1\hat{t}\right) + F_2\left(\hat{\xi} + c_1\hat{t}\right) - \frac{c_2\hat{t}^2\hat{\xi}}{2} + \gamma t^2 a,$$

where $\hat{t} = t$, $\hat{\xi} = \xi + a$, and c_1 , c_2 are constants.

Some examples of the results (with chosen conditions) on invariant solutions of Eqs. (8) corresponding to the generators and pressure functions of **Table 1** are presented in **Table 2**.

TABLE 2. Results of invariant solutions of Eqs. (8).

	$P(\xi,\varphi_\xi)$	**Extensions**	**Invariant solution**
M1	$\xi\varphi_\xi$	$t\partial_t + 2\xi\partial_\xi + 2\varphi\partial_\varphi$	$\varphi = F_1(\hat{t})F_2(\hat{\xi})e^{-2a}$, $F_1''(\hat{t}) = c_1F_1(\hat{t})$, $F_2''(\hat{\xi}) = \frac{1}{\hat{\xi}}\left(c_1F_2(\hat{\xi}) - F_2'(\hat{\xi})\right)$, $\hat{t} = te^a$, $\hat{\xi} = \xi e^{2a}$
M2	$e^\xi\varphi_\xi$	$t\partial_t - 2\partial_\xi + 2\varphi\partial_\varphi$	$\varphi = F_1(\hat{t})F_2(\hat{\xi})e^{-2a}$, $F_1''(\hat{t}) = c_1F_1(\hat{t})$, $F_2''(\hat{\xi}) = -c_1e^{-\hat{\xi}}F_2(\hat{\xi}) - F_2'(\hat{\xi})$, $\hat{t} = te^a$, $\hat{\xi} = \xi - 2a$
M3	$\phi(\varphi_\xi) + \beta\xi + \gamma\xi^2$	$\partial_\xi - \gamma t^2\partial_\varphi$	$\varphi = F_1(\hat{\xi} - c_1\hat{t}) + F_2(\hat{\xi} + c_1\hat{t}) - \frac{c_2\hat{t}^2\hat{\xi}}{2} + \gamma t^2 a$, $\hat{t} = t$, $\hat{\xi} = \xi + a$
M4	$\phi(\varphi_\xi)$	$t\partial_t + \xi\partial_\xi + \varphi\partial_\varphi$	$\varphi = F_1(\hat{\xi} - c_1\hat{t}) + F_2(\hat{\xi} + c_1\hat{t})$, $\hat{t} = te^a$, $\hat{\xi} = \xi e^a$
M5	$e^\xi\varphi_\xi^2$	$\partial_\xi - \varphi\partial_\varphi$	$\varphi = F_1(\hat{t})F_2(\hat{\xi})e^a$, $F_1''(\hat{t}) = c_1F_1^2(\hat{t})$, $F_2''(\hat{\xi}) = -\left(\frac{c_1F_2(\hat{\xi})e^{-\hat{\xi}} + \left(F_2'(\hat{\xi})\right)^2}{2F_2'(\hat{\xi})}\right)$, $\hat{t} = t$, $\hat{\xi} = \xi + a$
M6	$\xi\varphi_\xi^\gamma + \xi^2$	$t\partial_t + 2\gamma\xi\partial_\xi + 2(\gamma+1)\varphi\partial_\varphi$	$\varphi = \left(F_1(\hat{t}) + F_2(\hat{\xi})\right)e^{-2(\gamma+1)a}$, $F_1''(\hat{t}) = c_1$, $F_2''(\hat{\xi}) = \left(F_2'(\hat{\xi})\right)^{1-\gamma}\left(2 - \frac{c_1}{\gamma\xi}\right) - \frac{1}{\gamma}F_2'(\hat{\xi})$, $\hat{t} = te^a$, $\hat{\xi} = \xi e^{2\gamma a}$
M7	$e^{\beta\xi}\varphi_\xi^\gamma + k_1\xi^2$ $P(\xi,\varphi_\xi)$	$-\beta t\partial_t + 2\gamma\partial_\xi - 2\beta\varphi\partial_\varphi$ **Extensions**	$\varphi = \left(F_1(\hat{t}) + F_2(\hat{\xi})\right)e^{\beta a}$, $F_1''(\hat{t}) = c_1$, $F_2''(\hat{\xi}) = -\frac{e^{-\beta\hat{\xi}}}{\gamma}\left(F_2'(\hat{\xi})\right)^{1-\gamma}(c_1 + 2k_1\hat{\xi}) - \frac{\beta}{\gamma}F_2'(\hat{\xi})$, $\hat{t} = te^{-\beta a}$, $\hat{\xi} = \xi e^{2\gamma a}$ **Invariant solution**
M8	$\varphi_\xi^\gamma + \xi^2$	$\partial_\xi - t^2\partial_\varphi$	$\varphi = F_1(\hat{t}) + F_2(\hat{\xi}) + at^2$, $F_1''(\hat{t}) = c_1$, $F_2''(\hat{\xi}) = -\frac{1}{\gamma}\left(F_2'(\hat{\xi})\right)^{1-\gamma}(c_1 + 2\hat{\xi})$, $\hat{t} = t$, $\hat{\xi} = \xi + a$
M9	$\beta\varphi_\xi^\gamma$	$(\gamma - 1)t\partial_t - 2\varphi\partial_\varphi$	$\varphi = \left(F_1(\hat{t}) + F_2(\hat{\xi})\right)e^{2a}$, $F_1''(\hat{t}) = c_1$, $F_2''(\hat{\xi}) = -\frac{c_1}{\beta\gamma}\left(F_2'(\hat{\xi})\right)^{1-\gamma}$, $\hat{t} = te^{(\gamma-1)a}$, $\hat{\xi} = \xi$
M10	$b(\xi)\varphi_\xi^{-3}$	$t^2\partial_t + t\varphi\partial_\varphi$ $2t\partial_t + \varphi\partial_\varphi$	$\varphi = f(\hat{t},\hat{\xi})(1 - at)$, $\hat{t} = \frac{t}{1 - at}$, $\hat{\xi} = \xi$ $\varphi = f(\hat{t},\hat{\xi})e^{-a}$, $\hat{t} = te^{2a}$, $\hat{\xi} = \xi$

4. CONSERVATION LAWS IN LAGRANGIAN COORDINATES

Using the groups of transformations presented in **Table 1** and Noether's theorem, one can derive the conserved vector, T^1 and T^2, in Lagrangian coordinates for each type of the pressure functions, $P(\xi, \varphi_\xi)$ in Eqs. (8).

The conservation laws are considered in the form

$$D_t T^1 + D_\xi T^2 = 0, \tag{17}$$

where the functions $T^1 = T^1(t, \xi, \varphi, \varphi_t, \varphi_\xi)$ and $T^2 = T^2(t, \xi, \varphi, \varphi_t, \varphi_\xi)$ are listed in **Table** 3.

Calculations show that the generator $X = t\partial_t + \xi\partial_\xi + \varphi\partial_\varphi$ in model M_4, the generator $X = (\gamma-1)\partial_\xi - \beta\varphi\partial_\varphi$ in model M_5, the generator $(2l-1)t\partial_t + 2l\gamma\xi\partial_\xi + 2(2l-1+l\gamma)\varphi\partial_\varphi$ in M_6 and the generator $X = -\beta t\partial_t + 2\gamma\partial_\xi - 2\beta\varphi\partial_\varphi$ in model M_7 are not variational and do not allow the conservation laws to be derived.

TABLE 3. The conserved vectors in Lagrangian coordinate of Eqs. (8).

	T^1 **and** T^2	**Remarks**
M1	$T^1 = -t\xi\varphi_\xi h'(\xi) + (\alpha+1)\left(\varphi\varphi_t - t\varphi_\xi h(\xi)\right) - (\alpha+3/2)t\varphi_t^2 - \xi\varphi_t\varphi_\xi + (2\alpha+3)t\xi^{-2\alpha-4}\phi\left(\xi^{-\alpha}\varphi_\xi\right),$ $T^2 = t\xi\varphi_t h'(\xi) + (\alpha+1)(\varphi + t\varphi_t)h(\xi) + \xi\varphi_t^2/2 + \xi^{-2\alpha-3}\phi\left(\xi^{-\alpha}\varphi_\xi\right)$ $+\left((\alpha+1)\varphi - (2\alpha+3)t\varphi_t - \xi\varphi_\xi\right)\xi^{-3\alpha-4}\phi'\left(\xi^{-\alpha}\varphi_\xi\right).$	$\gamma \neq -2\alpha-2$
M2	$T^1 = -t\varphi_\xi h'(\xi) - 2\alpha t e^{2\alpha\xi}\phi\left(e^{\alpha\xi}\varphi_\xi\right) - \alpha\left(\varphi\varphi_t - t\varphi_\xi h(\xi) - t\varphi_t^2\right) - \varphi_t\varphi_\xi,$ $T^2 = t\varphi_t h'(\xi) - \alpha(\varphi + t\varphi_t)h(\xi) + \varphi_t^2/2 + e^{2\alpha\xi}\phi\left(e^{\alpha\xi}\varphi_\xi\right) - \alpha\left(\varphi - 2t\varphi_t + \frac{1}{\alpha}\varphi_\xi\right)e^{3\alpha\xi}\phi'\left(e^{\alpha\xi}\varphi_\xi\right).$	$\beta = 2\alpha$
M3	$T^1 = \varphi_t\varphi_\xi + \gamma t^2\varphi_t + 2\gamma t\xi\varphi_\xi,$ $T^2 = \beta\left(\xi + \gamma t^2\xi\right) - \varphi_t^2/2 - 2\gamma t\varphi_t + \gamma^2 t^2\xi^2 - \phi\left(\varphi_\xi\right) + \left(\varphi_\xi + \gamma t^2\right)\phi'\left(\varphi_\xi\right).$	
M4	$X = \partial_\xi \quad T^1 = \varphi_t\varphi_\xi, \quad T^2 = -\varphi_t^2/2 - \phi\left(\varphi_\xi\right) + \varphi_\xi\phi'\left(\varphi_\xi\right).$	
M5	$X = (\gamma-1)t\partial_\varphi \quad T^1 = \frac{2}{\varphi_\xi^2}\left(-t\gamma e^{\beta\xi} + \varphi\varphi_\xi^2 - t\varphi_t\varphi_\xi^2\right), \quad T^2 = \frac{2\gamma e^{\beta\xi}(\varphi - 2t\varphi_t)}{\varphi_\xi^3}.$	$\gamma \neq -1$
M8	$X = t\partial_t + \gamma\xi\partial_\xi + (\gamma+2)\varphi\partial_\varphi$ $T^1 = \beta t\left(2 - \ln\varphi_\xi\right) - \varphi\varphi_t + \frac{t}{2}\varphi_t^2 - \xi\varphi_t\varphi_\xi - k_1 t\xi^2\varphi_\xi,$ $T^2 = \beta\xi\left(\ln\varphi_\xi - 1\right) - \frac{\beta}{\varphi_\xi}(\varphi - t\varphi_t) + k_1\xi^2(t\varphi_t - \varphi) + \frac{\xi}{2}\varphi_t^2,$	$\gamma = -1$
M9	$X = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi$ $T^1 = 2\beta t\left(\ln\varphi_\xi - 1\right) + 2\varphi\varphi_t - t\varphi_t^2 + 2\xi\varphi_t\varphi_\xi, \quad T^2 = -2\beta\xi\ln\varphi_\xi + \frac{2\beta}{\varphi_\xi}(\varphi - t\varphi_t) - \xi\varphi_t^2,$ $X = (\gamma-1)\xi\partial_\xi + (\gamma+1)\varphi\partial_\varphi \quad T^1 = -2\beta t, \quad T^2 = 2\beta\xi$ $X = \partial_\xi \quad T^1 = \varphi_t\varphi_\xi, \quad T^2 = \beta\left(1 - \ln\varphi_\xi\right) - \frac{1}{2}\varphi_t^2,$	$\gamma = -1$
M10	$X = t^2\partial_t + t\varphi\partial_\varphi, \; T^1 = \frac{t^2 b(\xi)}{2\varphi_\xi^2} + \frac{\varphi^2}{2} - t\varphi\varphi_\xi + \frac{t^2}{2}\varphi_t^2, \quad T^2 = \frac{tb(\xi)(t\varphi_t - \varphi)}{\varphi_\xi^3}.$ $X = 2t\partial_t + \varphi\partial_\varphi, \; T^1 = \frac{tb(\xi)}{\varphi_\xi^2} - \varphi\varphi_\xi + t\varphi_t^2, \quad T^2 = \frac{b(\xi)(2t\varphi_t - \varphi)}{\varphi_\xi^3}.$	

4.1 Relations between conservation laws in Lagrangian and Eulerian coordinates

The operators of the total derivatives in Lagrangian and Eulerian coordinates are related as follows

$$D_\xi = \varphi_\xi D_x, \quad D_t = \varphi_t D_x + D_{\tilde{t}}, \tag{18}$$

where " ~ " is used in order to distinguish time in Eulerian coordinates from time in Lagrangian coordinates. Because the variables $\rho(t,x)$ and $u(t,x)$ are considered in Eulerian coordinates, omitting tilde ~ in further study is not misleading.

For T^1 and T^2 are the conserved vectors in Lagrangian coordinates and are considered in the form

$$D_t T^1 + D_\xi T^2 = 0.$$

By the definition of velocity $u = \varphi_t$ and density $\rho = \varphi_\xi^{-1}$, one has that

$$u_x = \varphi_{t\xi}\varphi_\xi^{-1}, \tag{19}$$

and

$$\begin{aligned}
D_t T^1 + D_\xi T^2 &= D_t\left(\varphi_\xi \rho T^1\right) + D_\xi T^2 \\
&= \varphi_{t\xi}\left(\rho T^1\right) + \varphi_\xi D_t\left(\rho T^1\right) + D_\xi T^2 \\
&= \varphi_{t\xi}\varphi_\xi^{-1} T^1 + \varphi_\xi\left(u D_x\left(\rho T^1\right) + D_{\tilde{t}}\left(\rho T^1\right)\right) + \varphi_\xi D_x T^2 \\
&= \varphi_{t\xi}\varphi_\xi^{-1} T^1 + \varphi_\xi\left(D_x\left(\rho u T^1\right) - u_x\left(\rho T^1\right) + D_{\tilde{t}}\left(\rho T^1\right)\right) + \varphi_\xi D_x T^2 \\
&= \left(\varphi_{t\xi}\varphi_\xi^{-1} - u_x\right) T^1 + \varphi_\xi\left(D_x\left(\rho u T^1 + T^2\right) + D_{\tilde{t}}\left(\rho T^1\right)\right) \\
&= \varphi_\xi D_x\left(\rho u T^1 + T^2\right) + D_{\tilde{t}}\left(\rho T^1\right).
\end{aligned}$$

Thus, the conserved vector in Eulerian coordinates $\left(T^t, T^x\right)$ can be derived as

$$T^t = \rho T^1, \quad T^x = \rho u T^1 + T^2.$$

Some of corresponding conserved vectors in Eulerian coordinates become:

For model M_4, the generator $X = \partial_\xi$,

$$T^t = u, \;\; T^x = \frac{u^2}{2} + \rho^{-1}\phi'\left(\rho^{-1}\right) - \phi\left(\rho^{-1}\right).$$

For model M_9, the generator $X = \partial_\xi$,

$$T^t = u, \;\; T^x = \frac{u^2}{2} + \beta\left(1 + \ln(\rho)\right),$$

the conservation laws of M_4 and M_9 were discussed in [16].

For model M_5, the generator $X = (\gamma - 1)t\partial_\varphi$,

$$T^t = 2\left(\rho(x - tu) - \gamma t \rho^3 e^{\beta\xi}\right),$$

$$T^x = 2\left(\rho u(x - tu) + \gamma\rho^3 e^{\beta\xi}(u - 3tu)\right),$$

with the restriction of $\gamma = 0$ this conservation law obtained in [17] and it is called the center of mass conservation law.

5. CONCLUSION

The gas dynamics equations are studied here and with a natural Lagrangian in the Lagrangian coordinates having the form of the Euler-Lagrange equation.

A group classification of the Euler-Lagrange equation in the Lagrangian coordinates with respect to the pressure function $P(\xi, \varphi_\xi)$ under the conditions $P_\xi \neq 0, P_{\varphi_\xi} < 0$ and $P_{\varphi_\xi \varphi_\xi} \neq 0$ is performed. The kernel of admitted Lie algebras $X_1 = \partial_t$, $X_2 = t\partial_\varphi$ and $X_3 = \partial_\varphi$ makes it possible to derive the known conservation laws, i.e. the energy, center of mass and momentum conservation laws (the mass conservation law is satisfied by virtue of choosing the Lagrangian coordinates). The extensions of the kernel of admitted lie algebras are found in the group classification with respect to many types of pressure function $P(\xi, \varphi_\xi)$, one allows the application of Noether's theorem to derived additional conservation laws in Lagrangian coordinates. Moreover the possibilities extend an area of using Group analysis for constructing invariant solutions of the Euler-Lagrange equation. A result of this study some exact invariant solutions corresponding to some subalgebras are presented.

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